Solutions to Problem Sheet 5. MSM3A05/MSM4A05.

Question 2.

We wish to determine a two term expansion of

$$\epsilon y'' + (x+1)y' + y = 2x, \quad 0 \le x \le 1, \quad 0 < \epsilon \ll 1, \quad y(0) = 1, \quad y(1) = 2.$$

(a)We begin by considering the two term outer expansion

$$y = y_0(x) + \epsilon y_1(x)$$

substitution into the differential equation yields

$$(x+1)y_0' + y_0 = 2x, \qquad y_0(1) = 2, \tag{1}$$

$$(x+1)y_1' + y_1 = -y_0'', \qquad y_1(1) = 0. \tag{2}$$

At leading order we have

$$y_0 = \frac{x^2+3}{x+1}.$$

At next order the equation may be written as

$$\frac{d}{dx}((x+1)y_1)' = -\frac{d}{dx}\left(\frac{2x}{x+1} - \frac{x^2+3}{(x+1)^2}\right),$$

which when used with the boundary condition we have

$$y_1(x) = -\frac{2x}{(x+1)^2} + \frac{x^2+3}{(x+1)^3}.$$

Consequently the two term outer solution is given by

$$y(x) = \frac{x^2+3}{x+1} + \epsilon\left[-\frac{2x}{(x+1)^2} + \frac{x^2+3}{(x+1)^3}\right].$$

(b)The appropriate scaling is $x = \epsilon^\lambda X$ and we can show that $\lambda = 1$ from which we have

$$Y_{XX} + (1+\epsilon X)Y_X + \epsilon Y = 2\epsilon^2 X.$$

Using a two term inner expansion we have

$$Y(X) = Y_0(X) + \epsilon Y_1(X) + \cdots$$

whence we have

$$Y_{0XX} + Y_{0X} = 0, \qquad Y_0(0) = 1, \tag{3}$$

$$y_{1XX} + Y_{1X} = -XY_{0X} - Y_0, \qquad Y_1(0) = 0. \tag{4}$$

The solution of the $O(1)$ is given by $Y_0 = A + (1-A)e^{-X}$. The $O(\epsilon)$ equation is

$$y_{1XX} + Y_{1X} = X(1-A)e^{-X} - A - (1-A)e^{-X}.$$

The complementary function is $C + De^{-X}$ and the particular integral will contain the sum αXe^{-X}, βX^2e^{-X} and γX. Substitution into the differential equation yields $\alpha = 0$, $\beta = (A-1)/2$ and $\gamma = -A$. The general solution is then

$$Y_1 = C + De^{-X} - AX + \frac{1}{2}(A-1)X^2e^{-X}.$$

The boundary condition $Y_1(0) = 0$ requires $D = -C$ from which the two term inner expansion is given by

$$Y = A + (1-A)e^{-X} + \epsilon\left[C(1-e^{-X}) - AX + \frac{1}{2}(A-1)X^2e^{-X}\right].$$

(c) We use the matching technique we are familiar with and we have

$$\begin{aligned}
Y &= A + (1-A)e^{-X} + \epsilon\left[C(1-e^{-X}) - AX + \frac{1}{2}(A-1)X^2e^{-X}\right], && (5)\\
Y &= A + (1-A)e^{-x/\epsilon} + \epsilon\left[C(1-e^{-x/\epsilon}) - \frac{Ax}{\epsilon} + \frac{1}{2}(A-1)\left(\frac{x}{\epsilon}\right)^2 e^{-x/\epsilon}\right], && (6)\\
Y &= A + \epsilon\left[C - \frac{Ax}{\epsilon}\right]. && (7)
\end{aligned}$$

and similarly

$$\begin{aligned}
y(x) &= \frac{x^2+3}{x+1} + \epsilon\left[-\frac{2x}{(x+1)^2} + \frac{x^2+3}{(x+1)^3}\right], && (8)\\
y(x) &= \frac{(\epsilon X)^2+3}{(\epsilon X)+1} + \epsilon\left[-\frac{2\epsilon X}{((\epsilon X)+1)^2} + \frac{(\epsilon X)^2+3}{((\epsilon X)+1)^3}\right], && (9)\\
y(x) &= 3 - 3\epsilon X + 3\epsilon + \cdots. && (10)
\end{aligned}$$

We compare (7) and (10) for which we need to change x in (7) into X whence we have

$$A + \epsilon\left[C - \frac{Ax}{\epsilon}\right] = A + C\epsilon - A\epsilon X = 3 - 3\epsilon X + 3\epsilon.$$

We see that $A = 3$ and $C = 3$. Consequently the two term composite solution is

$$\begin{aligned}
y^c = {} & \frac{x^2+3}{x+1} + \epsilon\left[-\frac{2x}{(x+1)^2} + \frac{x^2+3}{(x+1)^3}\right] + 3 - 2e^{-x/\epsilon} + \\
& \epsilon\left(3(1-e^{-x/\epsilon}) - \frac{3x}{\epsilon} + \frac{x^2}{\epsilon^2}e^{-x/\epsilon}\right) - 3 + 3x - 3\epsilon. && (11)
\end{aligned}$$

JU 05/11/08.