

# On perfect packings in dense graphs

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## Abstract

We say that a graph  $G$  has a perfect  $H$ -packing if there exists a set of vertex-disjoint copies of  $H$  which cover all the vertices in  $G$ . We consider various problems concerning perfect  $H$ -packings: Given  $n, r, D \in \mathbb{N}$ , we characterise the edge density threshold that ensures a perfect  $K_r$ -packing in any graph  $G$  on  $n$  vertices and with minimum degree  $\delta(G) \geq D$ . We also give two conjectures concerning degree sequence conditions which force a graph to contain a perfect  $H$ -packing. Other related embedding problems are also considered. Indeed, we give a structural result concerning  $K_r$ -free graphs that satisfy a certain degree sequence condition.

## 1 Introduction

Given two graphs  $H$  and  $G$ , a *perfect  $H$ -packing* in  $G$  is a collection of vertex-disjoint copies of  $H$  which cover all the vertices in  $G$ . Perfect  $H$ -packings are also referred to as  *$H$ -factors* or *perfect  $H$ -tilings*. Hell and Kirkpatrick [10] showed that the decision problem whether a graph  $G$  has a perfect  $H$ -packing is NP-complete precisely when  $H$  has a component consisting of at least 3 vertices. So for such graphs  $H$ , it is unlikely that there is a complete characterisation of those graphs containing a perfect  $H$ -packing. Thus, there has been significant attention on obtaining sufficient conditions that ensure a graph  $G$  contains a perfect  $H$ -packing.

A seminal result in the area is the Hajnal-Szemerédi theorem [9] which states that a graph  $G$  whose order  $n$  is divisible by  $r$  has a perfect  $K_r$ -packing provided that  $\delta(G) \geq (r-1)n/r$ . Kühn and Osthus [15, 16] characterised, up to an additive constant, the minimum degree which ensures a graph  $G$  contains a perfect  $H$ -packing for an arbitrary graph  $H$ .

It is easy to see that the minimum degree condition in the Hajnal-Szemerédi theorem cannot be lowered. Of course, this does not mean that one cannot strengthen this result. *Ore-type* degree conditions consider the sum of the degrees of non-adjacent vertices in a graph. The following Ore-type result of Kierstead and Kostochka [12] implies the Hajnal-Szemerédi theorem.

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**Theorem 1 (Kierstead and Kostochka [12])** *Let  $n, r \in \mathbb{N}$  such that  $r$  divides  $n$ . Suppose that  $G$  is a graph on  $n$  vertices such that for all non-adjacent  $x \neq y \in V(G)$ ,*

$$d(x) + d(y) \geq 2(1 - 1/r)n - 1.$$

*Then  $G$  contains a perfect  $K_r$ -packing.*

Kühn, Osthus and Treglown [17] characterised, asymptotically, the Ore-type degree condition which ensures a graph  $G$  contains a perfect  $H$ -packing for an arbitrary graph  $H$ .

### 1.1 Perfect packings in dense graphs of low minimum degree

It is easy to characterise the edge density that forces a graph  $G$  to contain a perfect  $K_r$ -packing when there are no other restrictions. Indeed, given  $n, r \in \mathbb{N}$  such that  $r \geq 2$  divides  $n$ , if  $G$  is a graph on  $n$  vertices and  $e(G) \geq \binom{n}{2} - n + r$  then  $G$  contains a perfect  $K_r$ -packing. Moreover, if  $G$  is a copy  $K$  of  $K_{n-1}$  together with a vertex which sends precisely  $r - 2$  edges to  $K$ , then  $e(G) = \binom{n}{2} - n + r - 1$  and  $G$  does not contain a perfect  $K_r$ -packing. The following result of Akiyama and Frankl [1] refines this observation.

**Theorem 2 (Akiyama and Frankl [1])** *Let  $n, r \in \mathbb{N}$  such that  $r$  divides  $n$ . Suppose  $G$  is a graph on  $n$  vertices and  $e(\overline{G}) \leq \min\{\binom{n/r+1}{2}, n - r + 1\}$ . Then  $G$  has a perfect  $K_r$ -packing unless  $\overline{G}$  is isomorphic to one of the following graphs:*

- (i) *A copy of  $K_{n/r+1}$  together with  $(1 - 1/r)n - 1$  isolated vertices;*
- (ii) *The disjoint union of  $K_{1, n-r-j+1}$ ,  $j$  edges and  $r-j-2$  isolated vertices, for some  $1 \leq j \leq r-2$ .*

When (for example)  $n \geq r^3$ ,  $\binom{n/r+1}{2} > n - r + 1$ . Hence, in this case Theorem 2 is equivalent to the following: If  $G$  is a graph on  $n$  vertices and  $e(G) \geq \binom{n}{2} - n + r - 1$  then either  $G$  contains a perfect  $K_r$ -packing or  $\overline{G}$  is isomorphic to a graph as in (ii).

In Sections 2 and 3 we consider the following natural problem: Let  $n, r \in \mathbb{N}$  such that  $r$  divides  $n$ . Given some  $D \in \mathbb{N}$ , what edge density condition ensures that any graph  $G$  on  $n$  vertices and of minimum degree  $\delta(G) \geq D$  contains a perfect  $K_r$ -packing?

We fully resolve the problem, and our answers for  $r = 2$  and  $r \geq 3$  differ.

**Theorem 3** *For an even positive  $n$  and integer  $1 \leq d < n/2$ , let  $h(n, d) := \binom{n-d-1}{2} + d(d+1)$  and let  $f(2, n, d)$  denote the maximum integer  $c$  such that some  $n$ -vertex graph with minimum degree at least  $d$  and at least  $c$  edges has no perfect matching. Then*

$$f(2, n, d) = \max\{h(n, d), h(n, 0.5n - 1)\}.$$

**Theorem 4** *Let  $n, r \in \mathbb{N}$  such that  $r \geq 3$  and  $r$  divides  $n$ . Given any  $D \in \mathbb{N}$  such that  $r - 1 \leq D \leq (r - 1)n/r - 1$  define*

$$g(n, r, D) := \max\left\{\binom{n}{2} - \binom{n/r+1}{2}, D(n - D) + \binom{n-1-D}{2} + e(T(D, r-2))\right\}.$$

*Suppose that  $G$  is a graph on  $n$  vertices with  $\delta(G) \geq D$  and  $e(G) > g(n, r, D)$ . Then  $G$  contains a perfect  $K_r$ -packing. Moreover, there exists a graph  $G'$  on  $n$  vertices with  $\delta(G') \geq D$  and  $e(G') = g(n, r, D)$  but such that  $G'$  does not contain a perfect  $K_r$ -packing.*

Clearly a graph  $G$  of minimum degree  $\delta(G) < r - 1$  cannot contain a perfect  $K_r$ -packing. Further, regardless of edge density, every graph  $G$  whose order  $n$  is divisible by  $r$  and with  $\delta(G) \geq (r - 1)n/r$  contains a perfect  $K_r$ -packing. Thus, Theorem 4 covers all values of  $D$  where our problem was not solved previously.

An *equitable  $k$ -colouring* of a graph  $G$  is a proper  $k$ -colouring of  $G$  such that any two colour classes differ in size by at most one. Let  $n, r \in \mathbb{N}$  such that  $r$  divides  $n$ . Notice that a graph  $G$  on  $n$  vertices has a perfect  $K_r$ -packing if and only if the complement  $\overline{G}$  of  $G$  has an equitable  $n/r$ -colouring. So, for example, the Hajnal-Szemerédi theorem can be stated in terms of equitable colourings: Let  $G$  be a graph on  $n$  vertices such that  $r$  divides  $n$ . If  $\Delta(G) \leq n/r - 1$  then  $G$  has an equitable  $n/r$ -colouring.

It is often easier to work in the language of equitable colourings compared to perfect packings. Indeed, rather than prove Theorem 1 directly, Kierstead and Kostochka proved the equivalent statement for equitable colourings. Here we also find it more convenient to work with equitable colourings. Thus, instead of proving Theorem 4 directly we prove the following equivalent result.

**Theorem 5** *Let  $n, r \in \mathbb{N}$  such that  $r \geq 3$  and  $r$  divides  $n$ . Recall that  $T(n, r)$  denotes the Turán graph. Given any  $D \in \mathbb{N}$  such that  $n/r \leq D \leq n - r$  define*

$$f(n, r, D) := \min \left\{ \binom{n/r + 1}{2}, D + e(\overline{T}(n - D - 1, r - 2)) \right\}.$$

*Suppose that  $G$  is a graph on  $n$  vertices with  $\Delta(G) \leq D$  and  $e(G) < f(n, r, D)$ . Then  $G$  has an equitable  $n/r$ -colouring. Moreover, there exists a graph  $G'$  on  $n$  vertices with  $\Delta(G') \leq D$  and  $e(G') = f(n, r, D)$  but such that  $G'$  does not have an equitable  $n/r$ -colouring.*

We prove Theorem 3 and describe extremal constructions for Theorems 4 and 5 in Section 2. That is, we show that the edge density condition in Theorem 4 is best possible for all values of  $D$  such that  $r - 1 \leq D \leq (r - 1)n/r - 1$ . Section 3 contains a proof of Theorem 5.

## 1.2 Degree sequence conditions forcing a perfect packing

Chvátal [4] gave a condition on the degree sequence of a graph which ensures Hamiltonicity: Suppose that  $G$  is a graph on  $n$  vertices and that the degrees of the graph are  $d_1 \leq \dots \leq d_n$ . If  $n \geq 3$  and  $d_i \geq i + 1$  or  $d_{n-i} \geq n - i$  for all  $i < n/2$  then  $G$  is Hamiltonian. The following is a simple consequence of Chvátal's theorem.

**Theorem 6 (Chvátal [4])** *Suppose that  $G$  is a graph on  $n \geq 2$  vertices and the degrees of the graph are  $d_1 \leq \dots \leq d_n$ . If*

$$d_i \geq i \quad \text{or} \quad d_{n-i+1} \geq n - i \quad \text{for all} \quad 1 \leq i \leq n/2$$

*then  $G$  contains a Hamilton path.*

We propose the following conjecture on the degree sequence of a graph which forces a perfect  $K_r$ -packing.

**Conjecture 7** *Let  $n, r \in \mathbb{N}$  such that  $r$  divides  $n$ . Suppose that  $G$  is a graph on  $n$  vertices with degree sequence  $d_1 \leq \dots \leq d_n$  such that:*

( $\alpha$ )  $d_i \geq (r-2)n/r + i$  for all  $i < n/r$ ;

( $\beta$ )  $d_{n/r+1} \geq (r-1)n/r$ .

Then  $G$  contains a perfect  $K_r$ -packing.

Note that Conjecture 7, if true, is much stronger than the Hajnal-Szemerédi theorem since the degree condition allows for  $n/r$  vertices to have degree less than  $(r-1)n/r$ . Further, Proposition 17 in Section 4 shows that the condition on the degree sequence in Conjecture 7 is essentially “best possible”. It is easy to see that Theorem 6 implies Conjecture 7 in the case when  $r = 2$ . We prove the conjecture in the case when  $G$  is additionally  $K_{r+1}$ -free (see Section 5).

If one can prove Conjecture 7, it seems likely it can be used to prove the next conjecture.

**Conjecture 8** *Suppose  $\gamma > 0$  and  $H$  is a graph with  $\chi(H) = r$ . Then there exists an integer  $n_0 = n_0(\gamma, H)$  such that the following holds. If  $G$  is a graph whose order  $n \geq n_0$  is divisible by  $|H|$ , and whose degree sequence  $d_1 \leq \dots \leq d_n$  satisfies*

- $d_i \geq (r-2)n/r + i + \gamma n$  for all  $i < n/r$ ,

then  $G$  contains a perfect  $H$ -packing.

Since first submitting this paper, the third author and Knox [13] have proven Conjecture 8 in the case when  $r = 2$ . (In fact, they have proven a much more general result concerning embedding spanning bipartite graphs of small bandwidth.)

The following result of Erdős [8] characterises those degree sequences which force a copy of  $K_r$  in a graph  $G$ .

**Theorem 9 (Erdős [8])** *Let  $G$  be a graph on  $n$  vertices with degree sequence  $d_1 \leq \dots \leq d_n$ . If  $G$  is  $K_{r+1}$ -free then there is an  $r$ -partite graph  $G'$  on  $n$  vertices whose degree sequence  $d'_1 \leq \dots \leq d'_n$  satisfies*

$$d_i \leq d'_i \text{ for all } i \leq n.$$

In Section 6 we prove the following related structural theorem.

**Theorem 10** *Suppose that  $n, r \in \mathbb{N}$  such that  $n \geq r$  and so that  $r$  divides  $n$ . Let  $G$  be a  $K_{r+1}$ -free graph on  $n$  vertices whose degree sequence  $d_1 \leq \dots \leq d_n$  is such that  $d_{n/r} \geq (r-1)n/r$ . Then  $G \subseteq T(n, r)$ , where  $T(n, r)$  is the complete  $r$ -partite Turán graph on  $n$  vertices; so each vertex class has size  $\lceil n/r \rceil$  or  $\lfloor n/r \rfloor$ .*

## 2 The case $r = 2$ and extremal examples for $r \geq 3$

### 2.1 Perfect matchings in dense graphs

In this section we establish the density threshold that ensures every graph  $G$  on an even number  $n$  of vertices and of minimum degree  $\delta(G) \geq d$  contains a perfect matching. Note that we only consider values of  $d$  such that  $1 \leq d < n/2$ , since if  $\delta(G) \geq n/2$  then  $G$  has a perfect matching, regardless of the edge density.

Recall that  $h(n, d) := \binom{n-d-1}{2} + d(d+1)$ . Note that for a fixed even  $n$ ,  $h(n, d)$  decreases with  $d$  in the interval  $[0, n/3 - 5/6]$  and increases with  $d$  in  $[n/3 - 5/6, 0.5n - 1]$ .

For a positive even  $n$  and an integer  $0 \leq d < n/2$ , let  $A, B$  and  $C$  be disjoint sets with  $|A| = d+1$ ,  $|B| = d$ ,  $|C| = n - 2d - 1$ . Let  $H = H(n, d)$  be the graph with the vertex set  $A \cup B \cup C$  such that  $H[B \cup C] = K_{n-d-1}$ , and each vertex in  $A$  is adjacent to each vertex in  $B$  and to no vertex in  $C$ . So  $H$  does not contain a perfect matching and has exactly  $h(n, d)$  edges.

The examples of  $H(n, d)$  show that  $f(2, n, d) \geq \max\{h(n, d), h(n, 0.5n - 1)\}$ . Thus to derive Theorem 3, it suffices to prove that an  $n$ -vertex graph  $G$  with  $\delta(G) \geq d$  and  $e(G) > \max\{h(n, d), h(n, 0.5n - 1)\}$  has a perfect matching.

Consider such a graph  $G$ . Let  $d_1 \leq \dots \leq d_n$  denote the degree sequence of  $G$ . If  $d_i \geq i$  for all  $1 \leq i \leq n/2$  then Theorem 6 implies that  $G$  contains a perfect matching. Suppose for a contradiction that  $d_{i'} \leq i' - 1$  for some  $1 \leq i' \leq n/2$ . Note that  $i' > d$  as  $\delta(G) \geq d$ .

Let  $A$  denote the set of  $i'$  vertices in  $G$  that correspond to the first  $i'$  terms  $d_1, \dots, d_{i'}$  of the degree sequence. Set  $B := V(G) \setminus A$ . Then

$$e(G[B]) \geq e(G) - i'(i' - 1) > \max\{h(n, d), h(n, 0.5n - 1)\} - i'(i' - 1)$$

since  $d(x) \leq i' - 1$  for all  $x \in A$ . Note that  $\max\{h(n, d), h(n, 0.5n - 1)\} \geq h(n, i' - 1)$  since  $d < i' \leq n/2$ . Therefore,

$$e(G[B]) > \max\{h(n, d), h(n, 0.5n - 1)\} - i'(i' - 1) \geq h(n, i' - 1) - i'(i' - 1) = \binom{n - i'}{2},$$

a contradiction as  $|B| = n - i'$ . Thus,  $d_i \geq i$  for all  $1 \leq i \leq n/2$ , as desired.

## 2.2 Examples for $r \geq 3$

We will give the extremal examples for Theorem 5. Since Theorems 4 and 5 are equivalent, the complements of the extremal graphs for Theorem 5 are the extremal graphs for Theorem 4.

**Proposition 11** *Suppose that  $n, r \in \mathbb{N}$  such that  $r \geq 3$  and  $r$  divides  $n$ . Then there exists a graph  $G_1$  on  $n$  vertices such that  $\Delta(G_1) = n/r$ ,*

$$e(G_1) = \binom{n/r + 1}{2},$$

*but such that  $G_1$  does not have an equitable  $n/r$ -colouring.*

**Proof.** Let  $G_1$  denote the disjoint union of a clique  $V$  on  $n/r + 1$  vertices and an independent set  $W$  of  $(1 - 1/r)n - 1$  vertices. So every independent set in  $G_1$  contains at most one vertex from  $V$ . But since  $|V| = n/r + 1$ ,  $G_1$  does not have an equitable  $n/r$ -colouring. Further,  $\Delta(G_1) = n/r$  and  $e(G_1) = \binom{n/r + 1}{2}$ .  $\square$

**Proposition 12** *Suppose that  $n, r \in \mathbb{N}$  such that  $r \geq 3$  and  $n = kr$  for some  $k \geq 2$ . Further, let  $D \in \mathbb{N}$  such that  $n/(r - 1) \leq D \leq n - r$ . Then there exists a graph  $G_2$  on  $n$  vertices such that  $\Delta(G_2) = D$ ,*

$$e(G_2) = D + e(\overline{T}(n - D - 1, r - 2)),$$

*but such that  $G_2$  does not have an equitable  $n/r$ -colouring.*

**Proof.** Let  $G_2$  denote the disjoint union of a copy  $K$  of  $K_{1,D}$  and a copy of  $\overline{T}(n-D-1, r-2)$ . So  $|G| = n$ . Let  $v$  denote the vertex of degree  $D$  in  $K$ . The largest independent set in  $G_2$  that contains  $v$  is of size  $r-1$ . Thus,  $G_2$  does not have an equitable  $n/r$ -colouring. Further,  $e(G_2) = D + e(\overline{T}(n-D-1, r-2))$ .

Since  $n/(r-1) \leq D$  we have that  $n-1 \leq (r-1)D$ . Thus, every vertex in the copy of  $\overline{T}(n-D-1, r-2)$  has degree at most

$$\left\lceil \frac{n-D-1}{r-2} \right\rceil - 1 \leq \frac{n-D-1}{r-2} \leq D.$$

This implies that  $\Delta(G_2) = D$ . □

Clearly Propositions 11 and 12 show that one cannot lower the edge density condition in Theorem 5 in the case when  $n/(r-1) \leq D \leq n-r$ . The following result, together with Proposition 11, shows that Theorem 5 is best possible in the case when  $n/r \leq D \leq n/(r-1)$ .

**Proposition 13** *Let  $n, r \in \mathbb{N}$  such that  $r \geq 3$  and  $r$  divides  $n \geq 2r$ . Suppose that  $D \in \mathbb{N}$  such that  $n/r \leq D \leq n/(r-1)$ . Then*

$$f(n, r, D) = \binom{n/r + 1}{2}.$$

The following simple consequence of Turán's theorem will be used in the proof of Theorem 5.

**Fact 14** *Let  $n, r \in \mathbb{N}$  such that  $r \leq n$ . Then*

$$e(T(n, r)) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2} \quad \text{and thus} \quad e(\overline{T}(n, r)) \geq \frac{n^2}{2r} - \frac{n}{2}.$$

We will also require the following easy result.

**Lemma 15** *Let  $n, r \in \mathbb{N}$  such that  $r \geq 4$  and  $r$  divides  $n \geq 3r$ . Suppose that  $D \in \mathbb{N}$  such that  $n/r \leq D < (n+r)/(r-1)$ . Then*

$$f(n, r, D) = \binom{n/r + 1}{2}.$$

## 3 Proof of Theorem 5

### 3.1 Preliminaries

Suppose for a contradiction that the result is false. Let  $G$  be a counterexample with the fewest vertices. That is,  $n = |V(G)| = rk$  for some  $k \in \mathbb{N}$ ,  $\Delta(G) \leq D$  for some  $D \in \mathbb{N}$  such that  $n/r \leq D \leq n-r$ ,  $e(G) < f(n, r, D)$  and  $G$  has no equitable  $n/r$ -colouring. By the Hajnal-Szemerédi theorem,  $\Delta(G) \geq n/r$ . Notice that given fixed  $n$  and  $r$ ,  $f(n, r, D)$  is non-increasing with respect to  $D$ . Thus, we may assume that  $\Delta(G) = D$ .

We first show that  $k \geq 4$ . Indeed, if  $n = 2r$  then  $f(n, r, D) \leq \binom{3}{2} = 3$ . But it is easy to see that every graph  $G_1$  on  $2r$  vertices and with  $e(G_1) \leq 2$  has an equitable 2-colouring. If  $n = 3r$  then  $f(n, r, D) \leq \binom{4}{2} = 6$ . Consider any graph  $G_1$  on  $3r$  vertices with  $e(G_1) \leq 5$  and  $3 \leq \Delta(G_1) \leq 5$ . Let  $x$  denote the vertex in  $G_1$  where  $d_{G_1}(x) = \Delta(G_1)$ . Since  $3 \leq d_{G_1}(x) \leq 5$ ,  $x$  lies in an independent

set  $I$  in  $G_1$  of size  $r$ . But then  $G_1 - I$  contains  $2r$  vertices and at most 2 edges. So  $G_1 - I$  has an equitable 2-colouring and hence  $G_1$  has an equitable 3-colouring.

Let  $v \in V(G)$  such that  $d_G(v) = D$ . Set  $G^* := G - (N_G(v) \cup \{v\})$ . Since  $f(n, r, D) \leq D + e(\overline{T}(n - D - 1, r - 2))$  we have that  $e(G^*) < e(\overline{T}(n - D - 1, r - 2))$ . Thus, by Turán's theorem,  $G^*$  contains an independent set of size  $r - 1$ . Hence,  $v$  lies in an independent set in  $G$  of size  $r$ . Amongst all such independent sets of size  $r$  that contain  $v$ , choose a set  $I = \{v, x_1, \dots, x_{r-1}\}$  such that  $d_G(x_1) + \dots + d_G(x_{r-1})$  is maximised.

Set  $G' := G - I$ ,  $n' := |V(G')| = n - r$  and  $D' := \Delta(G') \leq D$ . Notice that  $D' \geq n'/r$ . (Indeed, if not, then by the Hajnal-Szemerédi theorem  $G'$  contains an equitable  $n'/r$ -colouring. Thus, as  $I$  is an independent set in  $G$  this gives us an equitable  $n/r$ -colouring of  $G$ , a contradiction.) Furthermore,  $D' \leq n' - r$ . If not then

$$e(G) \geq D + D' \geq 2D' \geq 2(n' - r + 1) = 2n - 4r + 2$$

and further,

$$\begin{aligned} e(G) &< f(n, r, D) \leq f(n, r, n - 2r + 1) \leq (n - 2r + 1) + e(\overline{T}(2r - 2, r - 2)) \\ &\leq (n - 2r + 1) + (r + 3) = n - r + 4. \end{aligned}$$

Therefore,  $2n - 4r + 2 < n - r + 4$  and so  $n < 3r + 2$  a contradiction since  $n = kr \geq 4r$ .

Since  $n'/r \leq D' \leq n' - r$ , if  $e(G') < f(n', r, D')$  then the minimality of  $G$  implies that  $G'$  has an equitable  $n'/r$ -colouring. This then implies that  $G$  has an equitable  $n/r$ -colouring, a contradiction. Thus,

$$e(G') \geq f(n', r, D'). \tag{1}$$

We now split our argument into three cases.

### 3.2 Case 1: $f(n', r, D') = \binom{n'/r+1}{2}$ .

By (1),  $e(G') \geq \binom{n'/r+1}{2} = \binom{n/r}{2}$ . Since  $d_G(v) = D \geq n/r$ ,

$$e(G) \geq \frac{n}{r} + \binom{n/r}{2} = \binom{n/r+1}{2} \geq f(n, r, D),$$

a contradiction, as desired.

### 3.3 Case 2: $D' \leq D - 1$ and $f(n', r, D') = D' + e(\overline{T}(n' - D' - 1, r - 2))$ .

The following claim will be useful.

**Claim 16**  $D' < \frac{r-1}{2r-3}n - \frac{(r^2-r+1)}{2r-3}$ .

**Proof.** Note that

$$D + D' + e(\overline{T}(n' - D' - 1, r - 2)) \stackrel{(1)}{\leq} e(G) < f(n, r, D) \leq D + e(\overline{T}(n - D - 1, r - 2)). \tag{2}$$

Since  $D' \leq D - 1$ , clearly  $e(\overline{T}(n' - D, r - 2)) \leq e(\overline{T}(n' - D' - 1, r - 2))$ . Thus, (2) implies that

$$D' + e(\overline{T}(n' - D, r - 2)) < e(\overline{T}(n - D - 1, r - 2)). \quad (3)$$

One can obtain  $\overline{T}(n - D - 1, r - 2)$  from  $\overline{T}(n' - D, r - 2)$  by adding  $r - 1$  vertices and at most

$$(n' - D) + \frac{n - D - 2}{r - 2} \text{ edges.} \quad (4)$$

Hence (3) and (4) give

$$D' < n' - D + \frac{n - D - 2}{r - 2}.$$

Rearranging, and using that  $D' \leq D - 1$  and  $n' = n - r$  we get that

$$\left(2 + \frac{1}{r - 2}\right) D' < \left(1 + \frac{1}{r - 2}\right) n - \frac{(r^2 - r + 1)}{r - 2}.$$

Thus,

$$D' < \frac{r - 1}{2r - 3} n - \frac{(r^2 - r + 1)}{2r - 3},$$

as desired.  $\square$

Since we are in Case 2 we have that

$$D' + e(\overline{T}(n - r - D' - 1, r - 2)) \leq \binom{n'/r + 1}{2} = \binom{n/r}{2}. \quad (5)$$

Notice that for fixed  $n$  and  $r$ ,  $D' + e(\overline{T}(n - r - D' - 1, r - 2))$  is non-increasing as  $D'$  increases. Hence, (5) and Claim 16 imply that

$$D'' + e(\overline{T}(n - r - D'' - 1, r - 2)) \leq \frac{n^2}{2r^2} - \frac{n}{2r} \quad (6)$$

where  $D'' := \lfloor (r - 1)n/(2r - 3) - (r^2 - r + 1)/(2r - 3) \rfloor$ . Note that

$$n - r - \frac{r - 1}{2r - 3} n + \frac{(r^2 - r + 1)}{2r - 3} - 1 = \frac{r - 2}{2r - 3} n + \frac{4 - r^2}{2r - 3}.$$

So Fact 14 and (6) imply that

$$\begin{aligned} & \left( \frac{r - 1}{2r - 3} n - \frac{(r^2 - r + 1)}{2r - 3} - \frac{(2r - 4)}{2r - 3} \right) + \frac{1}{2(r - 2)} \left( \frac{r - 2}{2r - 3} n + \frac{4 - r^2}{2r - 3} \right)^2 \\ & - \frac{1}{2} \left( \frac{r - 2}{2r - 3} n + \frac{4 - r^2}{2r - 3} \right) \leq \frac{n^2}{2r^2} - \frac{n}{2r}. \end{aligned}$$

Next we will move all terms from the previous equation to the left hand side and simplify. The coefficient of  $n^2$  is

$$\frac{r - 2}{2(2r - 3)^2} - \frac{1}{2r^2} = \frac{r^3 - 6r^2 + 12r - 9}{2r^2(2r - 3)^2}. \quad (7)$$



The coefficient of  $n$  is

$$\frac{r-1}{2r-3} - \frac{(r-2)}{2(2r-3)} + \frac{1}{2r} + \frac{(4-r^2)}{(2r-3)^2} = \frac{r^2-4r+9}{2r(2r-3)^2}. \quad (8)$$

The constant term is

$$-\frac{(r^2+r-3)}{2r-3} + \frac{(r^2-4)^2}{2(r-2)(2r-3)^2} + \frac{(r^2-4)}{2(2r-3)} = \frac{-r^4+3r^3+4r^2-26r+28}{2(r-2)(2r-3)^2}. \quad (9)$$

Since  $n \geq 4r$ , (7)–(9) imply that

$$\frac{8(r^3-6r^2+12r-9)}{(2r-3)^2} + \frac{2(r^2-4r+9)}{(2r-3)^2} + \frac{-r^4+3r^3+4r^2-26r+28}{2(r-2)(2r-3)^2} \leq 0. \quad (10)$$

Multiplying (10) by  $2(r-2)(2r-3)^2$  we get

$$15r^4 - 121r^3 + 364r^2 - 486r + 244 \leq 0$$

This yields a contradiction, since it is easy to check that

$$15r^4 - 121r^3 + 364r^2 - 486r + 244 > 0$$

for all  $r \in \mathbb{N}$  such that  $r \geq 3$ .

### 3.4 Case 3: $D' = D$ and $f(\mathbf{n}', \mathbf{r}, D') = D' + e(\overline{\mathbf{T}}(\mathbf{n}' - D' - \mathbf{1}, \mathbf{r} - \mathbf{2}))$ .

By (1) we have that

$$e(G') \geq f(\mathbf{n}', \mathbf{r}, D') = D' + e(\overline{\mathbf{T}}(\mathbf{n}' - D' - \mathbf{1}, \mathbf{r} - \mathbf{2})). \quad (11)$$

Consider any vertex  $x \in V(G')$  such that  $d_{G'}(x) = D' = D$ . Since  $\Delta(G) = D$ ,  $x$  is not adjacent to any vertex in  $I = \{v, x_1, \dots, x_{r-1}\}$ . Further,  $I$  was chosen such that  $d_G(x_1) + \dots + d_G(x_{r-1})$  is maximised. Thus,  $d_G(x_1) = \dots = d_G(x_{r-1}) = D$ . Together with (11) this implies that

$$e(G) \geq (r+1)D + e(\overline{\mathbf{T}}(\mathbf{n}' - D - \mathbf{1}, \mathbf{r} - \mathbf{2})). \quad (12)$$

Since  $e(G) < f(n, r, D) \leq D + e(\overline{\mathbf{T}}(n - D - \mathbf{1}, \mathbf{r} - \mathbf{2}))$ , (12) implies that

$$rD + e(\overline{\mathbf{T}}(\mathbf{n}' - D - \mathbf{1}, \mathbf{r} - \mathbf{2})) < e(\overline{\mathbf{T}}(n - D - \mathbf{1}, \mathbf{r} - \mathbf{2})). \quad (13)$$

One can obtain  $\overline{\mathbf{T}}(n - D - \mathbf{1}, \mathbf{r} - \mathbf{2})$  from  $\overline{\mathbf{T}}(\mathbf{n}' - D - \mathbf{1}, \mathbf{r} - \mathbf{2})$  by adding  $r$  vertices and at most

$$(n' - D - 1) + \frac{2(n - D - 3)}{r - 2} + 1 \text{ edges.} \quad (14)$$

Thus, (13) and (14) imply that

$$rD < n - r - D + \frac{2(n - D - 3)}{r - 2}$$

and so

$$\left(r + 1 + \frac{2}{r-2}\right)D < \left(1 + \frac{2}{r-2}\right)n + \frac{(-r^2 + 2r - 6)}{r-2} < \left(1 + \frac{2}{r-2}\right)n. \quad (15)$$

If  $r = 3$  then (15) implies that

$$D < \frac{n}{2}.$$

Since  $f(n', 3, D) = \min\{\binom{n'/3+1}{2}, D + \binom{n'-D-1}{2}\}$  it is easy to see that if  $f(n', 3, D) = D + \binom{n'-D-1}{2}$  then  $D \geq 2n'/3 + 1 = 2n/3 - 1$ . Thus,  $2n/3 - 1 \leq D < n/2$ , a contradiction since  $n \geq 4r = 12$ .

If  $r \geq 4$  then (15) implies that

$$D < \frac{n}{r-1} = \frac{n'}{r-1} + \frac{r}{r-1}.$$

Since  $n' \geq 3r$ , Lemma 15 implies that  $f(n', r, D') = \binom{n'/r+1}{2}$  and so we are in Case 1, which we have already dealt with.

## 4 The extremal examples for Conjecture 7

**Proposition 17** *Suppose that  $n, r, k \in \mathbb{N}$  such that  $r \geq 2$  divides  $n$  and  $1 \leq k < n/r$ . Then there exists a graph  $G$  on  $n$  vertices whose degree sequence  $d_1 \leq \dots \leq d_n$  satisfies*

- $d_i = (r-2)n/r + k - 1$  for all  $1 \leq i \leq k$ ;
- $d_i = (r-1)n/r$  for all  $k+1 \leq i \leq (r-2)n/r + k$ ;
- $d_i = n - k - 1$  for all  $(r-2)n/r + k + 1 \leq i \leq n - k + 1$ ;
- $d_i = n - 1$  for all  $n - k + 2 \leq i \leq n$ ,

but such that  $G$  does not contain a perfect  $K_r$ -packing.

**Proof.** Let  $G'$  denote the complete  $(r-2)$ -partite graph whose vertex classes  $V_1, \dots, V_{r-2}$  each have size  $n/r$ . Obtain  $G$  from  $G'$  by adding the following vertices and edges: Add a set  $V_{r-1}$  of  $2n/r - 2k + 1$  vertices to  $G'$ , a set  $V_r$  of  $k - 1$  vertices and a set  $V_0$  of  $k$  vertices. Add all edges from  $V_0 \cup V_{r-1} \cup V_r$  to  $V_1 \cup \dots \cup V_{r-2}$ . Further, add all edges with both endpoints in  $V_{r-1} \cup V_r$ . Add all possible edges between  $V_0$  and  $V_r$ .

So  $V_0$  is an independent set, and there are no edges between  $V_0$  and  $V_{r-1}$ . This implies that any copy of  $K_r$  in  $G$  containing a vertex from  $V_0$  must also contain at least one vertex from  $V_r$ . But since  $|V_0| > |V_r|$  this implies that  $G$  does not contain a perfect  $K_r$ -packing. Furthermore,  $G$  has our desired degree sequence.  $\square$

Notice that the graphs  $G$  considered in Proposition 17 satisfy  $(\beta)$  from Conjecture 7 and only fail to satisfy  $(\alpha)$  in the case when  $i = k$  (and in this case  $d_k = (r-2)n/r + k - 1$ ).

Let  $n, r \in \mathbb{N}$  such that  $r$  divides  $n$ . Denote by  $T^*(n, r)$  the complete  $r$ -partite graph on  $n$  vertices with  $r-2$  vertex classes of size  $n/r$ , one vertex class of size  $n/r - 1$  and one vertex class of size  $n/r + 1$ . Then  $T^*(n, r)$  does not contain a perfect  $K_r$ -packing. Furthermore,  $T^*(n, r)$  satisfies  $(\alpha)$  but condition  $(\beta)$  fails; we have that  $d_{n/r+1} = (r-1)n/r - 1$  here. Thus, together  $T^*(n, r)$  and Proposition 17 show that, if true, Conjecture 7 is essentially best possible.

## 5 A special case of Conjecture 7

We now give a simple proof of Conjecture 7 in the case when  $G$  is  $K_{r+1}$ -free.

**Theorem 18** *Let  $n, r \in \mathbb{N}$  such that  $r \geq 2$  divides  $n$ . Suppose that  $G$  is a graph on  $n$  vertices with degree sequence  $d_1 \leq \dots \leq d_n$  such that:*

- $d_i \geq (r-2)n/r + i$  for all  $i < n/r$ ;
- $d_{n/r+1} \geq (r-1)n/r$ .

*Further suppose that no vertex  $x \in V(G)$  of degree less than  $(r-1)n/r$  lies in a copy of  $K_{r+1}$ . Then  $G$  contains a perfect  $K_r$ -packing.*

**Proof.** We prove the theorem by induction on  $n$ . In the case when  $n = r$  then  $d_{n/r+1} = d_2 \geq (r-1)r/r = r-1$ . This implies that every vertex in  $G$  has degree  $r-1$ . Hence  $G = K_r$  as desired. So suppose that  $n > r$  and the result holds for smaller values of  $n$ . Let  $x_1 \in V(G)$  such that  $d_G(x_1) = d_1 \geq (r-2)n/r + 1$ . If  $d_G(x_1) \geq (r-1)n/r$  then  $\delta(G) \geq (r-1)n/r$ . Thus  $G$  contains a perfect  $K_r$ -packing by the Hajnal-Szemerédi theorem. So we may assume that  $(r-2)n/r + 1 \leq d_G(x_1) < (r-1)n/r$ . In particular,  $x_1$  does not lie in a copy of  $K_{r+1}$ . We first find a copy of  $K_r$  containing  $x_1$ . If  $r = 2$ ,  $x_1$  has a neighbour and so we have our desired copy of  $K_2$ . So assume that  $r \geq 3$ . Certainly  $N_G(x_1)$  contains a vertex  $x_2$  such that  $d_G(x_2) \geq (r-1)n/r$ . Thus,  $|N_G(x_1) \cap N_G(x_2)| \geq (r-3)n/r + 1 > 0$ . So if  $r = 3$  we obtain our desired copy of  $K_r$ . Otherwise, we can find a vertex  $x_3 \in N_G(x_1) \cap N_G(x_2)$  such that  $d_G(x_3) \geq (r-1)n/r$ . We can repeat this argument until we have obtained vertices  $x_1, \dots, x_r$  that together form a copy  $K'_r$  of  $K_r$ .

Let  $G' := G - V(K'_r)$  and set  $n' := n - r = |V(G')|$ . Since  $G$  does not contain a copy of  $K_{r+1}$  containing  $x_1$ , every vertex  $x \in V(G) \setminus V(K'_r)$  sends at most  $r-1$  edges to  $K'_r$  in  $G$ . Thus,  $d_{G'}(x) \geq d_G(x) - (r-1)$  for all  $x \in V(G')$ . So if  $d_G(x) \geq (r-1)n/r$  then  $d_{G'}(x) \geq (r-1)n/r - (r-1) = (r-1)n'/r$  for all  $x \in V(G')$ . If a vertex  $y \in V(G')$  does not lie in a copy of  $K_{r+1}$  in  $G$  then clearly  $y$  does not lie in a copy of  $K_{r+1}$  in  $G'$ . This means that no vertex  $y \in V(G')$  of degree less than  $(r-1)n'/r$  lies in a copy of  $K_{r+1}$ .

Let  $d'_1 \leq \dots \leq d'_{n'}$  denote the degree sequence of  $G'$ . It is easy to check that  $d'_i \geq (r-2)n'/r + i$  for all  $i < n'/r$  and that  $d'_{n'/r+1} \geq (r-1)n'/r$ . Indeed, since  $x_1 \in V(K'_r)$  where  $d_G(x_1) = d_1$ , we have that  $d'_i \geq d_{i+1} - (r-1)$  for all  $1 \leq i \leq n'$ . Thus, for all  $1 \leq i < n'/r = n/r - 1$ ,  $d'_i \geq d_{i+1} - (r-1) \geq (r-2)n/r + (i+1) - (r-1) = (r-2)n'/r + i$ . Similarly,  $d'_{n'/r+1} = d'_{n/r} \geq d_{n/r+1} - (r-1) \geq (r-1)n/r - (r-1) = (r-1)n'/r$ . Hence, by induction  $G'$  contains a perfect  $K_r$ -packing. Together with  $K'_r$  this gives us our desired perfect  $K_r$ -packing in  $G$ .  $\square$

## 6 Proof of Theorem 10

Consider any  $x_1 \in V(G)$  such that  $d_G(x_1) \geq (r-1)n/r$ . Since  $d_{n/r} \geq (r-1)n/r$  we can greedily select vertices  $x_2, \dots, x_{r-1}$  such that

- $x_1, \dots, x_{r-1}$  induce a copy of  $K_{r-1}$  in  $G$ ;
- $d_G(x_i) \geq (r-1)n/r$  for all  $1 \leq i \leq r-1$ .

Note that since  $G$  is  $K_{r+1}$ -free,  $\cap_{i=1}^{r-1} N_G(x_i)$  is an independent set. The choice of  $x_1, \dots, x_{r-1}$  implies that  $|\cap_{i=1}^{r-1} N_G(x_i)| \geq n/r$ . Let  $V_1$  denote a subset of  $\cap_{i=1}^{r-1} N_G(x_i)$  of size  $n/r$ . Thus  $V_1$  contains a vertex  $x_1^1$  of degree at least  $(r-1)n/r$ .

As before we can find vertices  $x_2^1, \dots, x_{r-1}^1$  such that

- $x_1^1, \dots, x_{r-1}^1$  induce a copy of  $K_{r-1}$  in  $G$ ;
- $d_G(x_i^1) \geq (r-1)n/r$  for all  $1 \leq i \leq r-1$ .

So  $\cap_{i=1}^{r-1} N_G(x_i^1)$  is an independent set of size at least  $n/r$ . Let  $V_2$  denote a subset of  $\cap_{i=1}^{r-1} N_G(x_i^1)$  of size  $n/r$ . Note that  $N_G(x_1^1) \cap V_1 = \emptyset$  since  $x_1^1 \in V_1$  and  $V_1$  is an independent set. Thus as  $V_2 \subseteq N_G(x_1^1)$ ,  $V_1 \cap V_2 = \emptyset$ .

Our aim is to find disjoint sets  $V_1, \dots, V_r \subseteq V(G)$  of size  $n/r$  and vertices  $x_1^1, \dots, x_{r-1}^1, x_1^2, \dots, x_{r-1}^2, \dots, x_1^{r-1}, \dots, x_{r-1}^{r-1}$  with the following properties:

- $G[V_j]$  is an independent set for all  $1 \leq j \leq r$ ;
- Given any  $1 \leq j \leq r-1$ ,  $x_k^j \in V_k$  for each  $1 \leq k \leq j$ ;
- $d_G(x_k^j) \geq (r-1)n/r$  for all  $1 \leq j \leq r-1$  and  $1 \leq k \leq r-1$ ;
- $x_1^j, \dots, x_{r-1}^j$  induce a copy of  $K_{r-1}$  in  $G$  for all  $1 \leq j \leq r-1$ .

Clearly finding such a partition  $V_1, \dots, V_r$  of  $V(G)$  implies that  $G \subseteq T(n, r)$ .

Suppose that for some  $1 < j < r$  we have defined sets  $V_1, \dots, V_j$  and vertices  $x_1^1, \dots, x_{r-1}^1, \dots, x_1^{j-1}, \dots, x_{r-1}^{j-1}$  with our desired properties. Since  $d_{n/r} \geq (r-1)n/r$  and  $V_1, \dots, V_j$  are independent sets of size  $n/r$  we can choose vertices  $x_1^j, \dots, x_{r-1}^j$  such that for all  $1 \leq k \leq j$ ,

- $x_k^j \in V_k$  and  $d_G(x_k^j) \geq (r-1)n/r$ .

This degree condition, together with the fact that  $x_1^j, \dots, x_{r-1}^j$  lie in different vertex classes, implies that these vertices form a copy of  $K_j$  in  $G$ . We now greedily select further vertices  $x_{j+1}^j, \dots, x_{r-1}^j$  such that

- $x_1^j, \dots, x_{r-1}^j$  induce a copy of  $K_{r-1}$  in  $G$ ;
- $d_G(x_k^j) \geq (r-1)n/r$  for all  $j+1 \leq k \leq r-1$ .

So  $\cap_{i=1}^{r-1} N_G(x_i^j)$  is an independent set of size at least  $n/r$ . Let  $V_{j+1}$  denote a subset of  $\cap_{i=1}^{r-1} N_G(x_i^j)$  of size  $n/r$ . Note that, for each  $1 \leq k \leq j$ ,  $N_G(x_k^j) \cap V_k = \emptyset$  since  $x_k^j \in V_k$  and  $V_k$  is an independent set. Thus as  $V_{j+1} \subseteq N_G(x_k^j)$  for each  $1 \leq k \leq j$ ,  $V_{j+1}$  is disjoint from  $V_1 \cup \dots \cup V_j$ .

Repeating this argument we obtain our desired sets  $V_1, \dots, V_r \subseteq V(G)$  and vertices  $x_1^1, \dots, x_{r-1}^1, x_1^2, \dots, x_{r-1}^2, \dots, x_1^{r-1}, \dots, x_{r-1}^{r-1}$ .

## 7 Possible extensions of Conjecture 7

We suspect that the following ‘Chvátal-type’ degree sequence condition forces a graph to contain a perfect  $K_r$ -packing.

**Question 19** Let  $n, r \in \mathbb{N}$  such that  $r \geq 2$  divides  $n$ . Suppose that  $G$  is a graph on  $n$  vertices with degree sequence  $d_1 \leq \dots \leq d_n$  such that for all  $i \leq n/r$ :

- $d_i \geq (r-2)n/r + i$  or  $d_{n-i(r-1)+1} \geq n-i$ .

Does this condition imply that  $G$  contains a perfect  $K_r$ -packing?

Note that Theorem 6 answers this question in the affirmative when  $r = 2$ . The following example shows that we cannot have a lower value in the second part of the condition in Question 19.

**Proposition 20** Suppose that  $n, r, k \in \mathbb{N}$  such that  $r \geq 2$  divides  $n$  and  $1 \leq k \leq n/r$ . Then there exists a graph  $G$  on  $n$  vertices whose degree sequence  $d_1 \leq \dots \leq d_n$  satisfies

- $d_{n-i(r-1)+1} \geq n-i$  for all  $i \in [n/r] \setminus \{k\}$ ;
- $d_{n-k(r-1)+1} = n-k-1$ ,

but such that  $G$  does not contain a perfect  $K_r$ -packing.

**Proof.** Let  $G$  be the graph on  $n$  vertices with vertex classes  $V_1, V_2$  and  $V_3$  of sizes  $k, (r-1)k-1$  and  $n-rk+1$  respectively and with the following edges: There are all possible edges between  $V_1$  and  $V_2$  and between  $V_2$  and  $V_3$ . Further add all possible edges in  $V_2$  and all edges in  $V_3$ . Thus,  $V_1$  is an independent set and there are no edges between  $V_1$  and  $V_3$ .

The degree sequence of  $G$  is

$$\underbrace{(r-1)k-1, \dots, (r-1)k-1}_{k \text{ times}}, \underbrace{n-k-1, \dots, n-k-1}_{n-rk+1 \text{ times}}, \underbrace{n-1, \dots, n-1}_{(r-1)k-1 \text{ times}}.$$

Hence  $G$  satisfies our desired degree sequence condition. Every copy  $K'_r$  of  $K_r$  in  $G$  that contains a vertex from  $V_1$  must contain  $r-1$  vertices from  $V_2$ . But since  $|V_1|(r-1) > |V_2|$  this implies that  $G$  does not contain a perfect  $K_r$ -packing.  $\square$

The  $r$ th power of a Hamilton cycle  $C$  is obtained from  $C$  by adding an edge between every pair of vertices of distance at most  $r$  on  $C$ . Seymour [18] conjectured the following strengthening of Dirac's theorem.

**Conjecture 21 (Pósa-Seymour, see [18])** Let  $G$  be a graph on  $n$  vertices. If  $\delta(G) \geq \frac{r}{r+1}n$  then  $G$  contains the  $r$ th power of a Hamilton cycle.

Pósa (see [7]) had earlier proposed the conjecture in the case of the square of a Hamilton cycle (that is, when  $r = 2$ ). Komlós, Sárközy and Szemerédi [14] proved Conjecture 21 for graphs whose order is sufficiently larger than  $r$ . More recently, Châu, DeBiasio and Kierstead [3] proved Pósa's conjecture for graphs of order at least  $2 \times 10^8$ .

In the case when  $r+1$  divides  $|G|$ , a necessary condition for a graph  $G$  to contain the  $r$ th power of a Hamilton cycle is that  $G$  contains a perfect  $K_{r+1}$ -packing. Further, notice that the minimum degree condition in Conjecture 21 is the same as the condition in the Hajnal-Szemerédi theorem with respect to perfect  $K_{r+1}$ -packings. Thus an obvious question is whether the condition in Conjecture 7 forces a graph to contain the  $(r-1)$ th power of a Hamilton cycle. Interestingly though, when  $r = 3$ , this is not the case.

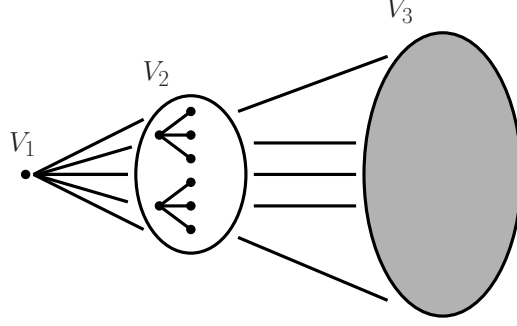


Figure 1: The example from Proposition 22 in the case when  $K = 2$  and  $|V_2| = 8$ .

**Proposition 22** *Suppose that  $C, n \in \mathbb{N}$  such that  $C \ll n$  and 3 divides  $n$ . Then there exists a graph  $G$  whose degree sequence  $d_1 \leq \dots \leq d_n$  satisfies*

$$d_i \geq \frac{n}{3} + C + i \quad \text{for all } 1 \leq i \leq \frac{n}{3}$$

*but such that  $G$  does not contain the square of a Hamilton cycle.*

**Proof.** Choose  $C, K, n \in \mathbb{N}$  so that  $C \ll K \ll n$ . Let  $G$  denote the graph on  $n$  vertices consisting of three vertex classes  $V_1 = \{v\}$ ,  $V_2$  and  $V_3$  where  $|V_2| = n/3 + C + 1$  and  $|V_3| = 2n/3 - C - 2$  which contains the following edges:

- All edges from  $v$  to  $V_2$ ;
- All edges between  $V_2$  and  $V_3$  and all possible edges in  $V_3$ ;
- There are  $K$  vertex-disjoint stars in  $V_2$ , each of size  $\lfloor |V_2|/K \rfloor, \lceil |V_2|/K \rceil$ , which cover all of  $V_2$  (see Figure 1).

Let  $d_1 \leq \dots \leq d_n$  denote the degree sequence of  $G$ . There are  $n/3 + C - K + 1 \leq n/3 - 2C - 1$  vertices in  $V_2$  of degree  $2n/3 - C$ . Since  $C \ll K \ll n$ , the remaining  $K$  vertices in  $V_2$  have degree at least  $2n/3 - C - 2 + \lfloor |V_2|/K \rfloor \geq 2n/3 + C + 1$ . Since  $d_G(v) = n/3 + C + 1$  and  $d_G(x) = n - 2$  for all  $x \in V_3$ , we have that  $d_i \geq \frac{n}{3} + C + i$  for all  $1 \leq i \leq \frac{n}{3}$ .

A necessary condition for a graph  $G$  to contain the square of a Hamilton cycle is that, for every  $x \in V(G)$ ,  $G[N(x)]$  contains a path of length 3. Note that  $N(v) = V_2$  and  $G[V_2]$  does not contain a path of length 3. So  $G$  does not contain the square of a Hamilton cycle.  $\square$

Notice that we can set  $C = o(\sqrt{n})$  in Proposition 22. We finish by raising the following question.

**Question 23** *What can be said about degree sequence conditions which force a graph to contain the  $r$ th power of a Hamilton cycle? In particular, can one establish a degree sequence condition that ensures a graph  $G$  on  $n$  vertices contains the  $r$ th power of a Hamilton cycle and which allows for “many” vertices of  $G$  to have degree “much less” than  $rn/(r + 1)$ ?*

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## References

- [1] A. Akiyama and P. Frankl, On the Size of Graphs with Complete-Factors, *J. Graph Theory* **9** (1985), 197–201.
- [2] P. Allen, J. Böttcher, J. Hladký and D. Piguet, A density Corrádi-Hajnal theorem, *Electronic Notes in Discrete Mathematics* **38** (2011), 31–36.
- [3] P. Châu, L. DeBiasio and H.A. Kierstead, Pósa’s Conjecture for graphs of order at least  $2 \times 10^8$ , *Random Structures and Algorithms* **39** (2011), 507–525.
- [4] V. Chvátal, On Hamilton’s ideals, *J. Combin. Theory B* **12** (1972), 163–168.
- [5] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, *Acta Math. Acad. Sci. Hungar.* **14** (1964), 423–439.
- [6] G.A. Dirac, Structural properties and circuits in graphs, in: C.St.-J.A. Nash-Williams, J. Sheehan (Eds.), Proceedings of the 5th British Combinatorial Conference, Congressus Numerantium, No. XV, *Utilitas Math.*, Winnipeg, Man., 1976, pp. 135–140.
- [7] P. Erdős, Problem 9, in: M. Fiedler (Ed.), *Theory of Graphs and its Applications*, Czech. Acad. Sci. Publ., Prague, 1964, p. 159.
- [8] P. Erdős, On the graph theorem of Turán, *Mat. Lapok.* **21** (1970), 249–251.
- [9] A. Hajnal and E. Szemerédi, Proof of a conjecture of Erdős, *Combinatorial Theory and its Applications vol. II* **4** (1970), 601–623.
- [10] P. Hell and D.G. Kirkpatrick, On the complexity of general graph factor problems, *SIAM J. Computing* **12** (1983), 601–609.
- [11] P. Justesen, On independent circuits in finite graphs and a conjecture of Erdős and Pósa, *Ann. Discrete Math.* **41** (1989), 299–306
- [12] H.A. Kierstead and A.V. Kostochka, An Ore-type Theorem on Equitable Coloring, *J. Combin. Theory B* **98** (2008), 226–234.
- [13] F. Knox and A. Treglown, Embedding spanning bipartite graphs of small bandwidth, *Combin. Probab. Comput.* **22** (2013), 71–96.
- [14] J. Komlós, G.N. Sárközy and E. Szemerédi, Proof of the Seymour conjecture for large graphs, *Annals of Combinatorics* **2** (1998), 43–60.

- [15] D. Kühn and D. Osthus, Critical chromatic number and the complexity of perfect packings in graphs, *17th ACM-SIAM Symposium on Discrete Algorithms (SODA 2006)*, 851–859.
- [16] D. Kühn and D. Osthus, The minimum degree threshold for perfect graph packings, *Combinatorica* **29** (2009), 65–107.
- [17] D. Kühn, D. Osthus and A. Treglown, An Ore-type theorem for perfect packings in graphs, *SIAM J. Disc. Math.* **23** (2009), 1335–1355.
- [18] P. Seymour, Problem section, in *Combinatorics: Proceedings of the British Combinatorial Conference 1973* (T.P. McDonough and V.C. Mavron eds.), 201–202, Cambridge University Press, 1974.

## Appendix

Here we give proofs of Proposition 13 and Lemma 15. The following fact will be used in both of these proofs.

**Fact 24** Fix  $n, r \in \mathbb{N}$  such that  $r \geq 3$  and  $r$  divides  $n \geq 2r$ . Define

$$h(x) := x + \frac{(n-x-1)^2}{2(r-2)} - \frac{1}{2}(n-x-1).$$

Then  $h(x)$  is a decreasing function for  $x \in [0, n/(r-1)]$ . Moreover, if  $n \geq 3r$  then  $h(x)$  is a decreasing function for  $x \in [0, (n+r)/(r-1)]$ .

**Proof.** Notice that

$$h'(x) = \frac{3}{2} - \frac{(n-x-1)}{r-2} = \frac{x}{r-2} + \frac{1-n}{r-2} + \frac{3}{2}.$$

So for  $x \leq n/(r-1)$ ,

$$h'(x) \leq \frac{n}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2}.$$

Note that  $3(r-1)/2 + (r-1)/(r-2) < n$  since  $n \geq 2r$  and  $r \geq 3$ . Thus,

$$h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2} < 0.$$

If  $x \leq (n+r)/(r-1)$  then

$$h'(x) \leq \frac{n+r}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2}.$$

If  $n \geq 3r$  then  $n > 3r/2 + 4$ . So  $n > 3(r-1)/2 + (2r-1)/(r-2)$ . Thus,

$$h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2} < 0,$$

as desired. □



**Proof of Proposition 13.** We need to show that, for all  $D \in \mathbb{N}$  such that  $n/r \leq D \leq n/(r-1)$ ,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \binom{n/r+1}{2} \leq D + e(\overline{T}(n-D-1, r-2)).$$

Since  $D \leq n/(r-1)$ , Facts 14 and 24 imply that

$$\begin{aligned} D + e(\overline{T}(n-D-1, r-2)) &\geq D + \frac{(n-D-1)^2}{2(r-2)} - \frac{(n-D-1)}{2} \\ &\geq \frac{n}{r-1} + \frac{1}{2(r-2)} \left[ \frac{(r-2)}{r-1} n - 1 \right]^2 - \frac{1}{2} \left[ \frac{(r-2)}{r-1} n - 1 \right] \\ &\geq \frac{(r-2)}{2(r-1)^2} n^2 - \frac{(r-2)}{2(r-1)} n. \end{aligned}$$

Thus, it suffices to show that

$$\frac{(r-2)}{2(r-1)^2} n - \frac{r-2}{2(r-1)} \geq \frac{n}{2r^2} + \frac{1}{2r}. \quad (16)$$

Notice that

$$\frac{r-2}{2(r-1)^2} - \frac{1}{2r^2} = \frac{(r-2)r^2 - (r-1)^2}{2r^2(r-1)^2} = \frac{r^3 - 3r^2 + 2r - 1}{2r^2(r-1)^2} \quad (17)$$

and

$$\frac{r-2}{2(r-1)} + \frac{1}{2r} = \frac{r^2 - r - 1}{2r(r-1)}.$$

Since  $n \geq 2r$ , (16) implies that it suffices to show that

$$\frac{r^3 - 3r^2 + 2r - 1}{r(r-1)^2} - \frac{r^2 - r - 1}{2r(r-1)} \geq 0. \quad (18)$$

Note that  $r^3 \geq 4r^2 - 4r + 3$  as  $r \geq 3$ . Thus,  $2(r^3 - 3r^2 + 2r - 1) \geq (r^2 - r - 1)(r - 1)$ . So indeed (18) is satisfied, as desired.  $\square$

**Proof of Lemma 15.** We need to show that, for all  $D \in \mathbb{N}$  such that  $n/r \leq D < (n+r)/(r-1)$ ,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \binom{n/r+1}{2} \leq D + e(\overline{T}(n-D-1, r-2)).$$

Since  $D < (n+r)/(r-1)$  we have that  $D \leq n/(r-1) + 1$ . So Facts 14 and 24 imply that

$$\begin{aligned} D + e(\overline{T}(n-D-1, r-2)) &\geq D + \frac{(n-D-1)^2}{2(r-2)} - \frac{(n-D-1)}{2} \\ &\geq \frac{n}{r-1} + 1 + \frac{1}{2(r-2)} \left[ \frac{(r-2)}{r-1} n - 2 \right]^2 - \frac{1}{2} \left[ \frac{(r-2)}{r-1} n - 2 \right] \\ &\geq \frac{(r-2)}{2(r-1)^2} n^2 - \frac{(r-2)}{2(r-1)} n - \frac{n}{r-1}. \end{aligned}$$

Thus, it suffices to show that

$$\frac{(r-2)}{2(r-1)^2}n - \frac{(r-2)}{2(r-1)} - \frac{1}{r-1} \geq \frac{n}{2r^2} + \frac{1}{2r}. \quad (19)$$

Notice that

$$\frac{r-2}{2(r-1)} + \frac{1}{r-1} + \frac{1}{2r} = \frac{r^2+r-1}{2r(r-1)}.$$

Since  $n \geq 3r$ , (17) and (19) imply that it suffices to show that

$$\frac{3(r^3 - 3r^2 + 2r - 1)}{2r(r-1)^2} - \frac{r^2 + r - 1}{2r(r-1)} \geq 0. \quad (20)$$

Note that  $2r^3 - 9r^2 + 8r - 4 \geq 0$  as  $r \geq 4$ . Thus,  $3(r^3 - 3r^2 + 2r - 1) \geq (r^2 + r - 1)(r - 1)$ . So indeed (20) is satisfied, as desired.  $\square$