Perfect packings in graphs and directed graphs

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Embedding subgraphs

Question

How many edges must a graph G contain to guarantee it contains a copy of H?

Theorem (Mantel 1907)

The densest triangle-free graph on n vertices is the complete balanced bipartite graph.

- Turán's theorem (1941) generalises this to all complete graphs.

The Erdős–Stone Theorem

Theorem (Erdős, Stone 1946)

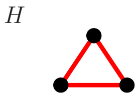
Given $\eta > 0$*, if* G *graph on sufficiently large* n *number of vertices and*

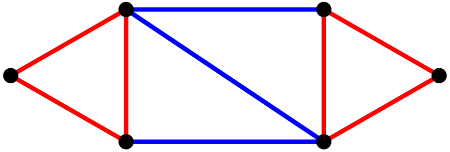
$$e(G) \geq \left(1 - \frac{1}{\chi(H) - 1} + \eta\right) \frac{n^2}{2}$$

then $H \subseteq G$.

Perfect packings in graphs

- An *H*-packing in *G* is a collection of vertex-disjoint copies of *H* in *G*.
- An *H*-packing is perfect if it covers all vertices in *G*.





perfect H-packing

Remarks

- Perfect H-packings sometimes called H-factors or perfect H-tilings.
- If $H = K_2$ then perfect H-packing $\iff$ perfect matching.
- Decision problem NP-complete (Hell and Kirkpatrick '83).
- Problem of determining largest H-packing APX-hard (Kann '94). (That is, impossible to approximate optimum solution within an arbitrary factor unless $P = NP$.)
- Sensible to look for simple sufficient conditions.

Perfect K_r-packings

Theorem (Hajnal, Szemerédi '70)

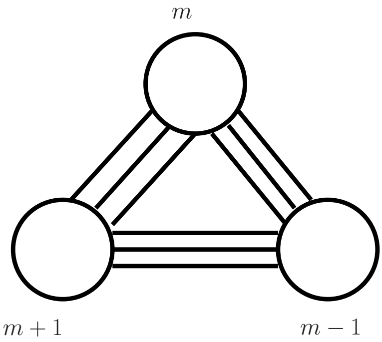
G graph, $|G| = n$ where $r|n$ and

$$\delta(G) \geq (r-1)\, n/r$$

$\Rightarrow$ G contains a perfect K_r-packing.

- Corrádi and Hajnal ('64) proved triangle case
- Kierstead, Kostochka, Mydlarz and Szemerédi '10 found 'fast' algorithmic proof ($O(rn^2)$ running time)

- Hajnal-Szemerédi theorem best possible.



$\delta(G) = 2m - 1 = 2n/3 - 1$ no perfect K_3-packing

- Although condition on $\delta(G)$ in Hajnal-Szemerédi is best possible, we can still ask for more general results!

Theorem (Kierstead, Kostochka '08)

G graph, $|G| = n$ where $r|n$ and

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{r}\right)n - 1 \ \ \forall \text{ non-adjacent } x, y$$

$\Rightarrow$ G contains a perfect K_r-packing.

- Result implies Hajnal-Szemerédi theorem.

- Theorem best possible.

Degree sequence conditions

Conjecture (Balogh, Kostochka and T. '13)

G graph, $|G| = n$ where $r|n$ with degree sequence $d_1 \leq \cdots \leq d_n$ such that:

(α) $d_i \geq (r-2)n/r + i$ *for all* $i < n/r$;

(β) $d_{n/r+1} \geq (r-1)n/r$.

$\Rightarrow$ *G contains a perfect K_r-packing.*

- If true, stronger than Hajnal-Szemerédi since n/r vertices allowed 'small' degree.
- If true, 'best possible'.

Results towards the conjecture

- Balogh, Kostochka, T.: true if no 'small' degree vertex lies in K_{r+1}.
- We also proved other related results.

Theorem (T. '14+)

G graph, $|G| = n$ where $r|n$ with degree sequence $d_1 \leq \cdots \leq d_n$ such that:

- $d_i \geq (r-2)n/r + i + o(1)n$ *for all* $i < n/r$.

$\Rightarrow$ *G contains a perfect K_r-packing.*

Perfect H-packings for general H

Theorem (Alon and Yuster '96)

Let H be a graph with $\chi(H) = r$. Suppose G graph, $|G| = n$ where $|H| | n$ and

$$\delta(G) \geq (1 - 1/r + o(1))n$$

$\Rightarrow$ *G contains a perfect H-packing.*

- Result best-possible up to error term $o(1)n$ for many graphs H.
- Proof algorithmic ($O(n^{2.376})$ running time)
- Komlós, Sárközy and Szemerédi '01 replaced error term with a constant dependent on H.
- Kühn and Osthus '09 characterised, up to an additive constant, $\delta(G)$ that forces perfect H-packing for *any* H.

A degree sequence Alon-Yuster theorem

Theorem (T. '14+)

Let H be a graph with $\chi(H) = r$. Suppose G graph, $|G| = n$ where $|H| | n$ and with degree sequence $d_1 \leq \cdots \leq d_n$ such that:

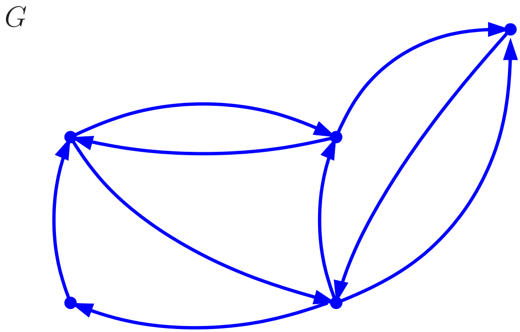
- $d_i \geq (r-2)n/r + i + o(1)n$ *for all* $i < n/r$.

$\Rightarrow$ *G contains a perfect H-packing.*

- Answers another conjecture of Balogh, Kostochka, T.
- Generalises the Alon-Yuster theorem
- For many H, degree sequence condition 'best possible'.

Versions of the Hajnal-Szemerédi theorem for directed graphs

Our digraphs are allowed "double edges".

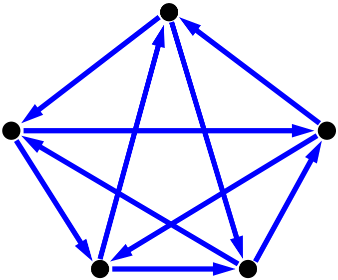


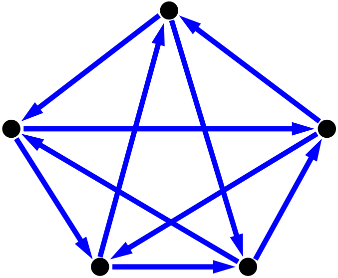
$\delta^+(G) = \delta^-(G) = 1$

Versions of the Hajnal-Szemerédi theorem for directed graphs

What is a natural analogue of the Hajnal-Szemerédi theorem for directed graphs?

- Minimum semi-degree $\delta^0(G) := \min\{\delta^+(G), \delta^-(G)\}$
- Tournament: orientation of a complete graph

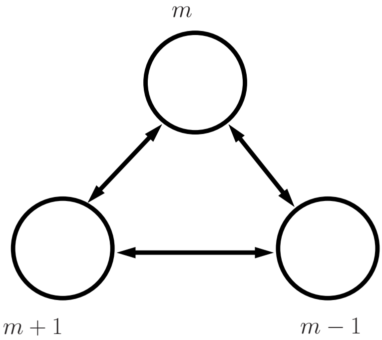




- T_r = transitive tournament on r vertices
- C_3 = cyclic triangle

A guess for an extremal example...

Let T be a tournament on 3 vertices.



$\delta^0(G) = 2m - 1 = 2n/3 - 1$

no perfect T-packing

A minimum semi-degree result

Theorem (T. +'13)

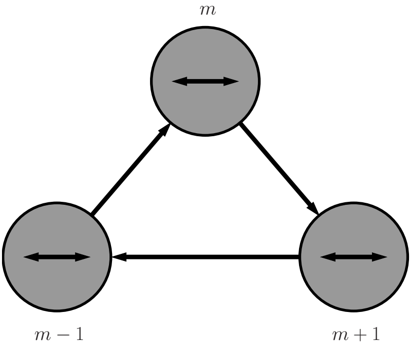
G large digraph, $|G| = n$ where $r|n$. Let T be tournament on r vertices.

$$\delta^0(G) \geq (1 - 1/r)n$$

$\Rightarrow$ G contains a perfect T-packing.

- Our guess was right in this case: minimum semi-degree condition best-possible.
- Earlier, Czygrinow, Kierstead and Molla gave approximate result when $T = C_3$.
- Result implies the Hajnal-Szemerédi theorem for large graphs.

Surprisingly, there is an extra extremal example when $T = C_3$.



$\delta^0(G) = 2m - 2 = 2n/3 - 2$ no perfect C_3-packing

Theorem

G large digraph, $|G| = n$ where $3|n$.

$$\delta^0(G) \geq 2n/3$$

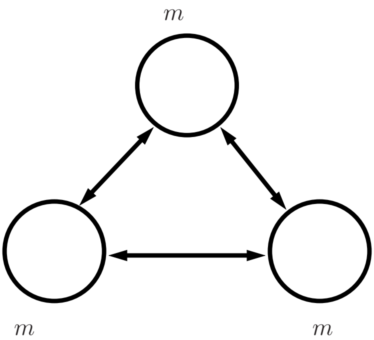
$\Rightarrow$ G contains a perfect T_3-packing.

Absorbing sets: A set $S \subseteq V(G)$ is an absorbing set for $Q \subseteq V(G)$ if both $G[S]$ and $G[S \cup Q]$ contain perfect T_3-packings.

- Suppose we find a 'small' set S that absorbs any 'very small' set $Q \subseteq V(G)$ (where $3||Q|$).
- Then it suffices to show $G \setminus S$ contains an 'almost' perfect T_3-packing.

Proof overview in T_3 case

Problem: May not be able to find such an absorbing set!



Cannot absorb any 3 vertices from same class

Proof overview in T_3 case

However, if G is 'non-extremal' then can find an absorbing set.

Lemma 1

OTFH:

(i) *G is extremal (contains an almost independent set of size $n/3$).*

(ii) *G contains an absorbing set S.*

Proof:

(0) Assume (i) doesn't hold.

(1) For each $x, y \in V(G)$, find many 'connecting structures' between x and y.

(2) Use these connecting structures to find 'local' absorbing sets.

(3) Randomly select 'local' absorbing sets to obtain S.

Proof overview in T_3 case

Lemma 2

G is extremal (contains an almost independent set of size $n/3$) $\Rightarrow$ G contains a perfect T_3-packing.

Proof: Easy!

Lemma 3

G is non-extremal $\Rightarrow$ $G \setminus S$ contains an 'almost' perfect T_3-packing.

Proof: Turn problem into one about hypergraph matchings

The proof in other cases

- Main work is in finding absorbing lemmas.
- Depending on structure of tournament T, we need different arguments.
- Hardest case is C_3 case as there are two extremal examples now.

Open problems

- Prove the degree sequence Hajnal–Szemerédi theorem exactly

Conjecture (Balogh, Kostochka and T. '13)

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$\Rightarrow$ *G contains a perfect K_r-packing.*

Open problems

Given any graph H define c_H to be the smallest number such that every graph G on n vertices with $\delta(G) \geq c_H n$ contains a perfect H-packing.

Question

Let $\gamma > 0$. Given a graph G on n vertices and with $\delta(G) \geq (c_H - \gamma)n$ is the decision problem whether G contains a perfect H-packing NP-complete?

- Kühn and Osthus answered question in affirmative for complete r-partite graphs.
- Look for analogue of Hajnal–Szemerédi theorem in the oriented graph setting.
- Balogh, Lo and Molla answered problem for transitive triangles.

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