Solutions to Problem Sheet 4. MSM3A05/MSM4A05.

Question 1.

We have that

$$J_0(x) = \frac{1}{\pi}\mathcal{R}e\left\{\int_0^\pi e^{ix\cos\theta}d\theta\right\}.$$

Here we have that $g(\theta) = \cos\theta$ and $g'(\theta) = -\sin\theta$ and so there are two stationary points, one at $\theta = 0$ and the other at $\theta = \pi$. We therefore have $g(\theta) = 1 - \theta^2/2 + \cdots$ and $g(\theta) = -1 + \frac{1}{2}(\theta-\pi)^2 + \cdots$ at the two stationary points respectively. Consequently, we have

$$J_0(x) \sim \frac{1}{\pi}\mathcal{R}e\left\{\int_0^\delta e^{ix(1-\theta^2/2+\cdots)}d\theta\right\} + \frac{1}{\pi}\mathcal{R}e\left\{\int_{\pi-\delta}^\pi e^{ix(-1+\frac{1}{2}(\theta-\pi)^2+\cdots)}d\theta\right\}.$$

We can simplify the above integrals by letting the limits go to infinity and minus infinity respectively and choosing $u = \sqrt{x/2}\theta$ and $v = \sqrt{x/2}(\theta - \pi)$ in the first and second integral respectively, we then have

$$J_0(x) \sim \frac{1}{\pi}\sqrt{\frac{2}{x}}\mathcal{R}e\left\{\int_0^\infty e^{ix-iu^2}du\right\} + \frac{1}{\pi}\sqrt{\frac{2}{x}}\mathcal{R}e\left\{\int_{-\infty}^0 e^{-ix+iv^2}dv\right\}.$$

Using the identities from our notes we have

$$J_0(x) \sim \frac{1}{\pi}\sqrt{\frac{2}{x}}\mathcal{R}e\left\{e^{ix}\frac{\sqrt{\pi}}{2}e^{-i\pi/4}\right\} + \frac{1}{\pi}\sqrt{\frac{2}{x}}\mathcal{R}e\left\{e^{-ix}\frac{\sqrt{\pi}}{2}e^{i\pi/4}\right\},$$

and thus

$$J_0(x) \sim \frac{1}{\sqrt{2\pi x}}\left[\cos\left(x - \frac{\pi}{4}\right) + \cos\left(-x + \frac{\pi}{4}\right)\right].$$

Since $\cos\theta = \cos(-\theta)$ we have

$$J_0(x) \sim \frac{1}{\sqrt{2\pi x}}\left[\cos\left(x - \frac{\pi}{4}\right) + \cos\left(x - \frac{\pi}{4}\right)\right] = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right).$$

Question 2.

We have

$$I(x) = \int_a^b f(t)e^{ixg(t)}dt.$$

Since $g(t)$ has a stationary point at $t = t_0$ and $g''(t_0) \neq 0$ we have

$$g(t) = g(t_0) + \frac{1}{6}g'''(t_0)(t-t_0)^3.$$

Thus we have

$$I(x) = \int_{t_0-\delta}^{t_0+\delta} f(t_0)e^{ixg(t_0)}e^{(ix/6)g'''(t_0)(t-t_0)^3}dt \sim f(t_0)e^{ixg(t_0)}\int_{-\infty}^\infty e^{(ix/6)g'''(t_0)(t-t_0)^3}dt.$$

If we let $\Omega = sgn(g'''(t_0))$ then we can choose a transformation $u = (t - t_0)\left((x/6)|g'''(t_0)|\right)^{\frac{1}{3}}$ to get

$$I(x) \sim \frac{f(t_0)e^{ixg(t_0)}}{\left((x/6)|g'''(t_0)|\right)^{\frac{1}{3}}}\int_{-\infty}^\infty e^{i\Omega u^3}du.$$

We have that

$$\int_{-\infty}^\infty e^{\pm iu^3}du = \frac{2}{3}e^{\pm i\pi/6}\Gamma\left(\frac{1}{3}\right).$$

(See examples class for explanation why).
We therefore have

$$I(x) \sim f(t_0)e^{ixg(t_0)}\left(\frac{6}{x|g'''(t_0)|}\right)^{\frac{1}{3}}\frac{2}{3}e^{\Omega i\pi/6}\Gamma\left(\frac{1}{3}\right).$$

Question 3.

Here we can use the ideas from Question 2. In this case we have $g(t) = \sin t - t$ and $g'(t) = \cos t - 1$ and so a stationary point occurs at $t = 0$ thus we have

$$g(t) = 0 + 0 + 0 - \frac{t^3}{6} + \cdots .$$

Hence we have

$$J_n(n) = \frac{1}{\pi}\mathcal{R}e\left\{\int_0^{\pi} e^{in(\sin t - t)}dt\right\} = \frac{1}{\pi}\mathcal{R}e\left\{\int_0^{\pi} e^{-int^3/6}dt\right\}.$$

If we use the substitution $u = t(n/6)^{\frac{1}{3}}$ we have

$$J_n(n) \sim \frac{1}{\pi}\left(\frac{6}{n}\right)^{\frac{1}{3}}\mathcal{R}e\left\{\int_0^{\infty} e^{-iu^3}du\right\} = \frac{1}{\pi}\left(\frac{6}{n}\right)^{\frac{1}{3}}\mathcal{R}e\left\{\frac{2}{3}e^{-i\pi/6}\Gamma\left(\frac{1}{3}\right)\right\} = \frac{\Gamma\left(\frac{1}{3}\right)}{3\pi}\left(\frac{6}{n}\right)^{\frac{1}{3}}\cos(\pi/6).$$

Question 4.

Substitution of the expansion into the differential equation leads to the following equations

$$\begin{aligned}
\epsilon^0 &: \quad (1+x)y_0' + y_0 = 0,\\
\epsilon^1 &: \quad y_0'' + (1+x)y_1' + y_1 = 0,\\
\epsilon^2 &: \quad y_1'' + (1+x)y_2' + y_2 = 0,
\end{aligned}$$

If we use the boundary condition $y(1) = 1$ we have the solutions

$$\begin{aligned}
y_0(x) &= \frac{2}{1+x},\\
y_1(x) &= \frac{2}{(1+x)^3} - \frac{1}{2(1+x)},\\
y_2(x) &= \frac{6}{(1+x)^5} - \frac{1}{2(1+x)^3} - \frac{1}{4(1+x)}.
\end{aligned}$$

JU 21/11/10.