Revision Notes For Perturbation Theory: Summer 2013

Order Symbols and Sequences

We say $f(x)$ is order or $O(g(x))$ as $x \to x_0$ if the limit

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = A,$$

where A is a nonzero finite constant.

Watson's Lemma

$$I(x) = \int_0^b e^{-xt} t^{\lambda} f(t) dt, \qquad (b > 0), \tag{1}$$

where

- (i) $\lambda > -1$,
- (ii) $f(t)$ is exponentially bounded in the interval $0 \leq t \leq b$,
- (iii) $f(t)$ possesses a Maclaurin series expansion.

If these conditions are met then we may say that

$$I(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{\Gamma(\lambda + n + 1)}{x^{\lambda+n+1}}, \quad \text{as} \quad x \to \infty. \tag{2}$$

Firstly, condition (i) above is the extent to which the integrand may be singular near $t = 0$ while ensuring the existence of the integral. Condition (ii) can be expressed as $|f(t)| < Ke^{ct}$ when $t \in [0, b]$ for some fixed constants K and c. This ensures that the term e^{-xt} dominates the integrand away from the lower limit (which is $t = 0$ here) when the x is large. This means that when $b = \infty$, we may have $f(t) = e^t, e^{10t}$, etc. but it does not[1] allow $f(t) = e^{t^2}$. An exponential bound also avoids having expressions of the form $f(t) = 1/(1-t)$ for $t \geq 1$. Condition (iii) allows $f(t)$ to be replaced by an expression which describes its behaviour in the region of dominant contribution ($t = 0$).

Expansion with a small parameter

Consider the function

$$f(x; \epsilon) = \frac{x\sqrt{1 + \epsilon x}}{x + \epsilon}, \qquad x \geq 0, \tag{3}$$

with $\epsilon \to 0^+$ for $x = O(1)$ we write

$$f(x; \epsilon) = \left(1 + \frac{\epsilon}{x}\right)^{-1} (1 + \epsilon x)^{\frac{1}{2}}. \tag{4}$$

Two applications of the binomial expansion yields

$$f(x; \epsilon) \sim 1 + \epsilon \left(\frac{x}{2} - \frac{1}{x}\right) \qquad \epsilon \to 0^+. \tag{5}$$

The domain of f is $x \geq 0$ and so we must consider $x \to 0$ and $x \to \infty$; in either case the asymptotic expansion (5) breaks down. For $x \to 0$ the breakdown occurs where $x = O(\epsilon)$ (which is taken from $\epsilon/x = O(1)$); for $x \to \infty$, the breakdown is where $x = O(\epsilon^{-1})$ (which is taken from $\epsilon x = O(1)$.

[1] This is because we would then have a term like $e^{-xt} \cdot e^{t^2}$ in which the term e^{t^2} dominates the term e^{-xt} as $t \to \infty$.

Laplace Method

Consider the integral

$$I(x) = \int_a^b e^{-\frac{1}{t-a}} e^{-x(t-a)^2} dt, \qquad a < b, \tag{6}$$

for large positive x. Now since $e^{-1/(t-a)}$ tends to zero much faster than any power of $(t-a)$, the contribution to the integral from the neighbourhood of $t = a$ is exponentially small so application of Watson's lemma *does not* yield its asymptotic form. We should not lose all hope since progress can be made if we first rewrite $I(x)$ as

$$I(x) = \int_a^b e^{h(x,t)} dt, \tag{7}$$

where $h(x,t) = -(t-a)^{-1} - x(t-a)^2$. Now since $h'(t) = (t-a)^{-2} - 2x(t-a)$, stationary values occur when $(t-a)^{-2} = 2x(t-a)$, i.e, $t = a + (2x)^{-\frac{1}{3}}$. The location of this maximum is a function of x so we need to transform the variable of integration so that the maximum of the exponent is independent of x. Let $t - a = x^{-\frac{1}{3}} s$ and so we can rewrite $I(x)$ as

$$I(x) = x^{-\frac{1}{3}} \int_0^{(b-a)x^{\frac{1}{3}}} e^{-x^{\frac{1}{3}}(s^2 + \frac{1}{s})} ds = x^{-\frac{1}{3}} \int_0^{(b-a)x^{\frac{1}{3}}} e^{-x^{\frac{1}{3}} h(s)} ds. \tag{8}$$

The maximum value of $h(s) = -(s^2 + \frac{1}{s})$ occurs at $s = 2^{-\frac{1}{3}}$. The details are as follows, $h'(s) = -2s + 1/s^2$ hence $h'(s) = 0$ when $2s = 1/s^2$ or $s^3 = \frac{1}{2}$, i.e., $s = 2^{-\frac{1}{3}}$, now $h''(s) = -2 - 2/s^3$, so $h''(2^{-\frac{1}{3}}) = -2 - 2/2^{-1} = -6$ and therefore $h(s)$ does indeed have a relative maximum at $s = 2^{-\frac{1}{3}}$.
We now expand $h(s)$ about this maximum so that we have

$$\begin{aligned} h(s) &= h(2^{-\frac{1}{3}}) + h''(2^{-\frac{1}{3}}) \frac{(s - 2^{-\frac{1}{3}})^2}{2!} + \cdots \\ &= -2^{\frac{1}{3}} \frac{3}{2} - 3(s - 2^{-\frac{1}{3}})^2 + \cdots \end{aligned} \tag{9}$$

We now let $\tau^2 = 3x^{\frac{1}{3}}(s - 2^{-\frac{1}{3}})^2$ so that $ds = d\tau/\sqrt{3x^{\frac{1}{3}}}$. After substituting into (8) and replacing upper and lower integration limits by ∞ and $-\infty$ we have

$$\begin{aligned} I(x) &\sim \frac{e^{-\frac{3}{2}(2x)^{\frac{1}{3}}}}{\sqrt{3x}} \int_{-\infty}^{\infty} e^{-\tau^2} d\tau \\ &\sim \left(\frac{\pi}{3x^{\frac{1}{3}}} \right)^{\frac{1}{2}} e^{-\frac{3}{2}(2x)^{\frac{1}{3}}} \end{aligned} \tag{10}$$

Boundary Layers

We begin with an example,
Consider the differential equation below

$$\epsilon y'' + xy' + y = 0, \qquad -4 \le x \le -2, \qquad 0 < \epsilon \ll 1, \quad y(-4) = 1, \quad y(-2) = 0.$$

(a) Assume that the boundary layer is located at $x = -2$. Write down a one term outer solution.
(b) Write down a one term inner expansion.
(c) Match these two expansions to find a one term composite solution.
This has the following solution... (a)

$$xy' + y = 0 \quad y(-4) = 1,$$

which has solution

$$y = \frac{-4}{x}.$$

(b) In this case we let $x = -2 + \epsilon^{\lambda} X$ and find that the equation now reads

$$\epsilon^{1-2\lambda} Y_{XX} + (-2 + \epsilon^{\lambda} X)\epsilon^{-\lambda} Y_X + Y = 0,$$

from which it follows that $\lambda = 1$. Therefore we have

$$Y_{XX} - 2Y_X = 0,$$

which may be solved at leading order to yield

$$Y_0 = Ae^{2X} + B,$$

which on using the inner boundary condition becomes $Y_0 = A(e^{2X} - 1)$.
(c) In order to match we note that

$$\lim_{x \to -2} \frac{-4}{x} = \lim_{X \to -\infty} A(e^{2X} - 1)$$

and so $2 = -A$ or $A = -2$ and hence we have the composite solution

$$y \sim \frac{-4}{x} - 2e^{2X} \sim \frac{-4}{x} - 2e^{\frac{2x+4}{\epsilon}}.$$

Next we will examine the following problem for this part of the course

$$\epsilon y'' + yy' - y = 0 \quad \text{for} \quad 0 \leq x \leq 1,$$

subject to

$$y(0) = \alpha, \qquad y(1) = \beta,$$

where α and β are constants with $\epsilon \ll 1$.

(a) Assume that a boundary layer exists at $x = 0$. Find the leading order outer and inner solution when $\alpha = 0$ and $\beta = 3$.
(b) Assume that an interior layer exists at $x = x_0$. Find the leading order outer and inner solution, and hence, show that $x_0 = 1/2$ when $\alpha = -1$ and $\beta = 1$.
Which has solution....
a) Assuming that a boundary layer exists at $x = 0$ we have for a one term outer solution the need for solving

$$y_0(y_0' - 1) = 0, \qquad y_0(1) = 3, \qquad \Rightarrow y_0 = x + 2.$$

For the inner solution we require the scaling $x = \epsilon^{\lambda} X$, $y = Y$ which gives the differential equation

$$\epsilon^{1-2\lambda} Y_{XX} + \epsilon^{-\lambda} Y Y_X - Y = 0. \tag{11}$$

Seeking a distinguished limit we find that to keep the most terms (including the highest derivative) we need $\lambda = 1$ for a leading order balance, and so at leading order we have

$$Y_{0XX} + YY_{0X} = 0. \qquad \Rightarrow Y_{0X} + \frac{Y_0^2}{2} = K, \tag{12}$$

where K is a constant. Since the outer solution has $y_0 \to 2$ as $x \to 0$ we need $Y_0 \to 2$ and $Y_{0X} \to 0$ as $X \to \infty$. This means that $K = 2$ and so

$$Y_{0X} + \frac{Y_0^2}{2} = 2, \tag{13}$$

and so

$$\frac{dY_0}{dX} = \frac{4 - Y_0^2}{2} \Rightarrow \frac{dY_0}{4 - Y_0^2} = \frac{1}{2} dX \Rightarrow \int \left(\frac{1}{4(2 - Y_0)} + \frac{1}{4(2 + Y_0)} \right) dY_0 = \frac{X}{2} + C,$$

where C is a constant, consequently we have

$$\ln \frac{2 + Y_0}{2 - Y_0} = 2X + C.$$

Using $Y_0(0) = 0$ we have $C = 0$ and so we may write

$$Y_0(X) = 2\frac{e^{2X} - 1}{e^{2X} + 1} = 2\tanh(X).$$

(b) If we are now given the boundary conditions $y(0) = -1, y(1) = 1$ and we assume that the boundary layer is located at $x_0 \in (0, 1)$ then we will have two outer solutions

$$y_0(x) = \begin{cases} x - 1 & : \quad 0 \le x < x_0, \\ x & : \quad x_0 < x \le 1, \end{cases} \tag{14}$$

For the inner solution we require the scaling $x = x_0 + \epsilon^\lambda X$, $y = Y$ which gives the differential equation

$$\epsilon^{1-2\lambda} Y_{XX} + \epsilon^{-\lambda} Y Y_X - Y = 0. \tag{15}$$

Seeking a distinguished limit we find that to keep the most terms (including the highest derivative) we need $\lambda = 1$ for a leading order balance, and so at leading order we have

$$Y_{0XX} + YY_{0X} = 0. \qquad \Rightarrow Y_{0X} + \frac{Y_0^2}{2} = K. \tag{16}$$

Now the outer solution has $y \to x_0$ as $x \to x_0^+$ and $y \to x_0 - 1$ as $x \to x_0^-$. Thus for matching we need as $X \to \infty$, $Y_0 \to x_0, Y_{0X} \to 0$ and as $X \to -\infty$ $Y_0 \to x_0 - 1$ and $Y_{0X} \to 0$. Thus we have

$$K = \frac{(x_0 - 1)^2}{2} = \frac{x_0^2}{2} \Rightarrow x_0 = \frac{1}{2}, K = \frac{1}{8}.$$

We thus have

$$Y_{0X} + \frac{Y_0^2}{2} = \frac{1}{8}, \tag{17}$$

and so

$$\frac{dY_0}{dX} = \frac{\frac{1}{4} - Y_0^2}{2} \Rightarrow \frac{dY_0}{\frac{1}{4} - Y_0^2} = \frac{1}{2}dX \Rightarrow \int \left(\frac{1}{(\frac{1}{2} - Y_0)} + \frac{1}{\frac{1}{2} + Y_0} \right) dY_0 = \frac{X}{2} + C,$$

where C is a constant, consequently we have

$$\ln \frac{\frac{1}{2} + Y_0}{\frac{1}{2} - Y_0} = \frac{X}{2} + C.$$

Using $Y_0(1/2) = 0$[2]we have $C = 0$ and so we may write

$$Y_0(X) = \frac{e^{X/2} - 1}{2e^{X/2} + 1} = \frac{1}{2}\tanh\left(\frac{X}{4}\right).$$

Now consider another problem given by Consider the problem

$$\epsilon \frac{d^2 y}{dx^2} + x^n \frac{dy}{dx} - x^m y = 0, \quad 0 < \epsilon \ll 1, \quad 0 < x < 1, \quad y(0) = \alpha, \quad y(1) = \beta, \tag{18}$$

where α, β, n and m are real constants.

Assume that a boundary layer exists at $x = 0$.

(a) Find a one term outer solution.

(b) Re-scale using an inner variable and obtain distinguished limits of equation (18) [Hint: there are three distinguished limits].

[2]If we have $Y_0 \to 1/2$ as $X \to \infty$ and $Y_0 \to -1/2$ as $X \to -\infty$ then in the limit $\epsilon \to 0$ the boundary layer will be very thin and will mean that $Y_0(0) = 0$

(c) Find the conditions under which no distinguished limit exists and therefore there is no boundary layer at $x = 0$.
Which has solution.......

(a)If we assume that a boundary layer exists at the origin then we can form an outer solution

$$y \sim y^o = y_0(x) + \epsilon y_1(x) + \cdots$$

which when substituted into our differential equation gives at leading order

$$x^n y_0' - x^m y_0 = 0. \tag{19}$$

Since this is the outer solution it must satisfy the boundary condition $y_0(1) = \beta$. The solution to this equation depends upon n and m and is given by

$$y_0 = \beta x, \quad \text{if} \quad n = m + 1, \tag{20}$$

or

$$y_0 = \beta \exp\left[\frac{x^{m-n+1} - 1}{m - n + 1}\right], \quad \text{if} \quad n \neq m + 1. \tag{21}$$

(b) To determine an inner expansion we seek a re-scaling such that we have $x = \epsilon^\lambda X$ where $\lambda > 0$. In the boundary layer we seek the expansion $y = Y = Y_0(X) + \epsilon Y_1(X) + \cdots$ whence our differential equation becomes

$$\epsilon^{1-2\lambda}\frac{d^2Y}{dX^2} + \epsilon^{(n-1)\lambda} X^n \frac{dY}{dX} - \epsilon^{m\lambda} X^m Y = 0. \tag{22}$$

We now search for distinguished limits of this equation (noting that in each case we are seeking to keep the highest derivative and as much terms as possible). It turns out there are three separate cases to consider

- $\lambda = (m+2)^{-1}$,

$$\frac{d^2Y}{dX^2} + X^{m+1}\frac{dY}{dX} - X^m Y = 0, \quad \text{if} \quad n - m = 1 \quad \text{and} \quad m \neq -2 \tag{23}$$

- $\lambda = (n+1)^{-1}$,

$$\frac{d^2Y}{dX^2} + X^n \frac{dY}{dX} = 0, \quad \text{if} \quad n - m < 1 \quad \text{and} \quad n \neq -1 \tag{24}$$

- $\lambda = (m+2)^{-1}$,

$$\frac{d^2Y}{dX^2} - X^m Y = 0, \quad \text{if} \quad n - m > 1 \quad \text{and} \quad m \neq -2 \tag{25}$$

(c) There are no distinguished limits (and therefore no boundary layer at the origin) when

- $n = -1$ and $m \in (-2, \infty)$
- $m = -2$ and $n \in [-1, \infty)$

Multiple Scales

We will go through the following question for this part of the course.
Consider the equation

$$\frac{d^2u}{dt^2} + \omega_0^2 u = \epsilon u \left(\frac{du}{dt}\right)^2, \quad 0 < \epsilon \ll 1, \quad t > 0, \tag{26}$$

where ω_0 is a real positive constant.
(a) Seek a straightforward expansion of the form $u = u_0(t) + \epsilon u_1(t) + \cdots$ and after substituting into (26) obtain a set of differential equations involving $u_0(t)$ and $u_1(t)$.

(b) Show that $u_0(t) = a\cos(\omega_0 t + b)$ satisfies your differential equation for $u_0(t)$ above (where a and b

are constants).

(c) Now, using the solution for $u_0(t)$ in (b) or otherwise, solve the differential equation in (a) to find an expression for $u_1(t)$. You may use the identity $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$.

(d) Hence, using $u_0(t)$ and $u_1(t)$ write down a two term expansion for $u(t)$ and comment on the region for which this expansion is not valid (i.e., the region of non-uniformity). Explain why this is so (one line answer needed only).

(e) Use the expansion $u = u_0(t,T) + \epsilon u_1(t,T) + \cdots$ where $T = \epsilon t$ and substitute into (26) and equate coefficients of ϵ to form a differential equation for $u_0(t,T)$ and $u_1(t,T)$.

(f) Show that $u_0(t,T) = A(T)e^{i\omega_0 t} + \bar{A}(T)e^{-i\omega_0 t}$ (where $\bar{A}$ is the complex conjugate of A) is a solution to the first differential equation obtained in (e).

(g) Using the second differential equation obtained in (e) find an expression involving A and $\bar{A}$ which must be satisfied in order to eliminate secular terms.

Answer: (a)Using the expansion given we find, after collecting terms in ϵ

$$\ddot{u}_0 + \omega_0^2 u_0 = 0, \tag{27}$$
$$\ddot{u}_1 + \omega_0^2 u_1 = \dot{u}_0^2 u_0. \tag{28}$$

(b) Simple exercise.
(c) We have

$$\begin{aligned} \ddot{u}_1 + \omega_0^2 u_1 &= a^3\omega_0^2 \cos(\omega_0 t + \beta)\sin^2(\omega_0 t + \beta) \\ &= \frac{a^3\omega_0^2}{4}\left[\cos(\omega_0 t + \beta) - \cos(3\omega_0 t + 3\beta)\right] \end{aligned} \tag{29}$$

A particular solution for u_1 is given by

$$u_1 = \frac{1}{8}a^3\omega_0 t \sin(\omega_0 t + \beta) + \frac{1}{32}a^3\cos(3\omega_0 t + 3\beta). \tag{30}$$

(d)

$$u = a\cos(\omega_0 t + \beta) + \frac{\epsilon a^3}{32}\left[4\omega_0 t \sin(\omega_0 t + \beta) + \cos(3\omega_0 t + 3\beta)\right] + \cdots \tag{31}$$

This equation is not valid for $t \geq O(\epsilon^{-1})$ since the second term becomes larger than the first in this interval.
(e) Using the expansion given we have

$$\frac{\partial^2 u_0}{\partial t^2} + \omega_0^2 u_0 = 0, \tag{32}$$
$$\frac{\partial^2 u_1}{\partial t^2} + \omega_0^2 u_1 = -2\frac{\partial^2 u_0}{\partial t \partial T} + \left(\frac{\partial u_0}{\partial t}\right) u_0. \tag{33}$$

(f) Again a simple exercise.
(g) Using (f) we have

$$\frac{\partial^2 u_1}{\partial t^2} + \omega_0^2 u_1 = -2i\omega_0 A'(T)e^{i\omega_0 t} + \omega_0^2 A^2(T)\bar{A}(T)e^{i\omega_0 t} - \omega_0^2 A^3(T)e^{3i\omega_0 t} + cc \tag{34}$$

We want to avoid secular terms which arise if the coefficient of $e^{i\omega_0 t}$ is non-zero. Hence we require,

$$2i\frac{dA}{dT} - \omega_0^2 A^2\bar{A} = 0. \tag{35}$$

Next Consider the differential equation

$$\ddot{u} + \omega_0^2 u = u^5, \qquad 0 <\ll 1.$$

Use the method of multiple scales to determine a first uniform approximation.
We begin by writing out the expansion of u in two variables

$$u = u_0(t, T) + \epsilon u_1(t, T) + \cdots,$$

where $T = \epsilon t$ and rewriting the differential equation using **two** independent variables such that we have at leading order

$$\frac{\partial^2 u_0}{\partial t^2} + \omega_0^2 u_0 = 0, \tag{36}$$

$$\frac{\partial^2 u_1}{\partial t^2} + \omega_0^2 u_1 = -2\frac{\partial^2 u_0}{\partial t \partial T} + \dot{u}_0^5. \tag{37}$$

A solution of the first equation may be written as

$$u_0 = A(T)e^{i\omega_0 t} + \bar{A}(T)e^{-i\omega_0 t},$$

whence the second equation above becomes

$$\frac{\partial^2 u_1}{\partial t^2} + \omega_0^2 u_1 = -2i\omega_0 A' e^{i\omega_0 t} + 10A^3\bar{A}^2 + c.c + \text{non secular terms.}$$

We seek to eliminate the secular terms so we require

$$i\omega_0 A' = 5A^3\bar{A}^2,$$

which when we substitute $A = R(T)e^{i\theta(T)}$ to separate out real and imaginary parts we have

$$\frac{\partial R}{\partial T} = 0,$$

$$-\omega_0 \frac{\partial \theta}{\partial T} = 5R^4.$$

From the first equation we have R must be a constant, say $R = \alpha$, whilst the second equation then reveals that

$$\theta = -\frac{5}{\omega_0}\alpha^4 T + \beta,$$

hence we have a first order uniform approximation as

$$\begin{aligned} u &\sim \alpha \exp\left(i\left(-\frac{5}{\omega_0}\alpha^4 T + \beta\right) + i\omega_0 t\right) + \alpha \exp\left(-i\left(-\frac{5}{\omega_0}\alpha^4 T + \beta\right) - i\omega_0 t\right), \\ &\sim \alpha \cos\left(\left(1 - \frac{5}{\omega_0^2}\epsilon\alpha^4\right)\omega_0 t + \beta\right). \end{aligned}$$