On generalisations of the Hajnal–Szemerédi theorem

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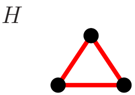
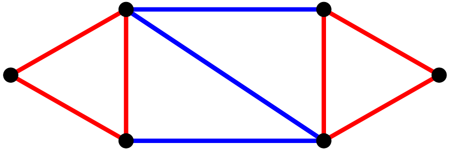
Perfect packings in graphs

- An H-packing in G is a collection of vertex-disjoint copies of H in G.
- An H-packing is perfect if it covers all vertices in G.





perfect H-packing

Remarks

- Perfect H-packings sometimes called H-factors or perfect H-tilings.
- If $H = K_2$ then perfect H-packing $\iff$ perfect matching.
- Decision problem NP-complete (Hell and Kirkpatrick '83).
- Sensible to look for simple sufficient conditions.

Perfect K_r-packings

Theorem (Hajnal, Szemerédi '70)

G graph, $|G| = n$ where $r|n$ and

$$\delta(G) \geq (r-1)\,n/r$$

$\Rightarrow$ G contains a perfect K_r-packing.

- Corrádi and Hajnal ('64) proved triangle case
- Easy to see minimum degree condition tight

- Although condition on $\delta(G)$ in Hajnal-Szemerédi is best possible, we can still ask for more general results!

Theorem (Kierstead, Kostochka '08)

G graph, $|G| = n$ where $r|n$ and

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{r}\right)n - 1 \;\; \forall \text{ non-adjacent } x, y$$

$\Rightarrow$ G contains a perfect K_r-packing.

- Result implies Hajnal-Szemerédi theorem.
- Theorem best possible.

Degree sequence conditions

Conjecture (Balogh, Kostochka and T.)

G graph, $|G| = n$ where $r|n$ with degree sequence $d_1 \leq \cdots \leq d_n$ such that:

(α) $d_i \geq (r-2)n/r + i$ *for all* $i < n/r$;

(β) $d_{n/r+1} \geq (r-1)n/r$.

$\Rightarrow$ *G contains a perfect K_r-packing.*

- If true, stronger than Hajnal-Szemerédi since n/r vertices allowed 'small' degree.
- If true, 'best possible'.

Towards the conjecture

Theorem (T. '14+)

G graph, $|G| = n$ where $r|n$ with degree sequence $d_1 \leq \cdots \leq d_n$ such that:

- $d_i \geq (r-2)n/r + i + o(1)n$ *for all* $i < n/r$.

$\Rightarrow$ *G contains a perfect K_r-packing.*

- Keevash and Knox also have a proof in K_3 case.

Perfect H-packings for general H

Theorem (Alon and Yuster '96)

Let H be a graph with $\chi(H) = r$. Suppose G graph, $|G| = n$ where $|H| | n$ and

$$\delta(G) \geq (1 - 1/r + o(1))n$$

$\Rightarrow$ *G contains a perfect H-packing.*

- Result best-possible up to error term $o(1)n$ for many graphs H.
- Komlós, Sárközy and Szemerédi '01 replaced error term with a constant dependent on H.
- Kühn and Osthus '09 characterised, up to an additive constant, $\delta(G)$ that forces perfect H-packing for *any* H.

A degree sequence Alon-Yuster theorem

Theorem (T. '14+)

Let H be a graph with $\chi(H) = r$. Suppose G graph, $|G| = n$ where $|H| | n$ and with degree sequence $d_1 \leq \cdots \leq d_n$ such that:

- $d_i \geq (r-2)n/r + i + o(1)n$ *for all* $i < n/r$.

$\Rightarrow$ *G contains a perfect H-packing.*

- Bipartite case proven earlier by Knox and T.
- Answers another conjecture of Balogh, Kostochka, T.
- Generalises the Alon-Yuster theorem
- For many H, degree sequence condition 'best possible'.

Theorem (T. '14+)

G graph, $|G| = n$ where $r|n$ with degree sequence $d_1 \leq \cdots \leq d_n$ such that:

- *$d_i \geq (r-2)n/r + i + o(1)n$ for all $i < n/r$.*

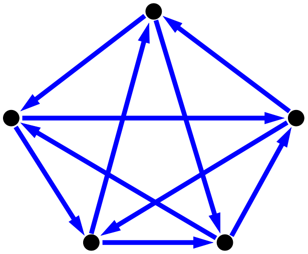
$\Rightarrow$ G contains a perfect K_r-packing.

- Proof uses Absorbing method of Rödl, Ruciński and Szemerédi.

Sketch of a proof sketch:

- Find an absorbing set M in G.
- Find an almost perfect K_r-packing in $G - M$.
- Use M to absorb remaining vertices to obtain a perfect K_r-packing.

- Minimum semi-degree $\delta^0(G) := \min\{\delta^+(G), \delta^-(G)\}$
- Minimum total degree $\delta(G) =$ minimum number of edges incident to a vertex in G
- Tournament: orientation of a complete graph



- T_r = transitive tournament on r vertices
- C_3 = cyclic triangle

Minimum total degree results

Theorem (Czygrinow, DeBiasio, Kierstead and Molla +'13)

G digraph, $|G| = n$ where $r|n$.

$$\delta(G) \geq 2(1 - 1/r)n - 1$$

$\Rightarrow$ G contains a perfect T_r-packing.

Theorem (Czygrinow, DeBiasio, Kierstead and Molla +'13)

G digraph, $|G| = n$ where $r|n$.

$$\delta^+(G) \geq (1 - 1/r)n$$

$\Rightarrow$ G contains a perfect T_r-packing.

- Both minimum degree conditions best-possible.
- Both results imply the Hajnal-Szemerédi theorem.

A minimum semi-degree result

Theorem (T. +'13)

G large digraph, $|G| = n$ where $r|n$. Let T be tournament on r vertices.

$$\delta^0(G) \geq (1 - 1/r)n$$

$\Rightarrow$ G contains a perfect T-packing.

- Minimum semi-degree condition best-possible.
- Earlier, Czygrinow, Kierstead and Molla gave approximate result when $T = C_3$.
- Result implies the Hajnal-Szemerédi theorem for large graphs.

A minimum semi-degree result

Theorem (T. +'13)

G large digraph, $|G| = n$ where $r|n$. Let T be tournament on r vertices.

$$\delta^0(G) \geq (1 - 1/r)n$$

$\Rightarrow$ *G contains a perfect T-packing.*

- Natural to ask if we can replace condition here with $\delta(G) \geq 2(1 - 1/r)n - 1$.
- However, a result of Wang shows we cannot do this for $T = C_3$.

Open problems

Question

G digraph, $|G| = n$ where $r|n$. Let T be tournament on r vertices s.t. $T \neq C_3$. Does

$$\delta(G) \geq 2(1 - 1/r)n - 1$$

$\Rightarrow$ G contains a perfect T-packing?

- Look for analogues in the oriented graph setting.
- Balogh, Lo and Molla have solved the $\delta^0(G)$ problem for transitive triangles.