Solutions Assessed Example Sheet 3 . MSM3A05/MSM4A05.

Question 1.

(a) We are considering the boundary value problem

$$\epsilon(2y + y'') + 2xy' - 4x^2 = 0 \quad \text{for} \quad -1 \le x \le 2, \tag{1}$$

subject to

$$y(-1) = 2, \quad y(2) = 7,$$

with $\epsilon \ll 1$.
We first consider whether there can be a boundary layer at $x = -1$. We re-scale so that we have $x = -1+\epsilon^\lambda X$ (where $\lambda > 0$ so that we are magnifying the region around $x = -1$). Our equation then becomes

$$\epsilon(2Y + \epsilon^{-2\lambda} Y_{XX}) + 2(-1 + \epsilon^\lambda)\epsilon^{-\lambda} Y_X - 4(-1 + \epsilon^\lambda X)^2 = 0. \tag{2}$$

Seeking a distinguished limit we find that to keep the most terms (including the highest derivative) we need $\lambda = 1$ for a leading order balance, and so

$$Y_{XX} - 2Y_X = 0 \Rightarrow Y \sim A + Be^{2X}. \tag{3}$$

However, this solution grows exponentially as $X \to \infty$ (since we are considering whether we can have a boundary layer at the left hand limit of x to match to our outer solution (which we haven't found yet) we will need to consider the limit $X \to \infty$). Thus, irrespective of the outer solution we cannot match this inner solution to it since Y becomes unbounded for large X. For the case $x = 2$ we follow a similar procedure except in this case we find that at leading order we have

$$Y_{XX} + 2Y_X = 0 \Rightarrow Y \sim A + Be^{-2X}, \tag{4}$$

which also grows exponentially as $X \to -\infty$ (note this time we need to let $X \to -\infty$ to match with the outer solution). Consequently, we cannot have a boundary layer at $x = -1$ or $x = 2$.
The re-scaling needed for an interior layer at $x = 0$ can be found by re-scaling $x = \epsilon^\lambda X$, $y = Y$ and substituting into the differential equation. Our equation becomes

$$\epsilon(2Y + \epsilon^{-2\lambda} Y_{XX}) + 2XY_X - 4\epsilon^{2\lambda} X^2 = 0. \tag{5}$$

In order to obtain a balance of terms we require $\lambda = \frac{1}{2}$ so that $x = \epsilon^{\frac{1}{2}} X$. Our equation now becomes

$$Y_{XX} + 2XY_X + 2\epsilon Y - 4\epsilon X^2 = 0. \tag{6}$$

(b) To find the leading order solution away from this interior layer we consider the leading order part of (9) which is

$$2xy' - 4x^2 = 0 \Rightarrow y' = 2x \Rightarrow y = x^2 + C \tag{7}$$

We will have different constant depending on whether we match to the left hand side or the right hand side boundary condition. These can be solved to find

$$y(x) = \begin{cases} x^2 + 1 & : \quad -1 \le x < 0, \\ x^2 + 3 & : \quad 0 < x < 2, \end{cases} \tag{8}$$

(c) The leading order inner solution requires us to solve (14) at leading order, that is

$$Y_{XX} + 2XY_X = 0 \Rightarrow \frac{d}{dX}\left(e^{X^2} Y_X\right) = 0 \Rightarrow Y_X = Ae^{-X^2} \Rightarrow Y = B + A\int_0^X e^{-s^2} ds \tag{9}$$

For $x > 0$ we know that $y \to 3$ as $x \to 0^+$ and for $x < 0$ we know that $y \to 1$ as $x \to 0^-$. Hence we can find these constants without resorting to the method used in lecture (although that way is safer). Hence we must have $Y \to 3$ as $X \to \infty$ and $Y \to 1$ and $X \to -\infty$ and so

$$Y = 1 + \frac{2}{\sqrt{\pi}} \int_{-\infty}^X e^{-s^2} ds. \tag{10}$$

(d) For this part of the question we want to know what $y(x)$ is at $x = 0$, hence we have

$$y(0) = Y(0) \sim 1 + \frac{2}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-s^2} ds = 2, \tag{11}$$

as required[1].

(e) A sketch of the leading order solution is given below. Note that the interior is so narrow it can hardly be seen (in lectures the boundary layer is drawn big to explain the details). This is in line with the whole idea of boundary layers where variables change over a small interval - vanishingly small.

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[1] There are other ways of getting this result.