Solutions Assessed Example Sheet 2. MSM3A05/MSM4A05.

Question 1.

$$I(x)=\int_0^{2\pi} e^{ix\sin\theta}d\theta=\int_0^{\pi} e^{ix\sin\theta}d\theta+\int_\pi^{2\pi} e^{ix\sin\theta}d\theta=I_1+I_2.$$

For I_1 we have a stationary point in the interval $[0,\pi]$ at $\theta=\pi/2$. Hence we seek an expansion of $\sin\theta$ about $\theta=\pi/2$. This is given by

$$\sin\theta=1-\frac{1}{2}(\theta-\pi/2)^2+\cdots.$$

Hence

$$I_1\sim\int_0^{\pi} e^{ix\left(1-\frac{1}{2}(\theta-\pi/2)^2+\cdots\right)}d\theta\sim e^{ix}\int_0^{\pi} e^{\frac{-ix}{2}(\theta-\pi/2)^2}d\theta.$$

if we let $u=\sqrt{(x/2)}(\theta-\pi/2)$ we have

$$I_1\sim e^{ix}\sqrt{\frac{2}{x}}\int_{-\pi/2}^{\pi/2} e^{-iu^2}du\sim e^{ix}\sqrt{\frac{2}{x}}\int_{-\infty}^{\infty} e^{-iu^2}du.$$

We have that $\int_{-\infty}^{\infty} e^{-iu^2}du=\sqrt{\pi}e^{-i\pi/4}$ and so

$$I_1\sim e^{i(x-\pi/4)}\sqrt{\frac{2\pi}{x}}\sim\sqrt{\frac{2\pi}{x}}\left(\cos(x-\pi/4)+i\sin(x-\pi/4)\right).$$

We can do a similar thing for I_2. in this case a stationary point in the interval $[\pi,2\pi]$ occurs at $\theta=3\pi/2$. Hence we seek an expansion of $\sin\theta$ about $\theta=3\pi/2$. This is given by

$$\sin\theta=-1+\frac{1}{2}(\theta-3\pi/2)^2+\cdots.$$

whence we have

$$I_2\sim\int_\pi^{2\pi} e^{ix\left(-1+\frac{1}{2}(\theta-3\pi/2)^2+\cdots\right)}d\theta\sim e^{-ix}\int_\pi^{2\pi} e^{\frac{ix}{2}(\theta-3\pi/2)^2}d\theta.$$

whence if we let $s=\sqrt{(x/2)}(\theta-3\pi/2)$ we have

$$I_2\sim e^{-ix}\sqrt{\frac{2}{x}}\int_{-\pi/2}^{\pi/2} e^{is^2}ds\sim e^{-ix}\sqrt{\frac{2}{x}}\int_{-\infty}^{\infty} e^{is^2}ds.$$

We have that $\int_{-\infty}^{\infty} e^{is^2}ds=\sqrt{\pi}e^{i\pi/4}$ and so

$$I_2\sim e^{i(-x+\pi/4)}\sqrt{\frac{2\pi}{x}}\sim\sqrt{\frac{2\pi}{x}}\left(\cos(-x+\pi/4)+i\sin(-x+\pi/4)\right).$$

By adding I_1 and I_2 we have

$$I\sim 2\sqrt{\frac{2\pi}{x}}\cos(x-\pi/4)\sim\sqrt{\frac{8\pi}{x}}\cos(x-\pi/4),$$

as required.

Question 2.

(a)If we assume that a boundary layer exists at the origin then we can form an outer solution

$$y\sim y^o=y_0(x)+\epsilon y_1(x)+\cdots$$

which when substituted into our differential equation gives at leading order

$$x^n y_0' - x^m y_0 = 0. \tag{1}$$

Since this is the outer solution it must satisfy the boundary condition $y_0(1) = \beta$. The solution to this equation depends upon n and m and is given by

$$y_0 = \beta x, \quad \text{if} \quad n = m + 1, \tag{2}$$

or

$$y_0 = \beta \exp\left[\frac{x^{m-n+1} - 1}{m - n + 1}\right], \quad \text{if} \quad n \neq m + 1. \tag{3}$$

(b) To determine an inner expansion we seek a re-scaling such that we have $x = \epsilon^\lambda X$ where $\lambda > 0$. In the boundary layer we seek the expansion $y = Y = Y_0(X) + \epsilon Y_1(X) + \cdots$ whence our differential equation becomes

$$\epsilon^{1-2\lambda}\frac{d^2Y}{dX^2} + \epsilon^{(n-1)\lambda}X^n\frac{dY}{dX} - \epsilon^{m\lambda}X^m Y = 0. \tag{4}$$

We now search for distinguished limits of this equation (noting that in each case we are seeking to keep the highest derivative and as much terms as possible). It turns out there are three separate cases to consider

- $\lambda = (m+2)^{-1}$,

$$\frac{d^2Y}{dX^2} + X^{m+1}\frac{dY}{dX} - X^m Y = 0, \quad \text{if} \quad n - m = 1 \quad \text{and} \quad m \neq -2 \tag{5}$$

- $\lambda = (n+1)^{-1}$,

$$\frac{d^2Y}{dX^2} + X^n\frac{dY}{dX} = 0, \quad \text{if} \quad n - m < 1 \quad \text{and} \quad n \neq -1 \tag{6}$$

- $\lambda = (m+2)^{-1}$,

$$\frac{d^2Y}{dX^2} - X^m Y = 0, \quad \text{if} \quad n - m > 1 \quad \text{and} \quad m \neq -2 \tag{7}$$

(c) There are no distinguished limits (and therefore no boundary layer at the origin) when

- $n = -1$ and $m \in (-2, \infty)$
- $m = -2$ and $n \in [-1, \infty)$

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