Abstracts

Recent developments in sharp restriction theory

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(joint work with Emanuel Carneiro, René Quilodrán, Mateus Sousa)

For the sake of concreteness, we start our discussion with the case of the unit sphere $\mathbb{S}^{d-1}$ equipped with surface measure σ, but the more general example of a smooth compact hypersurface should be kept in mind. Given $1 \leq p \leq 2$, for which exponents q does the *a priori* Fourier restriction inequality

$$\Big(\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^q d\sigma_\omega \Big)^{\frac{1}{q}} \leq C \|f\|_{L^p(\mathbb{R}^d)} \tag{1}$$

hold? One easily checks that, for any value of q, inequality (1) holds if $p = 1$ and fails if $p = 2$. By duality, estimate (1) is equivalent to the adjoint restriction, or extension, inequality

$$\Big(\int_{\mathbb{R}^d} |\widehat{g\sigma}(x)|^{p'} dx \Big)^{\frac{1}{p'}} \leq C \|g\|_{L^{q'}(\mathbb{S}^{d-1})}, \tag{2}$$

where $p' = \frac{p}{p-1}$ denotes the exponent dual to p, and similarly for q. A complete answer for $q = 2$ is given by the classical Tomas–Stein inequality, which establishes the restriction inequality (1) for $q = 2$ in the sharp range $1 \leq p \leq \frac{2(d+1)}{d+3}$. The question of what happens for values of $q < 2$ is the starting point for the celebrated Fourier restriction conjecture.

Tomas–Stein restriction estimates are very much related to Strichartz estimates for linear partial differential equations of dispersion type. Let us illustrate this point in one particular instance, that of solutions $u(x, t)$ with $(x, t) \in \mathbb{R}^{d+1}$ to the Schrödinger equation $iu_t = \Delta u$, with prescribed initial data. Strichartz established

$$\|u\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})} \leq C \|f\|_{L^2(\mathbb{R}^d)}, \tag{3}$$

provided that u is the solution of the Schrödinger equation satisfying $u(x, 0) = f(x)$. It turns out that Strichartz estimates for the Schrödinger equation correspond to extension estimates on the paraboloid, a non-compact manifold which exhibits some scale invariance properties that allow the reduction to the compact setup of the Tomas–Stein inequality.

For the past several years, I have been very interested in extremizers and optimal constants for sharp variants of restriction and Strichartz-type inequalities. Apart from their intrinsic mathematical interest and beauty, sharp inequalities often allow for various refinements of existing inequalities. The following are natural questions, which can be posed in the particular case of Fourier restriction inequalities:

- What is the value of the optimal constant?
- Do extremizers exist?

  - If so, are they unique, possibly after applying the symmetries of the problem?
  - If not, what is the mechanism responsible for this lack of compactness?
- How do extremizing sequences behave?
- What are some qualitative properties of extremizers?
- What are necessary and sufficient conditions for a function to be an extremizer?

Questions of this kind have been asked in a variety of situations, and in the context of classical inequalities from Euclidean harmonic analysis they go back at least to the early work of Beckner on the sharp Hausdorff–Young inequality, and of Lieb on the sharp Hardy–Littlewood–Sobolev inequality. In comparison, sharp Fourier restriction inequalities have a relatively short history, with the first works on the subject going back to Kunze [6], Foschi [2] and Hundertmark–Zharnitsky [4]. These works concern extremizers and sharp constants for inequality (3) in the low dimensional cases $d \in \{1, 2\}$. These are the cases for which the Strichartz exponent $2 + \frac{4}{d}$ is an even integer, and one can rewrite the left-hand side of inequality (3) as an L^2 norm, and invoke Plancherel in order to reduce the problem to a multilinear convolution estimate.

Sharp Fourier restriction theory is becoming increasingly more popular, as shown by the large body of work that appeared in the last decade, and in particular in the last few years. We mention a recent survey [3] on sharp Fourier restriction theory which may be consulted for information complementary to that on this abstract, and further references.

0.1. **Perturbed paraboloids.** Recent joint work with Quilodrán [7] focused on a family of sharp Strichartz estimates for higher order Schrödinger equations. More precisely, for an appropriate class of convex functions ϕ, we studied the Fourier extension operator on the surface

$$\{(\xi, \tau) \in \mathbb{R}^{2+1} : \tau = |\xi|^2 + \phi(\xi)\}. \tag{4}$$

One of our main tools was a new comparison principle for convolutions of certain singular measures supported on non-compact manifolds that holds in all dimensions. This is better illustrated in the following special case. Let μ_0 and μ_1 denote the projection measures on the surfaces given by (4) with $\phi(\xi) \equiv 0$ and $\phi(\xi) = |\xi|^4$, respectively. Then the pointwise inequality

$$(\mu_1 * \mu_1)\Big(\xi, \tau + \frac{|\xi|^2}{2} + \frac{|\xi|^4}{8}\Big) \leq (\mu_0 * \mu_0)\Big(\xi, \tau + \frac{|\xi|^2}{2}\Big)$$

holds for every $\tau > 0$ and $\xi \in \mathbb{R}^2$, and it is strict at almost every point of the support of the measure $\mu_1 * \mu_1$. This observation led to the exact determination of some optimal constants and to a proof that extremizers do not exist in this perturbed setting. Adapting ideas from the concentration-compactness principle of Lions, we further investigated the behaviour of general extremizing sequences. Generally speaking, the theory of concentration-compactness has proved a very efficient tool in exhibiting the precise mechanisms which are responsible for the loss of compactness in a variety of settings. In our concrete problem, extremizers fail to exist because extremizing sequences concentrate. Concentration can only occur at points where the convolution $\mu_1 * \mu_1$ attains its maximum value, or at spatial infinity. Last but not least, our methods further resolve a dichotomy from the recent literature [5] concerning the existence of extremizers for a family of fourth order Schrödinger equations.

0.2. **Hyperboloids.** In ongoing joint work with Carneiro and Sousa [1], we are investigating optimal constants and the existence of extremizers for the adjoint Fourier restriction inequality on hyperboloids. The $L^2 \to L^p$ adjoint restriction inequality on the d-dimensional hyperboloid $\mathbb{H}^d \subset \mathbb{R}^{d+1}$ holds provided $6 \leq p < \infty$, if $d = 1$, and $\frac{2(d+2)}{d} \leq p \leq \frac{2(d+1)}{d-1}$, if $d \geq 2$. Quilodrán [8] recently found the values of the optimal constants in the endpoint cases $(d,p) \in \{(2,4),(2,6),(3,4)\}$ and showed that the inequality does not have extremizers in these cases. We are able to answer two questions posed in [8], namely: (i) we find the explicit value of the optimal constant in the endpoint case $(d,p) = (1,6)$ (the remaining endpoint for which p is an even integer) and show that there are no extremizers in this case; and (ii) we establish the existence of extremizers in all non-endpoint cases in dimensions $d \in \{1,2\}$. This completes the qualitative description of this problem in low dimensions.

0.3. **An open problem.** To finish, we would like to mention the following open problem: Do Gaussians extremize inequality (3) in all dimensions? It is known that Gaussians are critical points of the associated Euler–Lagrange equation in all dimensions. If Gaussians were known to be extremizers, it would then be possible to establish the unconditional existence of extremizers for the corresponding problem on the unit sphere $\mathbb{S}^{d-1}$. The methods outlined above are not enough to tackle this problem when $d \geq 4$, and we intend to gear the direction of our research towards a better understanding of this fundamental question.

References

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