Unavoidable trees in tournaments

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Oriented tree $T$ on $n$ vertices, tournament $G$
Tournaments & Oriented Trees

Oriented tree $T$ on $n$ vertices, tournament $G$

Is there a copy of $T$ in $G$?

$|V(T)| = n \leq |V(G)|$
Oriented tree $T$ on $n$ vertices, tournament $G$

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Tournaments & Oriented Trees

Oriented tree $T$ on $n$ vertices, tournament $G$

Is there a copy of $T$ in $G$?

Definition (unavoidable trees)
A (oriented) tree $T$ with $|V(T)| = n$ is unavoidable if every tournament on $n$ vertices contains a copy of $T$. 

$|V(T)| = n \leq |V(G)|$
Unavoidable trees — examples

Directed paths (Rédei 1934)

\[
\bullet \rightarrow \bullet \rightarrow \bullet \cdots \bullet \rightarrow \bullet
\]
Unavoidable trees — examples

Directed paths (Rédei 1934)  

All large paths (Thomason '86)
Unavoidable trees — examples

Directed paths (Rédei 1934) \[ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \bullet \rightarrow \bullet \]

All large paths (Thomason ’86)

All paths, 3 exceptions (Havet & Thomassé ’98)
Unavoidable trees — examples

Directed paths (Rédei 1934)

All large paths (Thomason ’86)

All paths, 3 exceptions (Havet & Thomassé ’98)

Some claws (Saks & Sós 84; Lu ’93; Lu, Wang & Wong ’98)

\[
\left\{ \begin{array}{c}
\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \\
\bullet \rightarrow \bullet \rightarrow \cdots \\
\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \\
\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \\
\end{array} \right\} \leq \left( \frac{3}{8} + \frac{1}{200} \right) n \text{ branches}
\]
Examples — non-unavoidable trees

\[ n - 2 \]

And 5 vertices is not in 3-regular:

\[ 2 \cdot 5 - 3 \]
Examples — non-unavoidable trees

\[ n - 2 \]

is not in \[ n - 3 \]
Examples — non-unavoidable trees

is not in

And

5 vertices
Examples — non-unavoidable trees

\[ n - 2 \]

is not in \[ n - 3 \]

And

5 vertices

is not in

3-regular
Examples — non-unavoidable trees

\[ n - 2 \]

is not in \[ n - 3 \]

And

5 vertices

is not in

3-regular
Examples — non-unavoidable trees

$n - 2$ is not in $n - 3$

And

5 vertices is not in 3-regular
Examples — non-unavoidable trees

\[
\begin{align*}
&n - 2 \\
&\text{is not in} \\
&n - 3
\end{align*}
\]

And

\[
\begin{align*}
&\text{is not in} \\
&5 \text{ vertices}
\end{align*}
\]

3-regular
Examples — non-unavoidable trees

And

5 vertices

is not in

3-regular: $2 \cdot 5 - 3$ vertices

is not in
Conjecture and proofs

Sumner’s conjecture (1971)
Every oriented tree on \( n \) vertices is contained in every tournament on \( 2n - 2 \) vertices.
Conjecture and proofs

Sumner’s conjecture (1971)
Every oriented tree on \( n \) vertices is contained in every tournament on \( 2n - 2 \) vertices.

<table>
<thead>
<tr>
<th>publ.</th>
<th>who</th>
<th>tournament size</th>
</tr>
</thead>
<tbody>
<tr>
<td>1982</td>
<td>Chung</td>
<td>( n^{1+o(n)} )</td>
</tr>
<tr>
<td>1983</td>
<td>Wormald</td>
<td>( n \log_2(2n/e) )</td>
</tr>
<tr>
<td>1991</td>
<td>Häggkvist &amp; Thomason</td>
<td>( 12n ) and also ( (4 + o(n))n )</td>
</tr>
<tr>
<td>2002</td>
<td>Havet</td>
<td>( 38n/5 - 6 )</td>
</tr>
<tr>
<td>2000</td>
<td>Havet &amp; Thomassé</td>
<td>( (7n - 5)/2 )</td>
</tr>
<tr>
<td>2004</td>
<td>El Sahili</td>
<td>( 3n - 3 )</td>
</tr>
<tr>
<td>2011</td>
<td>Kühn, Mycroft &amp; Osthus</td>
<td>( 2n - 2 ) for large ( n )</td>
</tr>
</tbody>
</table>
Embedding bounded-degree trees

Theorem (Kühn, Mycroft & Osthus, 2011)

For all $\alpha, \Delta > 0$ there exists $n_0$ such that if $n > n_0$, each tournament on $(1 + \alpha)n$ vertices contains any tree $T$ on $n$ vertices with $\Delta(T) \leq \Delta$. 
When can we do better?

Question (Alon)
Which trees are unavoidable?
When can we do better?

**Question (Alon)**

Which trees are unavoidable?

Paths,
When can we do better?

Question (Alon)
Which trees are unavoidable?

Paths, some claws

\[ \leq \left( \frac{3}{8} + \frac{1}{200} \right) n \text{ branches} \]
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Paths, some claws

\[ \leq \left( \frac{3}{8} + \frac{1}{200} \right) n \text{ branches} \]
A family of examples – alternating trees

Alternating trees are rooted trees $B_\ell$

$B_1$:

```
  ●
  r(B_1)
```

Theorem (Mycroft, N. 2016): For $\ell$ large enough, $B_\ell$ is unavoidable.
A family of examples – alternating trees

Alternating trees are rooted trees $B_\ell$

$B_1: \quad r(B_1)$

$B_{i+1}: \quad r(B_i)$

$B_i$
A family of examples – alternating trees

Alternating trees are rooted trees $\mathcal{B}_\ell$

$\mathcal{B}_1$: $\bullet$ \quad $r(\mathcal{B}_1)$

$\mathcal{B}_i + 1$: $r(\mathcal{B}_i)$ $\mathcal{B}_i$ $r(\mathcal{B}_i)$

$\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are unavoidable:
A family of examples – alternating trees

Alternating trees are rooted trees $\mathcal{B}_\ell$

$\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are unavoidable:

Theorem (Mycroft, N. 2016$^+$)

For $\ell$ large enough, $\mathcal{B}_\ell$ is unavoidable.
More examples – balanced $q$-ary trees

$q$-ary tree are rooted trees $\mathbb{B}^q_{\ell}$ $q \in \mathbb{N}$

$\mathbb{B}^q_1$: \[ \bullet \]

$r(\mathbb{B}^q_1)$

Theorem (Mycroft, N. 2016)

For each $q \in \mathbb{N}$, if $\ell$ large enough then almost all orientations of $\mathbb{B}^q_{\ell}$ are unavoidable.

The method works a much wider class of trees.
More examples – balanced $q$-ary trees

$q$-ary tree are rooted trees $B^q_\ell \quad q \in \mathbb{N}$

$B^q_1$: \[ r(B^q_1) \]

$B^q_{i+1}$: \[ r(B^q_i) \]

Theorem (Mycroft, N. 2016) For each $q \in \mathbb{N}$, if $\ell$ large enough then almost all orientations of $B^q_\ell$ are unavoidable. The method works a much wider class of trees.
More examples – balanced $q$-ary trees

$q$-ary tree are rooted trees $B_{q}^{\ell} \quad q \in \mathbb{N}$

$B_{1}^{q}$: $r(B_{1}^{q})$

$B_{i+1}^{q}$: $r(B_{i+1}^{q})$

Theorem (Mycroft, N. 2016$^+$)

For each $q \in \mathbb{N}$, if $\ell$ large enough then almost all orientations of $B_{\ell}^{q}$ are unavoidable.
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The method works a much wider class of trees.
Some definitions and a property of $\mathbb{B}_\ell$

$\mathbb{B}_2$ is a cherry:
Some definitions and a property of $\mathcal{B}_\ell$

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Some definitions and a property of $\mathcal{B}_\ell$

$\mathcal{B}_2$ is a cherry:

\begin{center}
\begin{tikzpicture}
  \node (center) {centre};
  \node (inleaf) [below left of=center] {in-leaf};
  \node (outleaf) [below right of=center] {out-leaf};
  \draw (center) -- (inleaf);
  \draw (center) -- (outleaf);
\end{tikzpicture}
\end{center}
Some definitions and a property of $\mathcal{B}_\ell$

$\mathcal{B}_2$ is a cherry:

$\mathcal{B}_\ell$ has many pendant cherries
Some definitions and a property of $\mathcal{B}_\ell$

$\mathcal{B}_2$ is a cherry:

$\mathcal{B}_\ell$ has many pendant cherries

out cherry

in cherry
Characterization of large tournaments

Theorem (Kühn, Mycroft, Osthus 2011)

Large tournaments contain either a large strong cut or a large robust expander of linear minimum semidegree.
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Characterization of large tournaments

Theorem (Kühn, Mycroft, Osthus 2011)
Large tournaments contain either a large strong cut or a large robust expander of linear minimum semidegree.

\[ L \rightarrow R \quad \text{or} \quad \text{robust expander of linear semidegree} \]

bad
Characterization of large tournaments

Theorem (Kühn, Mycroft, Osthus 2011)

Large tournaments contain either a large strong cut or a large robust expander of linear minimum semidegree.

Theorem (Kühn, Osthus, Treglown 2010)

A large robust expander of linear minimum semidegree contains a regular cycle of cluster tournaments.
Embedding $\mathcal{B}_\ell$ to $G$ (general scheme)

1. Reserve a small set $S \subseteq G$.
2. Form $T' \subseteq B_\ell$ by removing a few leaves.
3. Embed $T'$ to $G - S$ (uses [KMO '11]).
4. Use $S$ to cover tricky vertices.
5. Use perfect matchings to complete the copy of $B_\ell$.
Embedding $B_\ell$ to $G$ (general scheme)

- reserve a small set $S \subseteq G$
Embedding $\mathcal{B}_\ell$ to $G$ (general scheme)

- reserve a small set $S \subseteq G$
- form $T' \subseteq \mathcal{B}_\ell$ removing a few leaves

$\mathcal{B}_\ell \quad T'$

$G$

$S$
Embedding $\mathcal{B}_\ell$ to $G$ (general scheme)

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Beyond binary trees

**Theorem (R. Mycroft, N., 2016⁺)**

*For all* $q > 0$ there exists $n_0$ such that if $n > n_0$ almost all orientations of every “roughly balanced” $q$-ary tree on $n$ vertices are unavoidable.*
Beyond binary trees

Theorem (R. Mycroft, N., 2016+)

For all $q > 0$ there exists $n_0$ such that if $n > n_0$
almost all orientations of every “roughly balanced” $q$-ary tree on $n$
vertices are unavoidable.

Work in progress

For all $\Delta > 0$ there exists $n_0$ such that for $n > n_0$ almost all
labelled trees $T$ on $n$ vertices with $\Delta(T) \leq \Delta$ are unavoidable.
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Work in progress

For all $\Delta > 0$ there exists $n_0$ such that for $n > n_0$ almost all labelled trees $T$ on $n$ vertices with $\Delta(T) \leq \Delta$ are unavoidable.

- most labelled undirected trees have pendant cherries
- most orientations of a labelled tree have good cherry orientations
Beyond binary trees

Theorem (R. Mycroft, N., 2016\textsuperscript{+})

For all \( q > 0 \) there exists \( n_0 \) such that if \( n > n_0 \)
almost all orientations of every “roughly balanced” \( q \)-ary tree on \( n \)
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Work in progress

For all \( \Delta > 0 \) there exists \( n_0 \) such that for \( n > n_0 \) almost all
labelled trees \( T \) on \( n \) vertices with \( \Delta(T) \leq \Delta \) are unavoidable.

- most labelled undirected trees have pendant cherries
- most orientations of a labelled tree have good cherry orientations

Questions

How about unbounded degree? (hopefully soon!)
How about the binary arborescence?
Quick Reference

Introduction
   Examples
   Sumner
   Back to the main question

Results
   Alternating trees
   $q$-ary trees
   Useful features these trees
   Characterization of Large Tournaments

Proof outline

Further extensions