

RAMSEY-TYPE PROBLEMS FOR TILINGS IN DENSE GRAPHS

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ABSTRACT. Given a graph H , the Ramsey number $R(H)$ is the smallest positive integer n such that every 2-edge-colouring of K_n yields a monochromatic copy of H . We write mH to denote the union of m vertex-disjoint copies of H . The members of the family $\{mH : m \geq 1\}$ are also known as H -tilings. A well-known result of Burr, Erdős and Spencer states that $R(mK_3) = 5m$ for every $m \geq 2$. On the other hand, Moon proved that every 2-edge-colouring of K_{3m+2} yields a K_3 -tiling consisting of m monochromatic copies of K_3 , for every $m \geq 2$. Crucially, in Moon's result, distinct copies of K_3 might receive different colours.

In this paper, we investigate the analogous questions where the complete host graph is replaced by a graph of large minimum degree. We determine the (asymptotic) minimum degree threshold for forcing a K_3 -tiling covering a prescribed proportion of the vertices in a 2-edge-coloured graph such that every copy of K_3 in the tiling is monochromatic. We also determine the largest size of a monochromatic K_3 -tiling one can guarantee in any 2-edge-coloured graph of large minimum degree. These results therefore provide dense generalisations of the theorems of Moon and Burr–Erdős–Spencer.

It is also natural to consider generalisations of these problems to r -edge-colourings (for $r \geq 2$) and for H -tilings (for arbitrary graphs H). We prove some results in this direction and propose several open questions.

1. INTRODUCTION

Ramsey theory is a central research topic in combinatorics. Ramsey's original theorem [28] asserts that for every $r \in \mathbb{N}$ and every graph H , there exists an $n \in \mathbb{N}$ such that every r -edge-colouring of the complete graph K_n on n vertices yields a monochromatic copy of H . We write $R_r(H)$ to denote the smallest n for which the above holds, and set $R(H) := R_2(H)$.

In general, determining $R(H)$ is a very difficult problem and there are relatively few graphs H for which the exact value of $R(H)$ is known. An interesting class of graphs whose Ramsey behaviour is quite well-understood are so-called tilings. For a fixed graph H , an H -tiling is a collection of vertex-disjoint copies of H . For $m \in \mathbb{N}$, we write mH to denote an H -tiling consisting of m copies of H . Erdős [12, Problem 9] raised the question of determining $R(mK_\ell)$ for $\ell \geq 3$. The following result of Burr, Erdős and Spencer [8] resolves this question for $\ell = 3$.

Theorem 1.1 (Burr, Erdős and Spencer [8]). *For every integer $m \geq 2$, we have $R(mK_3) = 5m$.*

More generally, Burr, Erdős and Spencer [8] proved that for a fixed graph H there exist constants c and m_0 such that $R(mH) = (2|H| - \alpha(H))m + c$ provided $m \geq m_0$, where $\alpha(H)$ is the independence

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number of H . Burr [6], and subsequently Bucić and Sudakov [5], provided methods for computing c exactly. Bucić and Sudakov [5] also obtained the current best bounds for m_0 . In the case of K_ℓ -tilings, their work states that there is a constant $C > 0$ such that $R(mK_\ell) = (2\ell - 1)m + R(\ell - 1) - 2$ provided $m \geq 2^{C\ell}$. Moreover, the bound on m is essentially tight; see [5].

Although not a Ramsey-type question in the classical sense, it is also natural to ask how large a complete r -edge-coloured graph needs to be to ensure there exists an H -tiling of a given size such that every copy of H is monochromatic. Crucially, in this setting, different copies of H in the tiling are allowed to receive different colours. This problem was studied prior to the work of Burr, Erdős and Spencer [8]. Indeed, the following result of Moon [25] settles the $H = K_3$ case of this problem.

Theorem 1.2 (Moon [25]). *For every integer $m \geq 2$, every 2-edge-colouring of K_{3m+2} yields a K_3 -tiling consisting of m monochromatic copies of K_3 . Furthermore, the term $3m + 2$ cannot be replaced by a smaller integer.*

Burr, Erdős and Spencer [8] proved an analogue of Theorem 1.2 for larger cliques: any 2-edge-colouring of $K_{\ell m + R(\ell, \ell - 1) - 1}$ yields m vertex-disjoint monochromatic copies of K_ℓ provided m is sufficiently large with respect to ℓ ; again the $\ell m + R(\ell, \ell - 1) - 1$ term is best possible. Here, we write $R(r, b)$ to denote the smallest n such that any red/blue edge-colouring of K_n yields a red K_r or a blue K_b .

We remark that there are many other Ramsey-type results concerning finding many vertex-disjoint monochromatic copies of graphs from a given graph family. Paths and cycles have received particular attention. Erdős and Gyárfás [13] proved that the vertex set of a 2-edge-coloured complete graph K_n can be partitioned into at most $2\sqrt{n}$ vertex-disjoint monochromatic paths, all of the same colour, and conjectured the $2\sqrt{n}$ term can be replaced by \sqrt{n} . This was proved recently by Pokrovskiy, Versteegen and Williams [27] for sufficiently large n . Gerencsér and Gyárfás [14] observed that the vertex set of any 2-edge-coloured complete graph can be partitioned into two vertex-disjoint monochromatic paths of different colours. In the 1970s, Lehel conjectured that the vertex set of any 2-edge-coloured complete graph K_n can be partitioned into two monochromatic cycles of different colours (see, e.g., [2]). Lehel's conjecture was proved for large n by Łuczak, Rödl and Szemerédi [24]. The bound on n was later improved by Allen [1], and finally Bessy and Thomassé [4] provided a full resolution of Lehel's conjecture.

Schelp [29] (see also [23]) proposed the study of Ramsey-type questions where the host graph, rather than being complete, can be any graph satisfying a given minimum degree condition. Several results have been proved in this direction. For example, Schelp [29] conjectured that any 2-edge-coloured n -vertex graph G with minimum degree $\delta(G) \geq 3n/4$ contains a monochromatic path of length at least $2n/3$. This conjecture was verified asymptotically by Gyárfás and Sárközy [15]. Balogh, Barát, Gerbner, Gyárfás and Sárközy [3] conjectured that the conclusion of Lehel's conjecture still holds for any 2-edge-coloured n -vertex graph G with minimum degree $\delta(G) \geq 3n/4$, and proved an asymptotic version of this statement. A stronger asymptotic result was proved by DeBiasio and Nelsen [11] and an exact statement (for n sufficiently large) was proved by Letzter [22].

Motivated by this line of research, in this paper we consider the natural generalisations of the aforementioned classical Ramsey-type results about tilings to the dense setting. The works of Burr–Erdős–Spencer and Moon suggest the following two problems. In the former, one is interested in finding a large monochromatic H -tiling in a graph with given minimum degree. The latter problem is the same except we only insist that individual copies of H in the H -tiling are monochromatic, but different copies of H may receive different colours.

Problem 1.3. *Let H be a fixed graph and $n, r, \delta \in \mathbb{N}$. Determine the largest $m \in \mathbb{N}$ such that any r -edge-coloured n -vertex graph G with minimum degree $\delta(G) \geq \delta$ contains a monochromatic copy of mH .*

Problem 1.4. *Let H be a fixed graph and $n, r, \delta \in \mathbb{N}$. Determine the largest $m \in \mathbb{N}$ such that any r -edge-coloured n -vertex graph G with minimum degree $\delta(G) \geq \delta$ contains an H -tiling consisting of m monochromatic copies of H .*

Note that Theorems 1.1 and 1.2 provide a full resolution of the case $H = K_3$, $r = 2$ and $\delta = n - 1$ of Problems 1.3 and 1.4 respectively.

The $r = 1$ case of both Problems 1.3 and 1.4 is equivalent to determining the largest H -tiling one can guarantee in any n -vertex graph G with $\delta(G) \geq \delta$. By itself, this case of the problem has received considerable attention, and has motivated a fruitful line of research. An H -tiling in a graph G is *perfect* if it contains all the vertices of G . Corrádi and Hajnal [10] determined the minimum degree threshold that guarantees the existence of a perfect K_3 -tiling. This result was further generalised to perfect K_t -tilings (for every $t \in \mathbb{N}$) by Hajnal and Szemerédi [17] and to perfect H -tilings (for every fixed graph H) by Kühn and Osthus [21]. Combining the Hajnal–Szemerédi theorem with an elementary interpolation argument, one can easily determine the minimum degree threshold to force a K_t -tiling covering a fixed proportion of the vertices (see, e.g., Theorem 3.2 in this paper). The same elementary strategy fails for H -tilings where H is an arbitrary fixed graph H . Komlós [19] determined (asymptotically) the minimum degree threshold that guarantees the existence of an H -tiling covering a fixed proportion of the vertices of the host graph, for any fixed graph H , provided the fixed proportion is less than 1. Therefore, the $r = 1$ case of Problems 1.3 and 1.4 is (asymptotically) fully understood.

The $H = K_2$ case of both Problems 1.3 and 1.4 has also been resolved. Indeed, the case $H = K_2$ of Problem 1.4 is equivalent to determining the largest K_2 -tiling in a graph with given minimum degree, and thus it is covered by, for example, the Hajnal–Szemerédi theorem. The case $H = K_2$ of Problem 1.3 has a more interesting history. Given graphs H_1, \dots, H_r , we write $R_r(H_1, \dots, H_r)$ to denote the smallest integer n such that any r -edge-colouring of K_n using colours c_1, \dots, c_r yields a monochromatic copy of H_i in colour c_i , for some i . Cockayne and Lorimer [9] proved that $R_r(mK_2) = (r + 1)(m - 1) + 2$, resolving the case $H = K_2$, $\delta = n - 1$ of Problem 1.3. Gyárfás and Sárközy [15] determined $R(mK_2, mK_2, S_t)$ for all $t, m \in \mathbb{N}$, where S_t is the star on $t + 1$ vertices. The connection of this purely Ramsey-type result to Problem 1.3 is that a red/blue/green edge-coloured K_n which does not contain a green monochromatic copy of S_t can be seen as a red/blue edge-coloured n -vertex graph G with $\delta(G) \geq n - t$. Therefore, Gyárfás and Sárközy’s result resolves the case $H = K_2$, $r = 2$ of Problem 1.3. Finally, Omidi, Raeisi and Rahimi [26] computed $R_r(mK_2, \dots, mK_2, S_t)$ for all $r, t, m \in \mathbb{N}$, thus resolving the case $H = K_2$ of Problem 1.3 in full.

1.1. Main results. In this paper, our main focus is to study the case $H = K_3$, $r = 2$ of Problems 1.3 and 1.4. Observe that the case $\delta \leq 4n/5$ is uninteresting, as one cannot guarantee a single monochromatic copy of K_3 . Indeed, consider a balanced complete n -vertex 5-partite graph G with classes V_1, \dots, V_5 . Clearly $\delta(G) = \lfloor 4n/5 \rfloor$. Colour all edges between V_i and V_{i+1} red, where the indices are taken modulo 5. All remaining edges are blue. Thus, G does not contain a monochromatic copy of K_3 .

For the case $H = K_3$, $r = 2$ of Problem 1.3, the following theorem provides an exact answer when δ is a bit larger than $4n/5$ or a bit smaller than $n - 1$.

Theorem 1.5. *Let $n \in \mathbb{N}$ and G be a 2-edge-coloured n -vertex graph. Then G contains a monochromatic copy of mK_3 where m is equal to*

$$(B.1) \quad \left\lfloor \frac{\delta(G)+1}{5} \right\rfloor \quad \text{if} \quad \frac{65n}{66} \leq \delta(G),$$

$$(B.2) \quad \left\lfloor \frac{5\delta(G)-4n}{2} \right\rfloor \quad \text{if} \quad \frac{4n}{5} \leq \delta(G) \leq \frac{5n}{6}.$$

Furthermore, parts (B.1) and (B.2) are best possible, in the sense that the statement of the theorem does not hold if m is replaced by a larger number.

Case (B.1) of Theorem 1.5 can be seen as a dense generalisation of the Burr–Erdős–Spencer result, as Theorem 1.1 corresponds precisely to the case $n = 5m$, $\delta(G) = n - 1$ of Theorem 1.5.

Theorem 1.5 does not cover graphs with minimum degree between $5n/6$ and $65n/66$, however, we raise the following question.

Question 1.6. *Is the following true? Let $n \geq 25$ be an integer and G be a 2-edge-coloured n -vertex graph. Then G contains a monochromatic copy of mK_3 , where m is equal to*

$$(C.1) \quad \left\lfloor \frac{\delta(G)+1}{5} \right\rfloor \quad \text{if} \quad \frac{15n}{17} \leq \delta(G),$$

$$(C.2) \quad \left\lfloor \frac{4\delta(G)-3n+1}{3} \right\rfloor \quad \text{if} \quad \frac{6n}{7} \leq \delta(G) \leq \frac{15n}{17},$$

$$(C.3) \quad \left\lfloor \frac{5\delta(G)-4n}{2} \right\rfloor \quad \text{if} \quad \frac{4n}{5} \leq \delta(G) \leq \frac{6n}{7}.$$

In Section 2.2 we provide extremal examples that show the bounds on m in Question 1.6 cannot be increased. Note that we put the condition $n \geq 25$ in Question 1.6 to ensure that we can separate into three cases and also to ensure that we have matching extremal examples. It may be possible that there is an affirmative answer to the question with a smaller lower bound on n .

When $H = K_3$ and $r = 2$, Problem 1.4 turns out to be much more tractable. The following theorem provides an (asymptotic) resolution of this case.

Theorem 1.7. *Let $n \in \mathbb{N}$ and G be a 2-edge-coloured n -vertex graph. Then there exists a K_3 -tiling in G such that every copy of K_3 is monochromatic and the number of copies of K_3 in the tiling is at least*

$$(M.1) \quad \left\lfloor \frac{2\delta(G)-n}{3} \right\rfloor \quad \text{if} \quad \frac{7n}{8} \leq \delta(G),$$

$$(M.2) \quad \left\lfloor \frac{4\delta(G)-3n}{2} \right\rfloor - o(n) \quad \text{if} \quad \frac{5n}{6} \leq \delta(G) \leq \frac{7n}{8},$$

$$(M.3) \quad 5\delta(G) - 4n \quad \text{if} \quad \frac{4n}{5} \leq \delta(G) \leq \frac{5n}{6}.$$

Furthermore, parts (M.1) and (M.3) are best possible and part (M.2) is best possible up to the $o(n)$ term.

Note that for $n \geq 8$, (M.1) deals with the case when the host graph G is complete, and so generalises Theorem 1.2. On the other hand, (M.3) for $n = 5$ reiterates that there exists a 2-edge-coloured K_5 without a monochromatic K_3 ; the $n = 6$ case reiterates that every 2-edge-coloured K_6 contains a monochromatic K_3 .

1.2. Organisation of the paper and notation. In the next section, we present the extremal examples showing the sharpness of Theorems 1.5 and 1.7, and the bounds in Question 1.6. The third and fourth sections cover the proofs of Theorems 1.7 and 1.5, respectively. In the final section we discuss some further results and research directions. We conclude this section with a list of the notation used throughout the paper.

Given $n \in \mathbb{N}$, we set $[n] := \{1, 2, \dots, n\}$. Given two sets A and B , we write $A \dot{\cup} B$ to denote the disjoint union of A and B . For a graph G , we write $|G|$ to denote the number of vertices in G . A set of vertices $S \subseteq V(G)$ is *independent* if no edge lies in it. A subgraph H of G is *spanning*

if $V(H) = V(G)$. Given a set $X \subseteq V(G)$, we write $G[X]$ for the *induced subgraph of G on X* , that is, the subgraph with vertex set X which contains all edges of G lying in X . Set $G \setminus X := G[V(G) \setminus X]$.

Given a partition V_1, \dots, V_k of $V(G)$, we write $G[V_1, \dots, V_k]$ to denote the spanning subgraph of G containing all edges of G except those lying within a class V_i , for any $i \geq 1$. Given a graph H and $k \in \mathbb{N}$, we write $H(k)$ to denote the blow-up of H where every vertex is replaced by a class of k vertices. A *blow-up* of an edge-coloured graph G is an edge-coloured $|G|$ -partite graph with vertex classes $\{V_v : v \in V(G)\}$ such that if $xy \in E(G)$ then all edges between V_x and V_y are present and have the same colour as xy , whereas if $xy \notin E(G)$ then there is no edge between V_x and V_y .

We say that a 2-edge-coloured K_5 is *badly coloured* if the edges of each colour form a cycle of length 5; so a badly coloured K_5 does not contain a monochromatic copy of K_3 .

2. EXTREMAL EXAMPLES FOR THEOREMS 1.5, THEOREM 1.7 AND QUESTION 1.6

In this section we present extremal examples that show the bounds on the size of the K_3 -tilings in Theorems 1.5 and 1.7, and Question 1.6 cannot be increased.

2.1. Extremal examples for Theorem 1.7. The following construction shows the sharpness of Theorem 1.7 for all its cases.

Extremal Example 1. For every $n, \delta \in \mathbb{N}$ such that $4n/5 \leq \delta \leq n - 1$, we write $\text{EX}_\Delta(n, \delta)$ to denote the following red/blue edge-coloured graph. We have $V(\text{EX}_\Delta(n, \delta)) = V_0 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_5$ where $|V_i| = n - \delta \geq 1$ for every $i \geq 1$ and $|V_0| = 5\delta - 4n \geq 0$. The sets V_1, \dots, V_5 are independent; all other pairs of vertices form an edge. The subgraph $\text{EX}_\Delta(n, \delta)[V_0 \cup V_1, V_2, V_3, V_4, V_5]$ is a blow-up of a badly coloured K_5 . The edges lying in V_0 and the edges incident to both V_0 and V_1 are red.

In the next lemma, we determine an upper bound for the largest K_3 -tiling in $\text{EX}_\Delta(n, \delta)$ consisting of monochromatic copies of K_3 .

Lemma 2.1. Let $n, \delta \in \mathbb{N}$ such that $4n/5 \leq \delta \leq n - 1$. Then $\text{EX}_\Delta(n, \delta)$ is an n -vertex graph with minimum degree $\delta(\text{EX}_\Delta(n, \delta)) = \delta$. Furthermore, for any collection \mathcal{F} of vertex-disjoint monochromatic copies of K_3 in $\text{EX}_\Delta(n, \delta)$, we have

$$|\mathcal{F}| \leq \min \left\{ 5\delta - 4n, \frac{4\delta - 3n}{2}, \frac{2\delta - n}{3} \right\}.$$

Proof. We have $|\text{EX}_\Delta(n, \delta)| = |V_0| + \dots + |V_5| = (5\delta - 4n) + 5(n - \delta) = n$. Every vertex in V_0 has degree $n - 1$, all other vertices have degree δ . In particular, $\delta(\text{EX}_\Delta(n, \delta)) = \delta$. Let \mathcal{F} be a collection of vertex-disjoint monochromatic copies of K_3 in $\text{EX}_\Delta(n, \delta)$.

As $\text{EX}_\Delta(n, \delta)[V_0 \cup V_1, V_2, V_3, V_4, V_5]$ is a blow-up of a badly coloured K_5 , it does not contain a monochromatic copy of K_3 . It follows that every monochromatic copy of K_3 contains an edge lying in $V_0 \cup V_1$, and thus it must be red. In particular, every monochromatic copy of K_3 (i) has at least one vertex in V_0 (since V_1 is independent) and (ii) at least two vertices in $V_0 \cup V_1$.

Recall that the blue edges of a badly coloured K_5 form a cycle of length 5 and so each vertex of K_5 is incident to two blue edges. As $\text{EX}_\Delta(n, \delta)[V_0 \cup V_1, V_2, V_3, V_4, V_5]$ is a blow-up of a badly coloured K_5 , we may assume without loss of generality that all edges between $V_0 \cup V_1$ and $V_2 \cup V_3$ are blue. It follows that (iii) no red monochromatic copy of K_3 intersects $V_2 \cup V_3$. Property (i) implies $|\mathcal{F}| \leq |V_0| = 5\delta - 4n$. Property (ii) implies $|\mathcal{F}| \leq |V_0 \cup V_1|/2 = (4\delta - 3n)/2$. Property (iii) implies $|\mathcal{F}| \leq (n - |V_2 \cup V_3|)/3 = (2\delta - n)/3$. \square

It is easy to check that

$$(1) \quad \min \left\{ 5\delta - 4n, \frac{4\delta - 3n}{2}, \frac{2\delta - n}{3} \right\} = \begin{cases} (2\delta - n)/3 & \text{if } 7n/8 \leq \delta; \\ (4\delta - 3n)/2 & \text{if } 5n/6 \leq \delta \leq 7n/8; \\ 5\delta - 4n & \text{if } 4n/5 \leq \delta \leq 5n/6. \end{cases}$$

Lemma 2.1 and equation (1) imply parts (M.1) and (M.3) of Theorem 1.7 are best possible, and part (M.2) is best possible up to the $o(n)$ term.

The next construction is an alternative extremal example for part (M.1) of Theorem 1.7.

Extremal Example 2. For every $n, \delta \in \mathbb{N}$ such that $7n/8 \leq \delta \leq n-1$, we write $EX_{\Delta}^{Alt}(n, \delta)$ to denote the following red/blue edge-coloured graph. We have $V(EX_{\Delta}^{Alt}(n, \delta)) = S_1 \dot{\cup} S_2 \dot{\cup} R$ where $|S_1| = |S_2| = n - \delta \geq 1$ and $|R| = 2\delta - n \geq 1$. The sets S_1 and S_2 are independent. All other pairs of vertices form an edge. All edges with one vertex in $S_1 \cup S_2$ and the other in R are blue. All remaining edges are red.

The next lemma provides an upper bound for the largest K_3 -tiling in $EX_{\Delta}^{Alt}(n, \delta)$ consisting of monochromatic copies of K_3 ; this bound matches exactly part (M.1) of Theorem 1.7.

Lemma 2.2. Let $n, \delta \in \mathbb{N}$ such that $7n/8 \leq \delta \leq n-1$. Then $EX_{\Delta}^{Alt}(n, \delta)$ is an n -vertex graph with minimum degree $\delta(EX_{\Delta}^{Alt}(n, \delta)) = \delta$. Furthermore, for any collection \mathcal{F} of vertex-disjoint monochromatic copies of K_3 in $EX_{\Delta}^{Alt}(n, \delta)$, we have $|\mathcal{F}| \leq (2\delta - n)/3$.

Proof. We have $|EX_{\Delta}^{Alt}(n, \delta)| = |S_1| + |S_2| + |R| = 2(n - \delta) + (2\delta - n) = n$. Every vertex in $S_1 \cup S_2$ has degree δ ; all remaining vertices have degree $n - 1$. In particular, $\delta(EX_{\Delta}^{Alt}(n, \delta)) = \delta$.

Let \mathcal{F} be a collection of vertex-disjoint monochromatic copies of K_3 in $EX_{\Delta}^{Alt}(n, \delta)$. Observe that the blue edges of $EX_{\Delta}^{Alt}(n, \delta)$ form a complete bipartite graph, hence there is no blue monochromatic copy of K_3 . Moreover, any red monochromatic copy of K_3 must lie in R . Since $|R| = 2\delta - n$, it follows that $|\mathcal{F}| \leq (2\delta - n)/3$. \square

2.2. Extremal examples for Theorem 1.5 and Question 1.6. We have three different constructions. We start with the one for large degree, which proves the sharpness of part (C.1) of Question 1.6 and thus of part (B.1) of Theorem 1.5.

Extremal Example 3. For every $n, \delta \in \mathbb{N}$ with $5 \leq \delta \leq n-1$,¹ we write $EX_{\blacktriangle}^1(n, \delta)$ to denote the following red/blue edge-coloured graph. We have $V(EX_{\blacktriangle}^1(n, \delta)) = R \dot{\cup} B \dot{\cup} S$ where $|S| = n - \delta \geq 1$, $|R| = 3\lceil(\delta + 1)/5\rceil + 2 \geq 5$ and $|B| = n - |S| - |R| \geq 0$. The set S is independent and all other pairs of vertices form an edge. All edges lying in R as well as the edges incident to both S and B are red. All edges lying in B as well as the edges incident to both R and $S \cup B$ are blue.

In the next lemma, we give an upper bound for the largest m such that $EX_{\blacktriangle}^1(n, \delta)$ contains a monochromatic mK_3 .

Lemma 2.3. Let $n, \delta \in \mathbb{N}$ with $5 \leq \delta \leq n-1$. Then $EX_{\blacktriangle}^1(n, \delta)$ is an n -vertex graph with minimum degree $\delta(EX_{\blacktriangle}^1(n, \delta)) = \delta$. Furthermore, for any monochromatic copy of mK_3 in $EX_{\blacktriangle}^1(n, \delta)$ we have $m \leq \lfloor (\delta + 1)/5 \rfloor$.

Proof. We have $|EX_{\blacktriangle}^1(n, \delta)| = |R| + |B| + |S| = n$. Every vertex in S has degree δ , all other vertices have degree $n - 1$, implying $\delta(EX_{\blacktriangle}^1(n, \delta)) = \delta$. Also, note that

$$(3\lceil(\delta + 1)/5\rceil + 2) + (2\lfloor(\delta + 1)/5\rfloor + 1) = 5\lfloor(\delta + 1)/5\rfloor + 3 \geq 5(\delta - 3)/5 + 3 = \delta.$$

Since $|B| = n - |S| - |R| = \delta - |R|$ it follows that $|B| \leq 2\lfloor(\delta + 1)/5\rfloor + 1$.

¹We need $5 \leq \delta$ here to ensure $|B| \geq 0$.

Observe that there is no monochromatic K_3 intersecting S , i.e., every monochromatic copy of K_3 must lie in $R \cup B$. In particular, a red copy of K_3 must lie completely in R , while a blue copy of K_3 must have at least two vertices in B . Therefore, if there is a monochromatic mK_3 then

$$m \leq \max \left\{ \left\lfloor \frac{|R|}{3} \right\rfloor, \left\lfloor \frac{|B|}{2} \right\rfloor \right\} \leq \max \left\{ \left\lfloor \frac{3\lfloor (\delta+1)/5 \rfloor + 2}{3} \right\rfloor, \left\lfloor \frac{2\lfloor (\delta+1)/5 \rfloor + 1}{2} \right\rfloor \right\} = \left\lfloor \frac{\delta+1}{5} \right\rfloor.$$

□

The next construction and lemma show that, if true, then (C.2) of Question 1.6 is sharp.

Extremal Example 4. For every $n, \delta \in \mathbb{N}$ such that $n \geq 25$ and $4n/5 \leq \delta \leq n-1$, we write $EX_{\blacktriangle}^2(n, \delta)$ to denote the following red/blue edge-coloured graph. We have $V(EX_{\blacktriangle}^2(n, \delta)) = V_1 \dot{\cup} \dots \dot{\cup} V_5$ where $|V_i| = n - \delta$ for every $2 \leq i \leq 5$ and $|V_1| = 4\delta - 3n$. Furthermore, $V_1 = R \dot{\cup} B$ where $|R| = 2 \lfloor \frac{4\delta - 3n + 1}{3} \rfloor + 1$ and $|B| = |V_1| - |R| \geq 0$. The sets V_2, \dots, V_5 are independent, and all remaining pairs of vertices form an edge. The subgraph $EX_{\blacktriangle}^2(n, \delta)[V_1, V_2, V_3, V_4, V_5]$ is a blow-up of a badly coloured K_5 . The edges lying in R are red, and the edges lying in B and the edges incident to both B and R are blue.

Lemma 2.4. Let $n, \delta \in \mathbb{N}$ such that $n \geq 25$ and $4n/5 \leq \delta \leq n-1$. Then $EX_{\blacktriangle}^2(n, \delta)$ is an n -vertex graph with minimum degree $\delta(EX_{\blacktriangle}^2(n, \delta)) = \delta$. Furthermore, for any monochromatic copy of mK_3 in $EX_{\blacktriangle}^2(n, \delta)$ we have $m \leq \lfloor (4\delta - 3n + 1)/3 \rfloor$.

Proof. We have $|EX_{\blacktriangle}^2(n, \delta)| = |V_1| + \dots + |V_5| = (4\delta - 3n) + 4(n - \delta) = n$ and the minimum degree is attained by vertices not in V_1 , yielding $\delta(EX_{\blacktriangle}^2(n, \delta)) = \delta$. Also, note that

$$\begin{aligned} \left(2 \left\lfloor \frac{4\delta - 3n + 1}{3} \right\rfloor + 1 \right) + \left\lfloor \frac{4\delta - 3n + 1}{3} \right\rfloor &= 3 \left\lfloor \frac{4\delta - 3n + 1}{3} \right\rfloor + 1 \\ &\geq 3 \left(\frac{4\delta - 3n - 1}{3} \right) + 1 = 4\delta - 3n = |V_1|. \end{aligned}$$

Since $|R| + |B| = |V_1|$, it follows that $|B| \leq \lfloor (4\delta - 3n + 1)/3 \rfloor$.

As $EX_{\blacktriangle}^2(n, \delta)[V_1, V_2, V_3, V_4, V_5]$ is a blow-up of a badly coloured K_5 , every monochromatic K_3 in $EX_{\blacktriangle}^2(n, \delta)$ must have at least two vertices in V_1 . Therefore, m is bounded above by the size of the largest monochromatic matching in V_1 . Observe that all red edges in V_1 lie in R , whereas all blue edges in V_1 are incident to B . Therefore, if there is a monochromatic mK_3 then

$$m \leq \max \left\{ \left\lfloor \frac{|R|}{2} \right\rfloor, |B| \right\} = \left\lfloor \frac{4\delta - 3n + 1}{3} \right\rfloor,$$

as required. □

Our final construction shows the sharpness of part (C.3) of Question 1.6 and thus of part (B.2) of Theorem 1.5.

Extremal Example 5. For every $n, \delta \in \mathbb{N}$ such that $4n/5 \leq \delta \leq n-1$, we write $EX_{\blacktriangle}^3(n, \delta)$ to denote the following red/blue edge-coloured graph. We have $V(EX_{\blacktriangle}^3(n, \delta)) = V_1 \dot{\cup} \dots \dot{\cup} V_5$ where $|V_i| = n - \delta$ for every $2 \leq i \leq 5$ and $|V_1| = 4\delta - 3n$. Furthermore, $V_1 = R \dot{\cup} B \dot{\cup} S$ where $|S| = n - \delta$, $|R| = \lceil \frac{5\delta - 4n}{2} \rceil$ and $|B| = \lfloor \frac{5\delta - 4n}{2} \rfloor$. The sets V_2, \dots, V_5 and S are independent, and all other pairs of vertices form an edge. The subgraph $EX_{\blacktriangle}^3(n, \delta)[V_1, V_2, V_3, V_4, V_5]$ is a blow-up of a badly coloured K_5 . The edges in $R \cup S$ are red, and the edges incident to B in V_1 are blue.

Lemma 2.5. Let $n, \delta \in \mathbb{N}$ such that $4n/5 \leq \delta \leq n-1$. Then $EX_{\blacktriangle}^3(n, \delta)$ is an n -vertex graph with minimum degree $\delta(EX_{\blacktriangle}^3(n, \delta)) = \delta$. Furthermore, for any monochromatic copy of mK_3 in $EX_{\blacktriangle}^3(n, \delta)$ we have $m \leq \lceil (5\delta - 4n)/2 \rceil$.

Proof. We have $|\text{EX}_\blacktriangle^3(n, d)| = |V_1| + \dots + |V_5| = (4\delta - 3n) + 4(n - \delta) = n$, and the minimum degree is $\delta(\text{EX}(n, \delta)) = \delta$, attained by the vertices in $V_2 \cup \dots \cup V_5 \cup S$.

As $\text{EX}_\blacktriangle^3(n, \delta)[V_1, V_2, V_3, V_4, V_5]$ is a blow-up of a badly coloured K_5 , every monochromatic K_3 in $\text{EX}_\blacktriangle^3(n, \delta)$ must have at least two vertices in V_1 . Therefore, m is bounded above by the size of the largest monochromatic matching in V_0 . Observe that all red edges in V_1 are adjacent to R , whereas all blue edges in V_1 are incident to B . Hence, any monochromatic matching has size at most $\max\{|R|, |B|\} = \lceil \frac{5\delta - 4n}{2} \rceil$, implying $m \leq \lceil \frac{5\delta - 4n}{2} \rceil$ as required. \square

3. PROOF OF THEOREM 1.7

The proofs of parts (M.1), (M.2) and (M.3) of Theorem 1.7 follow a common strategy. We first partition the vertex set of the host graph into vertex-disjoint cliques of appropriate size and then find monochromatic copies of K_3 within each clique. The first step is essentially achieved by applying the Hajnal–Szemerédi theorem, which we now state.

Theorem 3.1 (Hajnal–Szemerédi theorem [17]). *Let $n, t \in \mathbb{N}$ such that t divides n . If G is a graph on n vertices with $\delta(G) \geq (1 - 1/t)n$ then G contains a perfect K_t -tiling.*

It will be convenient to use the following apparently stronger but in fact equivalent statement to the Hajnal–Szemerédi theorem. While it is a well-known statement, for the sake of completeness we show how to deduce it from the Hajnal–Szemerédi theorem.

Theorem 3.2. *Let $n, t \in \mathbb{N}$ and G be a graph on n vertices such that*

$$\left(1 - \frac{1}{t-1}\right)n \leq \delta(G) \leq \left(1 - \frac{1}{t}\right)n.$$

Then G contains a K_t -tiling consisting of $(t-1)\delta(G) - (t-2)n$ copies of K_t and a K_{t-1} -tiling consisting of $(t-1)n - t\delta(G)$ copies of K_{t-1} , such that the two tilings are vertex-disjoint.

Proof. Let G' be the graph obtained by adding a set S of $(t-1)n - t\delta(G) \geq 0$ new vertices to G and all edges with exactly one vertex in S . Then G' is a graph on $n' := n + |S| = t(n - \delta(G))$ vertices with minimum degree

$$\delta(G') = \min\{\delta(G) + |S|, n\} = \min\{(t-1)(n - \delta(G)), n\} = (t-1)(n - \delta(G)).$$

In particular, $\delta(G') = (1 - 1/t)n'$ and n' is divisible by t . By the Hajnal–Szemerédi theorem (Theorem 3.1), G' contains a perfect K_t -tiling consisting of n'/t copies of K_t . Observe that no edge lies inside S , thus each copy of K_t contains at most one vertex in S . In particular, $n'/t - |S|$ copies of K_t do not contain a vertex from S and $|S|$ copies of K_t contain exactly one vertex from S .

It follows that the original graph G contains a K_t -tiling consisting of $n'/t - |S| = (t-1)\delta(G) - (t-2)n$ copies of K_t and a K_{t-1} -tiling consisting of $|S| = (t-1)n - t\delta(G)$ copies of K_{t-1} , such that the two tilings are vertex-disjoint. \square

At various points of our proofs, we will invoke the following well-known fact.

Fact 3.3. *A 2-edge-coloured K_6 contains two monochromatic copies of K_3 .* \square

Note that in Fact 3.3 the two copies of K_3 are not necessarily vertex-disjoint. The next two lemmas assert that, for larger dense graphs, we can indeed guarantee the existence of multiple vertex-disjoint monochromatic copies of K_3 . The first lemma is new, while the second lemma is an immediate corollary of Theorem 1.2.

Lemma 3.4. *A 2-edge-coloured $K_7(2)$ contains three vertex-disjoint monochromatic copies of K_3 .*

Lemma 3.5 (Moon [25]). *A 2-edge-coloured K_8 contains two vertex-disjoint monochromatic copies of K_3 .*

To apply Lemma 3.4 in conjunction with Theorem 3.2, we will apply Szemerédi's Regularity Lemma [30] and the Blow-up Lemma [20]; the downside of using these techniques is that they cause the $o(n)$ error term to appear in part (M.2) of Theorem 1.7.

In the next three subsections we prove parts (M.3), (M.1) and (M.2) of Theorem 1.7, respectively. The proof of Lemma 3.4 appears at the end of the section.

3.1. Proof of Theorem 1.7(M.3). Let $n \in \mathbb{N}$ and let G be a 2-edge-coloured n -vertex graph with $4n/5 \leq \delta(G) \leq 5n/6$. By Theorem 3.2, G contains a K_6 -tiling consisting of $5\delta(G) - 4n$ copies of K_6 . By Fact 3.3, each K_6 contains a monochromatic copy of K_3 . It follows that G contains a K_3 -tiling consisting of $5\delta(G) - 4n$ monochromatic copies of K_3 , as required. Q.E.D.

3.2. Proof of Theorem 1.7(M.1). Recall that part (M.1) of Theorem 1.7 states that any 2-edge-coloured n -vertex graph G with $\delta(G) \geq 7n/8$ contains a K_3 -tiling consisting of $\lfloor (2\delta(G) - n)/3 \rfloor$ monochromatic copies of K_3 . We prove this by induction on n .

Before this, we prove the case when $7n/8 \leq \delta(G) \leq (7n + 2)/8$ (for all $n \in \mathbb{N}$). Note that any induced subgraph H of G with $|H| = 8(n - \delta(G)) \leq n$ satisfies $\delta(H) \geq \delta(G) - (|G| - |H|) = 7(n - \delta(G)) = 7|H|/8$. Theorem 3.1 implies H , and thus G , contains a K_8 -tiling consisting of $n - \delta(G)$ copies of K_8 . By Lemma 3.5, each K_8 contains two vertex-disjoint monochromatic copies of K_3 . Taking the union of all such copies yields a K_3 -tiling consisting of precisely $2(n - \delta(G))$ monochromatic copies of K_3 . This concludes the verification of this case, as

$$\left\lfloor \frac{2\delta(G) - n}{3} \right\rfloor = 2(n - \delta(G)) + \left\lfloor \frac{8\delta(G) - 7n}{3} \right\rfloor \leq 2(n - \delta(G)) + \left\lfloor \frac{2}{3} \right\rfloor = 2(n - \delta(G)).$$

Now we can proceed by induction on n . The base cases when $8 \leq n \leq 10$ are covered by the last paragraph. Next, we check the inductive step. Suppose G is an n -vertex graph where $n \geq 11$. By the previous paragraph we may assume that $\delta(G) \geq (7n + 3)/8$. It is easy to show that G contains a K_6 (e.g., by Theorem 3.2), which in turn contains a monochromatic copy T of K_3 by Fact 3.3. Let $G' := G \setminus V(T)$. Note that G' is a 2-edge-coloured graph on $n - 3$ vertices with minimum degree $\delta(G') \geq \delta(G) - 3 \geq 7(n - 3)/8$. By the inductive hypothesis, G' contains a K_3 -tiling consisting of $\lfloor (2\delta(G') - (n - 3))/3 \rfloor \geq \lfloor (2\delta(G) - n)/3 \rfloor - 1$ monochromatic copies of K_3 . Adding T to this tiling yields a K_3 -tiling in G consisting of at least $\lfloor (2\delta(G) - n)/3 \rfloor$ monochromatic copies of K_3 . This concludes the inductive step and the proof. Q.E.D.

3.3. Proof of Theorem 1.7(M.2). To prove this part of the theorem, it suffices to show the following: Let $\eta > 0$ and $n \in \mathbb{N}$ be sufficiently large. Let G be an n -vertex 2-edge-coloured graph with $5n/6 \leq \delta(G) \leq 7n/8$. Then G contains a K_3 -tiling consisting of at least $(4\delta(G) - 3n)/2 - \eta n$ monochromatic copies of K_3 .

Let G be as in this statement. First, we use part (M.3) of Theorem 1.7 to show that we may assume $\delta(G)$ is bounded away from $5n/6$.

Claim 3.6. *Either $\delta(G) \geq (5/6 + \eta/4)n$ or G contains a K_3 -tiling consisting of at least $(4\delta(G) - 3n)/2 - \eta n$ monochromatic copies of K_3 .*

Proof. Suppose $5n/6 \leq \delta(G) < (5/6 + \eta/4)n$. Then there exists a spanning subgraph F of G with $\delta(F) = \lfloor 5n/6 \rfloor = n - \lceil n/6 \rceil$. We can therefore apply part (M.3) of Theorem 1.7 to F . Thus, F (and so G) contains $n - 5\lceil n/6 \rceil \geq n/6 - 5$ vertex-disjoint monochromatic copies of K_3 . As $\delta(G) < (5/6 + \eta/4)n$, we have that $(4\delta(G) - 3n)/2 - \eta n < n/6 - 5$. Then indeed G contains a K_3 -tiling consisting of at least $(4\delta(G) - 3n)/2 - \eta n$ monochromatic copies of K_3 , as desired. \square

By Claim 3.6, we may assume $\delta(G) \geq (5/6 + \eta/4)n$. As mentioned at the beginning of this section, we now employ Szemerédi's Regularity Lemma [30] and the Blow-up Lemma [20]. Essentially, the former provides an auxiliary graph R (which is commonly referred to as the *reduced graph*)

which approximates G in the following sense: the vertex set of G can be partitioned into vertex classes $\{V_v : v \in V(R)\}$ and a small exceptional set V_0 such that if $xy \in E(R)$, then the edges between V_x and V_y are evenly distributed. One can then argue that, using the Blow-up Lemma, for our purposes such evenly distributed edges behave essentially as a complete bipartite graph. In particular, given a collection of vertex-disjoint cliques in R , one can find vertex-disjoint blow-ups of cliques in the original graph. Our strategy then is to find an appropriate collection of vertex-disjoint copies of K_6 , K_7 and K_8 in the reduced graph, using Theorem 3.2. This yields a collection of vertex-disjoint copies of $K_6(2)$, $K_7(2)$ and $K_8(2)$ in the original graph G . We then apply Fact 3.3, Lemma 3.5 and, crucially, Lemma 3.4.

The next result formalises the ‘‘embedding step’’ from the reduced graph to the original graph G described above. For simplicity of exposition, we avoid introducing the standard notation used for the Regularity Lemma and instead keep the technicalities to a minimum. We note that the proof of this result is standard; the proof of the statement can be found in the appendix.

Lemma 3.7 (Embedding step). *For every $\eta > 0$ there exists $n_0 = n_0(\eta) \in \mathbb{N}$ such that for every graph G on $n \geq n_0$ vertices the following holds. There exist $m, \ell \in \mathbb{N}$, a partition V_0, V_1, \dots, V_ℓ of $V(G)$ and a graph R with vertex set $\{V_i : i \geq 1\}$ such that the following properties hold:*

- (i) $\delta(G)/n - \eta/4 \leq \delta(R)/|R| \leq \delta(G)/n$;
- (ii) $|V_i| = m$ for every $i \geq 1$ and $|V_0| \leq \eta n/2$;
- (iii) *If the vertices $\{V_{i_1}, \dots, V_{i_r}\}$ in R form a clique and $r \leq 8$, then $G[V_{i_1}, \dots, V_{i_r}]$ contains a $K_r(2)$ -tiling consisting of at least $(1 - \eta/2)m/2$ copies of $K_r(2)$.*

Apply Lemma 3.7 to the graph G to obtain $m, \ell \in \mathbb{N}$, a partition V_0, V_1, \dots, V_ℓ of $V(G)$ and a graph R satisfying properties (i)–(iii) of Lemma 3.7.

Claim 3.8. *There exists a K_3 -tiling in G consisting of at least $(4\delta(R) - 3|R|) \cdot \frac{(1-\eta/2)m}{2}$ monochromatic copies of K_3 .*

Proof. By property (i) and the fact that $(5/6 + \eta/4)n \leq \delta(G) \leq 7n/8$, it follows that

$$5|R|/6 \leq \delta(R) \leq 7|R|/8.$$

Suppose first that $\delta(R) \leq 6|R|/7$. Then by Theorem 3.2, R contains a K_7 -tiling consisting of $6\delta(R) - 5|R|$ copies of K_7 and a K_6 -tiling consisting of $6|R| - 7\delta(R)$ copies of K_6 such that the two tilings are vertex-disjoint. By property (iii), G contains a $K_7(2)$ -tiling consisting of at least $(6\delta(R) - 5|R|) \frac{(1-\eta/2)m}{2}$ copies of $K_7(2)$ and a $K_6(2)$ -tiling consisting of at least $(6|R| - 7\delta(R)) \frac{(1-\eta/2)m}{2}$ copies of $K_6(2)$ such that the two tilings are vertex-disjoint. Fact 3.3 implies that every copy of $K_6(2)$ contains two vertex-disjoint monochromatic copies of K_3 , whereas Lemma 3.4 implies every copy of $K_7(2)$ contains three vertex-disjoint monochromatic copies of K_3 . It follows that G contains a K_3 -tiling consisting of at least

$$(3 \cdot (6\delta(R) - 5|R|) + 2 \cdot (6|R| - 7\delta(R))) \cdot \frac{(1 - \eta/2)m}{2} = (4\delta(R) - 3|R|) \cdot \frac{(1 - \eta/2)m}{2}$$

monochromatic copies of K_3 , as required.

The case $\delta(R) \geq 6|R|/7$ is very similar. By Theorem 3.2, R contains a K_8 -tiling consisting of $7\delta(R) - 6|R|$ copies of K_8 and a K_7 -tiling consisting of $7|R| - 8\delta(R)$ copies of K_7 such that the two tilings are vertex-disjoint. It follows that G contains a $K_8(2)$ -tiling consisting of at least $(7\delta(R) - 6|R|) \frac{(1-\eta/2)m}{2}$ copies of $K_8(2)$ and a $K_7(2)$ -tiling consisting of at least $(7|R| - 8\delta(R)) \frac{(1-\eta/2)m}{2}$ copies of $K_7(2)$ such that the two tilings are vertex-disjoint. Lemma 3.5 implies every copy of $K_8(2)$ contains four vertex-disjoint monochromatic copies of K_3 , whereas Lemma 3.4 implies every copy of $K_7(2)$ contains three vertex-disjoint monochromatic copies of K_3 . It follows that G contains

a K_3 -tiling consisting of at least

$$(3 \cdot (7|R| - 8\delta(R)) + 4 \cdot (7\delta(R) - 6|R|)) \cdot \frac{(1 - \eta/2)m}{2} = (4\delta(R) - 3|R|) \cdot \frac{(1 - \eta/2)m}{2}$$

monochromatic copies of K_3 . □

Note that $m = (n - |V_0|)/|R|$ and thus $m \geq (1 - \eta/2)n/|R|$ by property (ii). Furthermore, $\delta(R)/|R| \geq \delta(G)/n - \eta/4$ by property (i). Using these inequalities, we obtain

$$\begin{aligned} (4\delta(R) - 3|R|) \cdot \frac{(1 - \eta/2)m}{2} &\geq (4\delta(R) - 3|R|) \cdot \frac{(1 - \eta/2)^2 n}{2|R|} = (1 - \eta/2)^2 \cdot (4\delta(R)/|R| - 3) \frac{n}{2} \\ &\geq (1 - \eta/2)^2 \cdot \frac{(4\delta(G) - \eta n - 3n)}{2} \geq \frac{4\delta(G) - 3n}{2} - \eta n. \end{aligned}$$

Therefore, by Claim 3.8, G contains a K_3 -tiling consisting of at least $(4\delta(G) - 3n)/2 - \eta n$ monochromatic K_3 . This concludes the proof of case (M.2). Q.E.D.

3.4. Proof of Lemma 3.4. We start with the following claim.

Claim 3.9. *A 2-edge-coloured K_7 contains two monochromatic K_3 sharing at most one vertex.*

Proof. Suppose the statement of the claim does not hold. Let G be a 2-edge-coloured complete graph with $V(G) = \{v_1, \dots, v_7\}$ such that every pair of monochromatic copies of K_3 share two vertices. By Fact 3.3, there are two monochromatic copies of K_3 in G , without loss of generality $T_1 = v_1v_2v_3$ and $T_2 = v_1v_2v_4$. Note that the only copy of K_3 in $G \setminus \{v_1\}$ that shares two vertices with both T_1 and T_2 is $v_2v_3v_4$. In particular, $G \setminus \{v_1\}$ contains at most one monochromatic K_3 . Since $G \setminus \{v_1\}$ is a copy of K_6 , this contradicts Fact 3.3. □

Let G be a 2-edge-coloured $K_7(2)$ with vertex set $U \dot{\cup} V$ where $U = \{u_1, \dots, u_7\}$, $V = \{v_1, \dots, v_7\}$ and the non-edges of G are precisely the pairs $u_i v_i$ for $i \in [7]$. By Claim 3.9, there are two monochromatic copies of K_3 in $G[V]$ sharing at most one vertex. If they are vertex-disjoint we are done, thus we may assume that $v_1v_2v_3$ and $v_3v_4v_5$ are monochromatic copies of K_3 . By Fact 3.3, the graph $G[U \setminus \{u_3\}]$ contains a monochromatic copy of K_3 , say $u_i u_j u_k$. By the pigeonhole principle, we have either $|\{1, 2\} \cap \{i, j, k\}| \leq 1$ or $|\{4, 5\} \cap \{i, j, k\}| \leq 1$. Without loss of generality we may assume $|\{1, 2\} \cap \{i, j, k\}| \leq 1$ and in particular $1 \notin \{i, j, k\}$. Let $S := \{u_1, u_3, v_4, v_5, v_6, v_7\}$. Observe that S is vertex-disjoint to $v_1v_2v_3$ and $u_i u_j u_k$. Furthermore, $G[S]$ is a copy of K_6 and thus it contains a monochromatic copy T of K_3 by Fact 3.3. Note that T , $v_1v_2v_3$ and $u_i u_j u_k$ are three vertex-disjoint monochromatic copies of K_3 , as required. Q.E.D.

4. PROOF OF THEOREM 1.5

Let $n \in \mathbb{N}$ and G be a 2-edge-coloured n -vertex graph. If $4n/5 \leq \delta(G) \leq 5n/6$ then by part (M.3) of Theorem 1.7 there is a K_3 -tiling in G consisting of at least $5\delta(G) - 4n$ monochromatic K_3 . At least $m := \lceil (5\delta(G) - 4n)/2 \rceil$ of these triangles receive the same colour, and thus they form a monochromatic copy of mK_3 . This verifies part (B.2) of Theorem 1.5.

For part (B.1), a different approach is needed. We use the following definition which was introduced in [8] for the proof of Theorem 1.1.

Definition 4.1. A *bowtie* consists of two monochromatic copies of K_3 of different colours which share exactly one vertex.

A useful fact, observed in [8], is that if a complete graph contains two vertex-disjoint monochromatic copies of K_3 of different colours then it must contain a bowtie. The following lemma is a strengthening of this statement.

Lemma 4.2. *Suppose a 2-edge-coloured K_6 contains two vertex-disjoint monochromatic copies of K_3 of different colours. Then for every vertex $v \in V(K_6)$, there exists a bowtie containing v .*

Proof. Without loss of generality, we may assume $V(K_6) = [6]$, 123 is a red K_3 and 456 is a blue K_3 . By symmetry, it suffices to prove the statement of the lemma for $v = 1$.

If 1 is incident to two blue edges, 14 and 15 say, then the copies 123 and 145 of K_3 form a bowtie containing 1. Thus, 1 is incident to at most one blue edge. Similarly, 2 is incident to at most one blue edge. It follows that for some $i \in \{4, 5, 6\}$ the edges $1i$ and $2i$ are red. Then the copies 123 and 456 of K_3 form a bowtie containing 1. \square

Using Lemma 4.2, we obtain the following.

Lemma 4.3. *Suppose a 2-edge-coloured K_7 contains a bowtie. Then there exists another bowtie on a different vertex set.*

Proof. Let B be a bowtie in K_7 and let K_B denote the blue copy of K_3 in B . Let $\{x, y\} = V(K_7) \setminus V(B)$. It suffices to show that there exists a monochromatic copy K of K_3 containing either x or y (or both). Indeed, suppose such K exists and without loss of generality suppose that K is red. If K and K_B are disjoint, then Lemma 4.2 implies that there is a bowtie B' containing either x or y ; so B and B' have different vertex sets. If K and K_B intersect, then they must share precisely one vertex; so K and K_B form a bowtie on a different vertex set to B .

We now prove that K exists. Without loss of generality, we may assume xy is red. If there is a vertex $z \in V(K_B)$ such that xz and yz are red, we are done. Thus, for every $z \in V(K_B)$, there is a blue edge incident to z and xy ; so two vertices of K_B must be both adjacent via blue edges to some vertex $w \in \{x, y\}$, and so we are done. \square

We are now ready to prove part (B.1). Let $n \in \mathbb{N}$ and G be a 2-edge-coloured n -vertex graph with $\delta(G) \geq 65n/66$. Set $m := \lfloor (\delta(G) + 1)/5 \rfloor$.

Let \mathcal{B} and \mathcal{T} be two collections of subsets of $V(G)$ satisfying the following properties.

Properties:

- (i) For every distinct $X, Y \in \mathcal{B} \cup \mathcal{T}$ we have $X \cap Y = \emptyset$.
- (ii) Each $X \in \mathcal{B}$ induces a copy of K_5 in G that contains a bowtie.
- (iii) Each $X \in \mathcal{T}$ induces a monochromatic copy of K_3 in G . Moreover, all these monochromatic copies of K_3 have the same colour.
- (iv) $|\mathcal{B}|$ is as large as possible. Conditioned on this, $|\mathcal{T}|$ is as large as possible.

It is easy to see that G contains a monochromatic copy of $(|\mathcal{B}| + |\mathcal{T}|)K_3$: if the copies of K_3 obtained from \mathcal{T} are red say, then we select a red K_3 in each element of \mathcal{B} and \mathcal{T} and then take their disjoint union. Hence, it suffices to show that $|\mathcal{B}| + |\mathcal{T}| \geq m$. We assume for a contradiction that $|\mathcal{B}| + |\mathcal{T}| < m$. We abuse notation slightly and write $V(\mathcal{B})$, $V(\mathcal{T})$ and $V(\mathcal{B} \cup \mathcal{T})$ to denote the number of vertices covered by the elements of \mathcal{B} , \mathcal{T} and $\mathcal{B} \cup \mathcal{T}$ respectively.

We start by providing a lower bound on $|\mathcal{B}|$.

Claim 4.4. *We have $|\mathcal{B}| \geq 5n/33$.*

Proof. Suppose that $|\mathcal{B}| < 5n/33$. Then the number of vertices in $V(\mathcal{B} \cup \mathcal{T})$ is

$$5|\mathcal{B}| + 3|\mathcal{T}| \leq 5|\mathcal{B}| + 3(m - 1 - |\mathcal{B}|) = 3m + 2|\mathcal{B}| - 3 < \frac{3\delta(G)}{5} + \frac{10n}{33}.$$

First suppose that $\mathcal{T} = \emptyset$. Since $\delta(G) \geq 65n/66$, we have that $\delta(G \setminus V(\mathcal{B})) > 4|G \setminus V(\mathcal{B})|/5$, hence there exists a K_6 in $G \setminus V(\mathcal{B})$. However, this copy of K_6 must contain a monochromatic K_3 that is vertex-disjoint to $V(\mathcal{B})$, contradicting the assumption that $\mathcal{T} = \emptyset$.

Suppose now that $\mathcal{T} \neq \emptyset$, say \mathcal{T} contains a set inducing a red copy T of K_3 . Let $n' := n - |V(\mathcal{B} \cup \mathcal{T})|$. Since $\delta(G) \geq 65n/66$ and $|V(\mathcal{B} \cup \mathcal{T})| < 3\delta(G)/5 + 10n/33$, we have that every vertex $x \in V(G)$ has at least $\delta(G) - |V(\mathcal{B} \cup \mathcal{T})| > 6n'/7$ neighbours in G that lie in $V(G) \setminus V(\mathcal{B} \cup \mathcal{T})$. Thus, there exists a K_5 in $G \setminus V(\mathcal{B} \cup \mathcal{T})$ that together with T forms a K_8 in G . If this copy of K_8 contains a blue K_3 then it contains a bowtie by Lemma 4.2. This contradicts the maximality of \mathcal{B} , thus any monochromatic K_3 in this copy of K_8 must be red. By Lemma 3.5, this K_8 contains two vertex-disjoint red copies of K_3 . This contradicts the maximality of \mathcal{T} . \square

Combining the lower bound in Claim 4.4 with the minimum degree condition, we obtain the following claim.

Claim 4.5. *Let $S \subseteq V(G) \setminus V(\mathcal{B})$ such that $|S| \leq 10$. Then there exists $B \in \mathcal{B}$ such that the graph $G[B \cup \{s\}]$ is complete for every $s \in S$.*

Proof. Suppose for a contradiction the claim is false; so there does not exist a set $B \in \mathcal{B}$ such that, in G , the vertices in $S \subseteq V(H)$ are adjacent to every vertex in B . This implies that there is a vertex $s \in S$ that is non-adjacent to at least $|\mathcal{B}|/|S| \geq n/66$ vertices in G (and itself), a contradiction as $\delta(G) \geq 65n/66$. \square

We are now ready to combine all our lemmas and claims to conclude the proof.

Let $X := \emptyset$. If \mathcal{T} is non-empty, let T be an element of \mathcal{T} . Otherwise, set $T := \emptyset$. We iterate the following procedure as long as there is some edge uv in G non-incident to $V(\mathcal{B} \cup \mathcal{T}) \cup X$ and $|X| \leq 5$. During the procedure, we maintain the property that $G[T \cup X]$ is a clique.

Procedure: Apply Claim 4.5 with $S := T \cup X \cup \{u, v\}$ to find some $B \in \mathcal{B}$ such that $G[B \cup \{s\}]$ is complete for every $s \in S$. In particular, $G[B \cup \{u, v\}]$ is a copy of K_7 .

By Lemma 4.3, there exists a vertex set $B' \subseteq B \cup \{u, v\}$ such that B' spans a bowtie and $B' \neq B$. Let $z \in B \setminus B'$.

Set $\mathcal{B} := (\mathcal{B} \setminus \{B\}) \cup \{B'\}$ and $X := X \cup \{z\}$. Note that \mathcal{B} and \mathcal{T} still satisfy the initial properties. Furthermore, $G[T \cup X]$ is a clique.

END PROCEDURE

If at the end of this procedure $|X| = 6$, then define $Y := X$. If $|X| \leq 5$ and there does not exist $w \in V(G) \setminus (V(\mathcal{B} \cup \mathcal{T}) \cup X)$ such that $G[T \cup X \cup \{w\}]$ is a clique, then we set $Y := X$. Otherwise, there is a $w \in V(G) \setminus (V(\mathcal{B} \cup \mathcal{T}) \cup X)$ such that $G[T \cup X \cup \{w\}]$ is a clique, however, in G , w is not adjacent to any vertex in $V(G) \setminus (V(\mathcal{B} \cup \mathcal{T}) \cup X)$;² in this case we define $Y := X \cup \{w\}$.

Note that in all cases $G[T \cup Y]$ is a clique. Further, if $|Y| \leq 5$ then there are no edges in $G \setminus (V(\mathcal{B} \cup \mathcal{T}) \cup Y)$.

Claim 4.6. *Either (a) $|Y| = 6$ or (b) $|Y| = 5$ and $|\mathcal{B}| + |\mathcal{T}| = m - 1$.*

Proof. If $|Y| = 6$ we are done, so suppose that $|Y| \leq 5$. If $V(G) = V(\mathcal{B} \cup \mathcal{T}) \cup Y$, then $\delta(G) \leq |V(\mathcal{B} \cup \mathcal{T}) \cup Y| - 1$. Otherwise, by definition of Y , for every vertex x in $V(G) \setminus (V(\mathcal{B} \cup \mathcal{T}) \cup Y)$ we have $\delta(G) \leq d_G(x) \leq |V(\mathcal{B} \cup \mathcal{T}) \cup Y| - 1$.

In both cases we conclude that

$$\delta(G) \leq |V(\mathcal{B} \cup \mathcal{T}) \cup Y| - 1 \leq 5(|\mathcal{B}| + |\mathcal{T}|) + |Y| - 1.$$

Combining the above with $|\mathcal{B}| + |\mathcal{T}| \leq m - 1$ and $m \leq (\delta(G) + 1)/5$ we obtain

$$\delta(G) \leq 5m - 6 + |Y| \leq \delta(G) - 5 + |Y|.$$

In particular, we must have $|Y| \geq 5$. However, we assumed that $|Y| \leq 5$. Therefore, $|Y| = 5$ and all the above inequalities are in fact equalities. Thus, we have $|\mathcal{B}| + |\mathcal{T}| = m - 1$. \square

²This latter condition follows by definition of the procedure above.

First, suppose \mathcal{T} is non-empty, hence $G[T]$ is a monochromatic K_3 . We have that $G[T \cup Y]$ is a clique, and in particular it has at least $|T| + |Y| \geq 3 + 5 = 8$ vertices. If $G[T \cup Y]$ contains a monochromatic K_3 whose colour is different from the colour of $G[T]$, then by Lemma 4.2 it contains a bowtie. This contradicts the assumption that \mathcal{B} is maximal. Hence, all monochromatic copies of K_3 in $G[T \cup Y]$ must be of the same colour as $G[T]$. By Lemma 3.5, $G[T \cup Y]$ must contain two vertex-disjoint monochromatic copies of K_3 . This contradicts the assumption that \mathcal{T} is maximal.

Therefore, it must be the case that \mathcal{T} is empty. If $|Y| = 6$, then $G[Y]$ contains a monochromatic K_3 that does not intersect $V(\mathcal{B})$. This contradicts the maximality of \mathcal{T} . Hence, we may assume that $|Y| = 5$ and so $|\mathcal{B}| + |\mathcal{T}| = |\mathcal{B}| = m - 1$. By applying Claim 4.5 with $S = Y$, there is some B in \mathcal{B} such that $G[B \cup Y]$ is a clique. Since $|B| + |Y| = 5 + 5 = 10$, by Theorem 1.1, $G[B \cup Y]$ contains a monochromatic copy of $2K_3$, say red. It follows that there are $|\mathcal{B} \setminus \{B\}| + 2$ vertex-disjoint red copies of K_3 . Since $|\mathcal{B}| = m - 1$, it follows that there is a red copy of mK_3 , as required. Q.E.D.

5. FURTHER RESULTS AND CONCLUDING REMARKS

5.1. Further results on Problem 1.4. In this subsection, we discuss some further results related to Problem 1.4. First, similarly to Theorems 1.5 and 1.7, if δ is not large enough then we might not be able to ensure even a single monochromatic copy of the sought structure. This is formalised via the *chromatic Ramsey number*, a parameter introduced in [7].

Definition 5.1. For $r \in \mathbb{N}$ and a graph H , we say a graph G is (H, r) -Ramsey if every r -edge-colouring of G contains a monochromatic copy of H . The *chromatic Ramsey number* $R_\chi(H, r)$ is the least $m \in \mathbb{N}$ such that there exists an (H, r) -Ramsey graph of chromatic number m .

For example, it is simple to see that $R_\chi(K_\ell, r) = R_r(K_\ell)$ for all $\ell, r \geq 2$.

Note that by definition of $R_\chi(H, r)$, there is an r -edge-colouring of the $(R_\chi(H, r) - 1)$ -partite Turán graph on n vertices (for any choice of $n \in \mathbb{N}$) that does not yield a monochromatic copy of H . Thus, Problems 1.3 and 1.4 are trivial if

$$(2) \quad \delta \leq \left\lfloor \left(1 - \frac{1}{R_\chi(H, r) - 1} \right) n \right\rfloor,$$

since, in this case, we cannot guarantee even a single monochromatic copy of H . On the other hand, the Erdős–Stone–Simonovits theorem implies that for any $\eta > 0$, every sufficiently large r -edge-coloured n -vertex graph G with $\delta(G) \geq (1 - \frac{1}{R_\chi(H, r) - 1} + \eta)n$ does contain a monochromatic copy of H .

The next simple result generalises part (M.3) of Theorem 1.7 for larger cliques and multiple colours. Set $R_r(\ell) := R_r(K_\ell)$.

Theorem 5.2. *Let $n, r \in \mathbb{N}$, $\ell \geq 2$, and let G be an r -edge-coloured n -vertex graph such that*

$$\left(1 - \frac{1}{R_r(\ell) - 1} \right) n \leq \delta(G) \leq \left(1 - \frac{1}{R_r(\ell)} \right) n.$$

Then there exists a K_ℓ -tiling in G such that every copy of K_ℓ is monochromatic and the number of copies of K_ℓ in the tiling is at least $(R_r(\ell) - 1)\delta(G) - (R_r(\ell) - 2)n$.

Proof. By Theorem 3.2, G contains a $K_{R_r(\ell)}$ -tiling consisting of $(R_r(\ell) - 1)\delta(G) - (R_r(\ell) - 2)n$ copies of $K_{R_r(\ell)}$. By definition, each $K_{R_r(\ell)}$ contains a monochromatic copy of K_ℓ . It follows that G contains a K_ℓ -tiling consisting of $(R_r(\ell) - 1)\delta(G) - (R_r(\ell) - 2)n$ monochromatic copies of K_ℓ , as required. \square

Note that Theorem 5.2 is best possible. Indeed, consider an r -edge-coloured $K_{R_r(\ell)}$ such that all monochromatic copies of K_ℓ have a common vertex v . Let $\delta \in \mathbb{N}$ so that $(1 - \frac{1}{R_r(\ell) - 1})n < \delta \leq$

$(1 - \frac{1}{R_r(\ell)})n$. Consider the n -vertex blow-up G of this $K_{R_r(\ell)}$ where v is replaced by a class V of size $(R_r(\ell) - 1)\delta - (R_r(\ell) - 2)n$ and all other vertices are replaced by a class of size $n - \delta$. Note that $\delta(G) = \delta$, and every monochromatic copy of K_ℓ in G must contain some vertex in V . Therefore, any collection of more than $|V|$ monochromatic copies of K_ℓ cannot form a K_ℓ -tiling.

The next theorem considers the case where $\delta(G)$ is close to $n - 1$. To state it, we need the following variant of the Ramsey number.

Definition 5.3. Let $r \geq 2$. A *special r -edge-colouring* of a graph G is an r -edge-colouring of G using colours c_1, \dots, c_r such that there exists a $v \in V(G)$ that is not incident to any edge of colour c_i , for some $i \in [r]$. The *special Ramsey number* $SR_r(H)$ is the smallest $n \in \mathbb{N}$ such that any special r -colouring of K_n yields a monochromatic copy of H . We write $SR_r(\ell) := SR_r(K_\ell)$.

Theorem 5.4. *For every $r, \ell \geq 2$, there exists $n_0 \in \mathbb{N}$ such that the following holds. Let G be an r -edge-coloured graph on $n \geq n_0$ vertices and with $\delta(G) \geq (1 - 1/n_0)n$. Then there exists a K_ℓ -tiling in G consisting of monochromatic K_ℓ and the number of copies of K_ℓ in the tiling is at least*

$$\left\lfloor \frac{(SR_r(\ell) - 2)\delta(G) - (SR_r(\ell) - 3)n}{\ell} \right\rfloor.$$

Observe that the bound on the number of copies of K_ℓ in Theorem 5.4 is optimal, as shown by the following construction. Firstly, the case $\ell = 2$ is trivial since $SR_r(2) = 2$ and an n -vertex graph can have at most $\lfloor n/2 \rfloor$ vertex-disjoint copies of K_2 . So fix $r, \ell \in \mathbb{N}$ with $r \geq 2$ and $\ell \geq 3$, and let $n_0 := SR_r(\ell) - 2 \geq 1$. Pick any $n, \delta \in \mathbb{N}$ with $n - 1 \geq \delta > (1 - 1/n_0)n$. Consider a special r -edge-coloured copy H of $K_{SR_r(\ell)-1}$ which does not contain a monochromatic copy of K_ℓ ; say the vertex v is not adjacent to any red edge. Let G be the r -edge-coloured graph obtained by blowing up each vertex in $V(H) \setminus \{v\}$ to a class of size $n - \delta \geq 1$ and v to a class U of size $n - (n - \delta)(SR_r(\ell) - 2)$. Observe that

$$n - (n - \delta)(SR_r(\ell) - 2) > n - (n - (1 - 1/n_0)n)(SR_r(\ell) - 2) = 0,$$

so the blow-up is well-defined. Finally, add all edges inside U and colour them red. Note that $\delta(G) = n - (n - \delta) = \delta$. Any monochromatic copy of K_ℓ in G must have at least two vertices in U , otherwise there would be a monochromatic K_ℓ in H , a contradiction. However, since all edges in U are red and all edges between U and $V(G) \setminus U$ are not red, it follows that every monochromatic copy of K_ℓ must lie completely in U . Therefore, if there are m vertex-disjoint monochromatic copies of K_ℓ in G then

$$m \leq \left\lfloor \frac{|U|}{\ell} \right\rfloor = \left\lfloor \frac{(SR_r(\ell) - 2)\delta - (SR_r(\ell) - 3)n}{\ell} \right\rfloor,$$

where the right-hand side of the above inequality matches the bound in Theorem 5.4.

Recall that the proof of case (M.1) of Theorem 1.7 combined Lemma 3.5 with the Hajnal–Szemerédi theorem. To prove Theorem 5.4, we need a generalisation of Lemma 3.5 to larger cliques and multiple colours. For two colours, such a generalisation was obtained by Burr, Erdős and Spencer [8], as mentioned in the introduction. We obtain a further generalisation for more colours.

Theorem 5.5. *For every $r, \ell \geq 2$, there exists some sufficiently large $t \in \mathbb{N}$ such that every r -edge-coloured $K_{t\ell + SR_r(\ell) - 2}$ contains t vertex-disjoint monochromatic copies of K_ℓ .*

Theorem 5.5 is sharp. Indeed, pick a special r -edge-colouring of $K_{SR_r(\ell)-1}$ that does not contain a monochromatic K_ℓ ; say no edge of colour c is adjacent to the vertex v in $K_{SR_r(\ell)-1}$. Then blow-up the vertex v into a class A of size $t\ell - 1$, and add all edges inside A and colour them c . The resulting graph is an r -edge-coloured $K_{t\ell + SR_r(\ell) - 3}$. Every monochromatic K_ℓ must lie in A , and thus there are no t vertex-disjoint monochromatic K_ℓ .

Proof of Theorem 5.5. Take $t_0 := (\ell - 1)R_r(\ell)$, $t := R_r(t_0)$ and $n := t\ell + SR_r(\ell) - 2$. It suffices to show that any r -edge-coloured copy G of K_n contains t vertex-disjoint monochromatic copies of K_ℓ . As $n \geq t = R_r(t_0)$, there exists a monochromatic copy K of K_{t_0} in G , say a red copy.

Next, we construct a collection \mathcal{H} of vertex-disjoint monochromatic copies of K_ℓ . This is achieved by repeatedly adding monochromatic copies of K_ℓ to \mathcal{H} according to certain rules. We write $V(\mathcal{H})$ to denote the set of vertices contained in some copy of K_ℓ in \mathcal{H} . We also define $C := V(K) \setminus V(\mathcal{H})$ and $V := V(G) \setminus (V(K) \cup V(\mathcal{H}))$. Note that, as we add copies of K_ℓ to \mathcal{H} , the sets V , C and $V(\mathcal{H})$ will be modified accordingly. It is convenient to think of V and C as the vertices in $V(G) \setminus V(K)$ and $V(K)$ that have not yet been used to form copies of K_ℓ in \mathcal{H} . Initially, we set $\mathcal{H} := \emptyset$ and so we have $V = V(G) \setminus V(K)$ and $C = V(K)$. We add copies of K_ℓ to \mathcal{H} in three phases.

Phase I. If V contains a monochromatic copy of K_ℓ , add it to \mathcal{H} . Iterate this as long as possible.

After Phase I is completed, we must have that there is no monochromatic copy of K_ℓ in V . In particular, we have $|V| \leq R_r(\ell)$. Note that no copy of K_ℓ added to \mathcal{H} so far intersects K , hence we still have $C = V(K)$ and so $|C| = (\ell - 1)R_r(\ell) \geq (\ell - 1)|V|$.

Phase II. If there is a vertex $v \in V$ and a set $X \subseteq C$ with $|X| = \ell - 1$ such that all edges between v and X are red, then $G[\{v\} \cup X]$ is a red copy of K_ℓ . Add it to \mathcal{H} and iterate as long as possible.

After Phase II is completed, if $V = \emptyset$ then $V(\mathcal{H}) \cup V(K) = V(G)$. Since K is monochromatic, it is easy to see that there exist t vertex-disjoint monochromatic copies of K_ℓ as required. Hence, suppose that $V \neq \emptyset$.

Observe that after each iteration of Phase II, the quantities $|V|$ and $|C|$ decrease by 1 and $\ell - 1$, respectively. Thus, after Phase II is completed, it is still the case that $|C| \geq (\ell - 1)|V| > 0$.

Since both V and C are non-empty, Phase II must have terminated because for every vertex $v \in V$ there are at most $\ell - 2$ red edges between v and C . This fact together with $|C| \geq (\ell - 1)|V|$ implies there exists a subset $S \subseteq C$ of size $|S| = |V|$ such that none of the edges between S and V are coloured red.

Phase III. If $|V| \geq SR_r(\ell) - 1$, then pick any vertex $v \in S \setminus V(\mathcal{H})$ and observe that $G[V \cup \{v\}]$ has at least $SR_r(\ell)$ vertices and is equipped with a special r -edge-colouring (namely, no edge incident to v is coloured red). Therefore, $G[V \cup \{v\}]$ contains a monochromatic copy of K_ℓ . Add this copy to \mathcal{H} , and iterate as long as it is possible.

Observe that after each iteration of Phase III, the quantity $|V|$ decreases by either $\ell - 1$ or ℓ while $|S \setminus V(\mathcal{H})|$ decreases by either 1 or 0. In particular, after Phase III has terminated we have $|V| \leq |S \setminus V(\mathcal{H})|$. Hence, it must be the case that Phase III terminated because $|V| \leq SR_r(\ell) - 2$.

It follows that $|V(\mathcal{H}) \cup V(K)| \geq t\ell$. Since K is monochromatic, it is easy to see that $G[V(\mathcal{H}) \cup V(K)]$ contains t vertex-disjoint monochromatic copies of K_ℓ , as required. \square

We are now ready to prove Theorem 5.4.

Proof of Theorem 5.4. Let $r, \ell \geq 2$ and $t \in \mathbb{N}$ be sufficiently large so that the statement of Theorem 5.5 holds. Let $n_0 := t\ell + SR_r(\ell) - 2$. It suffices to show that for any r -edge-coloured graph G on $n \geq n_0$ vertices and with $\delta(G) \geq (1 - 1/n_0)n$, there exists a K_ℓ -tiling in G such that every copy of K_ℓ is monochromatic and the number of copies of K_ℓ in the tiling is at least

$$\left\lfloor \frac{(SR_r(\ell) - 2)\delta(G) - (SR_r(\ell) - 3)n}{\ell} \right\rfloor.$$

We proceed by induction on n . Before this, we consider the case when $(1 - 1/n_0)n \leq \delta(G) \leq ((n_0 - 1)n + \ell - 1)/n_0$ (for all $n \geq n_0$). Note that any induced subgraph H of G with $|H| = n_0(n - \delta(G)) \leq n$ satisfies $\delta(H) \geq \delta(G) - (|G| - |H|) = (n_0 - 1)(n - \delta(G)) = (1 - 1/n_0)|H|$.

Theorem 3.1 implies H , and thus G , contains a K_{n_0} -tiling consisting of $n - \delta(G)$ copies of K_{n_0} . By Theorem 5.5, each K_{n_0} contains t vertex-disjoint monochromatic copies of K_ℓ . Taking the union of all such copies yields a K_ℓ -tiling consisting of precisely $t(n - \delta(G))$ monochromatic copies of K_ℓ . This concludes the verification of this case, as

$$\begin{aligned} \left\lfloor \frac{(SR_r(\ell) - 2)\delta(G) - (SR_r(\ell) - 3)n}{\ell} \right\rfloor &= t(n - \delta(G)) + \left\lfloor \frac{(SR_r(\ell) - 2 + t\ell)\delta(G) - (SR_r(\ell) - 3 + t\ell)n}{\ell} \right\rfloor \\ &= t(n - \delta(G)) + \left\lfloor \frac{n_0\delta(G) - (n_0 - 1)n}{\ell} \right\rfloor \leq t(n - \delta(G)) + \left\lfloor \frac{\ell - 1}{\ell} \right\rfloor = t(n - \delta(G)). \end{aligned}$$

Now we can proceed by induction on n . The base cases when $n_0 \leq n \leq n_0 + \ell - 1$ are covered by the last paragraph. Next, we check the inductive step. Suppose G is an n -vertex graph where $n \geq n_0 + \ell$. By the previous paragraph we may assume that $\delta(G) \geq ((n_0 - 1)n + \ell)/n_0$.

It is easy to show that G contains a K_{n_0} (e.g., by Theorem 3.2), which in turn contains a monochromatic copy T of K_ℓ by Theorem 5.5. Let $G' := G \setminus V(T)$. Note that G' is an r -edge-coloured graph on $n - \ell$ vertices with minimum degree $\delta(G') \geq \delta(G) - \ell \geq (1 - 1/n_0)(n - \ell)$. By the inductive hypothesis, G' contains a K_ℓ -tiling consisting of

$$\left\lfloor \frac{(SR_r(\ell) - 2)\delta(G') - (SR_r(\ell) - 3)(n - \ell)}{\ell} \right\rfloor \geq \left\lfloor \frac{(SR_r(\ell) - 2)\delta(G) - (SR_r(\ell) - 3)n}{\ell} \right\rfloor - 1$$

monochromatic copies of K_ℓ . Adding T to this tiling yields a K_ℓ -tiling in G with the required number of monochromatic copies of K_ℓ . This concludes the inductive step and the proof. \square

5.2. Open problems. In general, Problems 1.3 and 1.4 remain wide open. The main open problem which complements our work is Question 1.6; if true, this would fully generalise Theorem 1.1 to the minimum degree setting. It would also be interesting to improve the error term in case (M.2) of Theorem 1.7; we believe that the $o(n)$ term should not appear at all. Ideas used in the proof of the Hajnal–Szemerédi theorem (see also [18]) may be helpful.

Although our argument for case (B.1) of Theorem 1.5 does not immediately generalise to larger cliques and multiple colours, it seems likely that some of the ideas used should be useful. The main challenge is to prove analogues of Lemmas 4.2 and 4.3.

Another potentially interesting future direction is the asymmetric version of Problems 1.3 and 1.4; that is, the monochromatic tiling one seeks in each colour could be different now.

Finally, we remark that Ramsey-type results for tilings have been used in some interesting applications. For example, in [16] an analogue of Theorem 1.1 was proved where one insists that the monochromatic copy of mK_3 must lie in a connected subgraph of its own colour. This result was used to determine the Ramsey number of an ‘almost’ square of a cycle C_m . It would be interesting to see if our results in this paper have similar applications (perhaps via the regularity method).

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REFERENCES

- [1] P. Allen, Covering two-edge-coloured complete graphs with two disjoint monochromatic cycles. *Combin. Probab. Comput.* **17**(4) (2008), 471–486.
- [2] J. Ayel., Sur l’existence de deux cycles supplémentaires unicolores, disjoints et de couleurs différentes dans un graphe complet bicolore. PhD thesis, Université Joseph-Fourier-Grenoble I, 1979.

- [3] J. Balogh, J. Barát, D. Gerbner, A. Gyárfás, and G.N. Sárközy, Partitioning 2-edge-colored graphs by monochromatic paths and cycles. *Combinatorica* **34** (2014), 507–526.
- [4] S. Bessy and S. Thomassé, Partitioning a graph into a cycle and an anticycle, a proof of Lehel’s conjecture. *J. Combin. Theory Ser. B* **100**(2) (2010), 176–180.
- [5] M. Bucić and B. Sudakov, Tight Ramsey bounds for multiple copies of a graph. *Adv. Combin.* (2023), 22pp.
- [6] S.A. Burr, On the Ramsey numbers $r(G, nH)$ and $r(nG, nH)$ when n is large. *Discr. Math.* **65** (1987), 215–229.
- [7] S.A. Burr, P. Erdős and L. Lovász, On graphs of Ramsey type, *Ars Combinatoria* **1** (1976), 167–190.
- [8] S.A. Burr, P. Erdős, and J.H. Spencer, Ramsey theorems for multiple copies of graphs. *Trans. Amer. Math. Soc.* **209** (1975), 87–99.
- [9] E.J. Cockayne and P.J. Lorimer, The Ramsey number for stripes. *J. Aust. Math. Soc.* **19** (1975), 252–256.
- [10] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph. *Acta Math. Hungar.* **14**(3-4) (1963), 423–439.
- [11] L. DeBiasio and L.L. Nelsen, Monochromatic cycle partitions of graphs with large minimum degree. *J. Combin. Theory Ser. B* **122** (2017), 634–667.
- [12] P. Erdős, Some unsolved problems in graph theory and combinatorial analysis, in: *Combinatorial Mathematics and Its Applications (Proc. Conf. Oxford 1969)*, Academic Press, London, 1971, pp. 97–109.
- [13] P. Erdős and A. Gyárfás, Vertex covering with monochromatic paths. *Math. Pannon.* **6**(1) (1995), 7–10.
- [14] L. Gerencsér and A. Gyárfás, On Ramsey-type problems. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math* **10** (1967), 167–170.
- [15] A. Gyárfás and G.N. Sárközy, Star versus two stripes Ramsey numbers and a conjecture of Schelp. *Combin. Probab. Comput.* **21**(1-2) (2012), 179–186.
- [16] A. Gyárfás and G.N. Sárközy, Ramsey number of a connected triangle matching. *J. Graph Theory* **83**(2) (2016), 109–119.
- [17] A. Hajnal and E. Szemerédi, Proof of a conjecture of P. Erdős. *Combin. Theory Appl.* **2** (1970).
- [18] H.A. Kierstead and A.V. Kostochka, A short proof of the Hajnal–Szemerédi theorem on equitable colouring. *Combin. Probab. Comput.* **17**(2) (2008), 265–270.
- [19] J. Komlós, Tiling Turán theorems. *Combinatorica* **20**(2) (2000), 203–218.
- [20] J. Komlós, G.N. Sárközy and E. Szemerédi, Blow-up lemma, *Combinatorica* **17** (1997), 109–123.
- [21] D. Kühn and D. Osthus, The minimum degree threshold for perfect graph packings. *Combinatorica* **29** (2009), 65–107.
- [22] S. Letzter, Monochromatic cycle partitions of 2-coloured graphs with minimum degree $3n/4$. *Electron. J. Combin.* **26** (2019), P1.19.
- [23] H. Li, V. Nikiforov and R.H. Schelp, A new class of Ramsey–Turán problems, *Discr. Math.* **310**(24) (2010), 3579–3583.
- [24] T. Łuczak, V. Rödl and E. Szemerédi, Partitioning two-coloured complete graphs into two monochromatic cycles. *Combin. Probab. Comput.* **7**(4) (1998), 423–436.
- [25] J.W. Moon, Disjoint Triangles in Chromatic Graphs. *Mathematics Magazine* **39**(5) (1966), 259–261.
- [26] G.R. Omid, G. Raeisi and Z. Rahimi, Stars versus stripes Ramsey numbers. *Eur. J. Comb.* **67** (2018), 268–274.
- [27] A. Pokrovskiy, L. Versteegen and E. Williams, A proof of a conjecture of Erdős and Gyárfás on monochromatic path covers, arXiv:2409.03623.
- [28] F.P. Ramsey, On a problem of formal logic. *Proc. Lond. Math. Soc.* **30**(1) (1930), 264–286.
- [29] R.H. Schelp, Some Ramsey–Turán type problems and related questions. *Discr. Math.* **312**(14) (2012), 2158–2161.
- [30] E. Szemerédi, Regular partitions of graphs, *Problèmes Combinatoires et Théorie des Graphes Colloques Internationaux CNRS* **260** (1978), 399–401.

APPENDIX

In this appendix, we prove Lemma 3.7 using the regularity method. First, we introduce some notation. The *density* of a bipartite graph with vertex classes A and B is defined to be

$$d(A, B) := \frac{e(A, B)}{|A| \cdot |B|},$$

where here $e(A, B)$ is the number of edges between A and B . Given $\varepsilon > 0$, a graph G and two disjoint sets $A, B \subseteq V(G)$, we say that the pair $(A, B)_G$ is (ε, d) -regular if $d(A, B) \geq d$ and, for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, we have $|d(A, B) - d(X, Y)| < \varepsilon$. The pair $(A, B)_G$ is (ε, d) -super-regular if all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ satisfy $d(X, Y) \geq d$ and, furthermore, $d_G(a) \geq d|B|$ for all $a \in A$ and $d_G(b) \geq d|A|$ for all $b \in B$.

The following is the degree form of Szemerédi's Regularity Lemma.

Lemma 5.6 (Regularity Lemma [30]). *For every $\varepsilon > 0$ and every $\ell_0 \in \mathbb{N}$ there is an $M = M(\varepsilon, \ell_0)$ such that for every $d \in [0, 1)$ and for every graph G on $n \geq M$ vertices, there exists a partition V_0, V_1, \dots, V_ℓ of $V(G)$ and a spanning subgraph G' of G such that the following holds:*

- $\ell_0 \leq \ell \leq M$ and $|V_0| \leq \varepsilon n$,
- $|V_i| = |V_1|$ for every $i \in [\ell]$,
- $d_{G'}(x) \geq d_G(x) - (d + \varepsilon)n$ for all $x \in V(G)$,
- for all $i \in [\ell]$ the graph $G'[V_i]$ is empty,
- for all $1 \leq i < j \leq \ell$, $(V_i, V_j)_{G'}$ either has density 0 or is (ε, d) -regular.

The *reduced graph* R of G with parameters ε , ℓ_0 and d is the graph with vertex set $\{V_i : i \in [\ell]\}$ and in which $V_i V_j$ is an edge precisely when $(V_i, V_j)_{G'}$ is (ε, d) -regular. The following well-known consequence of the Regularity Lemma states that the reduced graph almost inherits the minimum degree of the original graph.

Proposition 5.7. *Let $\ell_0 \in \mathbb{N}$, let $0 < \varepsilon, d, k < 1$ and let G be an n -vertex graph with $\delta(G) \geq kn$. If R is the reduced graph of G obtained by applying Lemma 5.6 with parameters ε , ℓ_0 and d , then $\delta(R) \geq (k - 2\varepsilon - d)|R|$. \square*

Lemma 5.8 (Blow-up Lemma [20]). *Given a graph R of order ℓ and $d, \Delta > 0$, there exists an $\varepsilon > 0$ such that the following holds. Given any $m \in \mathbb{N}$, let V_1, V_2, \dots, V_ℓ denote the vertex classes of the blow-up $R(m)$ of R (so $|V_i| = m$ for all $i \in [\ell]$). Let G be a graph obtained from $R(m)$ as follows: for every $1 \leq i < j \leq \ell$ such that (V_i, V_j) induces a complete bipartite graph in $R(m)$, $(V_i, V_j)_G$ now forms an (ε, d) -super-regular pair. If a graph H with $\Delta(H) \leq \Delta$ lies in $R(m)$, then there is a copy of H in G .*

Lemma 3.7 now follows easily from Lemmas 5.6 and 5.8. In the proof below, constants in the displayed hierarchy are chosen from right to left.

Proof of Lemma 3.7. Given $\eta > 0$, choose constants

$$0 < 1/n_0 \ll 1/\ell_0 \ll \varepsilon \ll d \ll \eta, 1/8.$$

Given $n \geq n_0$, let G be an n -vertex graph as in the statement of the lemma. Apply Lemma 5.6 to the graph G with parameters ε , ℓ_0 and d to obtain $\ell \in \mathbb{N}$, a partition V_0, V_1, \dots, V_ℓ of G , a spanning subgraph G' of G , and a reduced graph R of G . Set $m := |V_1|$.

By Proposition 5.7 we have $\delta(R) \geq (\delta(G)/n - 2\varepsilon - d)|R| \geq (\delta(G)/n - \eta/4)|R|$, and so $\delta(R)/|R| \geq \delta(G)/n - \eta/4$. By greedily deleting edges, we may further assume that $\delta(R)/|R| \leq \delta(G)/n$ and so property (i) holds. Property (ii) also holds as $\varepsilon \leq \eta/2$. It remains to verify property (iii).

Let V_{i_1}, \dots, V_{i_r} form a clique in R with $r \leq 8$. By deleting 'small degree' vertices, for each $j \in [r]$, one obtains a set $V'_{i_j} \subseteq V_{i_j}$ such that $|V'_{i_j}| = \lceil (1 - r\varepsilon)m \rceil$ and so that $(V'_{i_j}, V'_{i_k})_{G'}$ is $(2\varepsilon, d - r\varepsilon)$ -super-regular for each distinct $j, k \in [r]$. Now by Lemma 5.8, $G'[V'_{i_1}, \dots, V'_{i_r}]$ (and thus $G[V_{i_1}, \dots, V_{i_r}]$) contains a $K_r(2)$ -tiling consisting of at least $(1 - \eta/2)m/2$ copies of $K_r(2)$; so indeed (iii) holds. \square