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# METRIC CHARACTERIZATION OF PURE UNRECTIFIABILITY

**Abstract**

We show that an analytic subset of the finite dimensional Euclidean space $\mathbb{R}^m$ is purely unrectifiable if and only if the image of any of its compact subsets under every local Lipschitz quotient function is a Lebesgue null. We also construct purely unrectifiable compact sets of Hausdorff dimension greater than 1 which are necessarily sent to finite sets by local Lipschitz quotient functions.

## 1 Introduction.

Let $X$ and $Y$ be metric spaces. A mapping $f$ defined on a subset $S$ of $X$ with values in $Y$ is called Lipschitz if there exists $L > 0$, such that for any $x_1, x_2 \in S$ the distance between $f(x_1)$ and $f(x_2)$ does not exceed the distance between $x_1$ and $x_2$ multiplied by $L$. The least such $L$ is called the Lipschitz constant of the mapping $f$.

A dual notion, the notion of co-Lipschitz mappings was introduced in several texts (e.g. [12], [6], [5]) but was first systematically studied in [1]. A

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mapping $f\colon X \to Y$ between two metric spaces is called co-Lipschitz, if there exists a constant $c > 0$ such that

$$f(B_r(p)) \supset B_{cr}(f(p)) \tag{1}$$

for any $p \in X$ and $r > 0$. (By $B_r(p)$ we mean the open ball in $X$ centered at $p$ of radius $r$.) The greatest such $c$ is called the co-Lipschitz constant of the mapping $f$.

In this paper, we consider a local analogue of the co-Lipschitz property. Namely, we say that $f\colon S \to Y$ is local co-Lipschitz, if instead of (1) one has

$$f(B_r(p) \cap S) \supset B_{cr}(f(p)) \cap f(S) \tag{2}$$

for any $p \in S$ and $r > 0$.

If $f\colon S \to Y$ is Lipschitz and local co-Lipschitz, then we say that $f$ is local Lipschitz quotient.

Note that the difference between co-Lipschitz and local co-Lipschitz mappings is more substantial than that between Lipschitz mappings defined on the whole space $X$ and Lipschitz mappings defined on a set $S \subset X$.

In the case of Lipschitz mappings, a restriction of a Lipschitz mapping defined on the whole space is Lipschitz on a set. Also, if $X$ and $Y$ are Euclidean spaces, then any mapping $f$ defined on a subset of $X$ may be extended to a Lipschitz mapping defined on the whole space $X$, which has the same Lipschitz constant as $f$ has. (This is a particular case of the Kirszbraun theorem, see [8], [3, 3, 2.10.43].)

Such statements do not hold for co-Lipschitz and local co-Lipschitz mappings. Consider the orthogonal projection $P\colon (x,y) \mapsto x$ from the plane onto the real line. Its co-Lipschitz constant is equal to 1, but for $S = \{(x,y)\colon x = 0 \text{ or } y = 0\}$ the mapping $P\colon S \to \mathbb{R}$ is not local co-Lipschitz (take, for example, $p = (x,y) = (0,2)$ and $r = 1$). On the other hand, it is clear that a local co-Lipschitz mapping even between Euclidean spaces need not have a co-Lipschitz extension to the whole space, since any constant mapping is local co-Lipschitz, but any co-Lipschitz mapping between spaces is open.

In this paper we investigate some properties of subsets of $\mathbb{R}^m$ which are studied using orthogonal projections. (An example of such property is pure unrectifiability defined below.) We would like to find an analogue of these properties in terms of the metric structure of the sets themselves rather than in terms of their embeddings in $\mathbb{R}^m$. In this context we consider local Lipschitz quotient mappings from subsets of $\mathbb{R}^m$ to $\mathbb{R}$, as candidates to replace linear projections.

Let us recall the definition of purely unrectifiable subsets of $\mathbb{R}^m$ and a theorem characterizing such sets in terms of linear projections (see [11]).

A set $E \subset \mathbb{R}^m$ is called 1-rectifiable if there exist Lipschitz mappings $f_i \colon \mathbb{R} \to \mathbb{R}^m$, $i = 1, 2, \dots$, such that $\mathcal{H}^1\big(E \setminus \bigcup_{i\geq 1} f_i(\mathbb{R})\big) = 0$. The $D$-dimensional Hausdorff measure $\mathcal{H}^D$ is defined as

$$\mathcal{H}^D(A) = \sup_{\delta>0} \inf \Big\{ \sum_{j\geq 1} \operatorname{diam}(C_j)^D \mid A \subset \bigcup_{j\geq 1} C_j, \operatorname{diam}(C_j) \leq \delta \Big\}.$$

A set $F \subset \mathbb{R}^m$ is called purely 1-unrectifiable (or purely unrectifiable) if $\mathcal{H}^1(E \cap F) = 0$ for every 1-rectifiable set $E \subset \mathbb{R}^m$.

**Remark 1.1.** Note that rectifiable sets may be equivalently defined with $C^1$ mappings $f_i$ replacing Lipschitz mappings (see [11, Theorem 15.21]).

The following theorem was proved first by Besicovitch [2] in the case $m = 2$. The general case was proved by Federer [4] (see also [11, Theorem 18.1]).

**Theorem 1.2 (Besicovitch-Federer).** *A set $F \subset \mathbb{R}^m$ with finite 1-dimensional Hausdorff measure is purely unrectifiable if and only if for almost every direction $e \in S^{m-1}$ the projection $P_e F$ has Lebesgue measure zero.*

In the present paper we show that for any compact purely unrectifiable set $F \subset \mathbb{R}^m$ and for any local Lipschitz quotient mapping $f \colon F \to \mathbb{R}$, the image $f(F)$ necessarily has Lebesgue measure zero. Yet this property alone does not characterize compact purely unrectifiable sets. We show that there is a rectifiable compact $K \subset \mathbb{R}^2$ of finite positive 1-dimensional Hausdorff measure, such that for any local Lipschitz quotient $f \colon K \to \mathbb{R}$, the image $f(K)$ is a single point.

Since an analytic set is purely unrectifiable if and only if all its compact subsets are purely unrectifiable, we conclude that a characteristic property should be as follows. An analytic set $F \subset \mathbb{R}^m$ is purely unrectifiable if and only if for any compact subset $F'$ of $F$ and any local Lipschitz quotient mapping $f \colon F' \to \mathbb{R}$, the image $f(F')$ has Lebesgue measure zero. This is proved in Theorem 2.1.

Furthermore, it turns out that unlike linear projections, local Lipschitz quotient mappings fail to distinguish the sets of Hausdorff dimension greater than 1 from those of dimension 1 or less. Recall the following theorem proved by Marstrand [10] (see also [11, Theorem 8.9 and Corollary 9.8]).

**Theorem 1.3 (Marstrand).** *Assume that the set $E \subset \mathbb{R}^m$ is compact and its Hausdorff dimension is greater than 1. Then for almost every direction $e \in S^{m-1}$ the projection $P_e E$ has positive one dimensional Lebesgue measure.*

We show that there exist planar sets of Hausdorff dimension up to 2 such that any local Lipschitz quotient mapping has finite image whenever the domain is one of those sets. As shown in Remark 3.7, the construction can be modified to give sets of Hausdorff dimension up to $m$ with this property in the Euclidean space $\mathbb{R}^m$.

Throughout the paper, by $M_\delta$ we mean the open $\delta$-neighborhood of a set $M$; i.e., $M_\delta = \bigcup_{p\in M} B_\delta(p)$. With the exception of Lemma 3.1, where we talk about general metric spaces, we always consider balls in the Euclidean norm. By $A_r(p)$ we denote the closed $\ell^2_\infty$ ball of radius $r$; i.e., the square centered at $p$ with side $2r$. We also use the notation $\mathcal{L}^k$ for the $k$-dimensional Lebesgue measure.

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## 2 Criterion of Pure Unrectifiability.

In this section we prove a characteristic property of analytic purely unrectifiable subsets of $\mathbb{R}^m$. Recall that a set is called analytic if it is a continuous image of a Borel set.

**Theorem 2.1.** *An analytic set $F \subset \mathbb{R}^m$ is purely unrectifiable if and only if for any compact subset $F' \subset F$ and any local Lipschitz quotient mapping $f\colon F' \to \mathbb{R}$, the image $f(F')$ has Lebesgue measure* 0*.*

Proof. Assume $F$ is not purely unrectifiable. Then by Remark 1.1 there exists a $C^1$ mapping $g\colon \mathbb{R} \to \mathbb{R}^m$, such that $\mathcal{H}^1(F \cap g(\mathbb{R})) > 0$. Let $A = g^{-1}(F \cap g(\mathbb{R}))$. Note that $A$ is analytic, and hence is Lebesgue measurable [9, Chapter III, § 39, II]. We conclude that $\mathcal{L}^1(A) > 0$, and choose a compact subset $C \subset A$ such that $\mathcal{L}^1(C) > 0$. By the Lebesgue density theorem, almost every point $p \in C$ is a density point (i.e., $\lim_{r\to 0} \frac{\mathcal{L}^1\left((p-r,p+r)\cap C\right)}{2r} = 1$). Consider any density point $p \in C$ such that the derivative $g'(p)$ is not 0, and introduce the coordinate system in $\mathbb{R}^m = \mathbb{R}[x_1] \oplus \mathbb{R}^{m-1}[x_2, \dots, x_m]$, with the $x_1$-axis in the direction of $g'(p)$. Let $t > 0$ be such that $\frac{\mathcal{L}^1\left([p-t,p+t]\cap C\right)}{2t} > 1/2$. Since $g$ is continuously differentiable at $p$, there is a small neighborhood $U$ of $p$, such that the curve $g(U)$ can be viewed as a graph of a $C^1$-smooth mapping from $\mathbb{R}[x_1]$ to $\mathbb{R}^{m-1}[x_2, \dots, x_m]$. Then the orthogonal projection from $F' = g(U \cap [p-t, p+t] \cap C)$ onto the $x_1$-axis is a local Lipschitz quotient mapping. But the image of $F'$ under this projection has positive Lebesgue measure.

For the proof of the "only if" part let us show that if $E \subset \mathbb{R}^m$ is a compact purely unrectifiable set and $f\colon E \to \mathbb{R}$ is local Lipschitz quotient, then $f(E)$ has Lebesgue measure zero.

Denote $C = f(E)$ and assume $\mathcal{L}^1(C) > 0$. The set $C \subset \mathbb{R}$ is separable, so there exists a countable dense subset $\{y_i\}_{i=1}^{\infty}$ of $C$.

For each $n \geq 1$, find points $\{x_{i,n}\}_{i=0}^{n}$ in $E$, so that $f(x_{i,n}) = y_i$ for each $i = 1, \ldots, n$ and $\|x_{i,n} - x_{j,n}\| \leq c^{-1}|y_i - y_j|$ for all $1 \leq i, j \leq n$, where $c$ is the co-Lipschitz constant of $f$. This can be done since we may put $y_i$, $1 \leq i \leq n$ in the increasing order and then lift them one after another starting from the preimage of $\min_{1\leq i\leq n} y_i$.

Since $\{x_{1,n}\} \subset E$, we can choose $n_s^1$ such that $\{x_{1,n_s^1}\}_{s=1}^{\infty}$ converges. Denote its limit by $x_1$. Note that $x_1 \in E$ and $f(x_1) = y_1$. Now consider the sequence $\{x_{2,n_s^1}\}_{s=1}^{\infty}$ and choose a subsequence which converges. Denote its limit by $x_2$. Note that $x_2 \in E$, $f(x_2) = y_2$ and $\|x_1 - x_2\| \leq c^{-1}|y_1 - y_2|$ (since $\|x_{1,n_s^2} - x_{2,n_s^2}\| \leq c^{-1}|y_1 - y_2|$ for all $s$). In the same way for each $k \geq 1$ we construct $x_k \in E$ such that $f(x_k) = y_k$ and $\|x_i - x_j\| \leq c^{-1}|y_i - y_j|$ for every $i, j \geq 1$.

Consider a mapping $g\colon \{y_i\}_{i=1}^{\infty} \to \mathbb{R}^m$ defined by the rule $g(y_i) = x_i$. Since $g$ is a Lipschitz mapping, we may extend it to the closure of $\{y_i\}$, which coincides with $C$. Note that then $E_1 = g(C) \subset E$ and $f(g(y)) = y$ for all $y \in C$. Let $G\colon \mathbb{R} \to \mathbb{R}^m$ be a Lipschitz extension of $g$. Since $E$ is purely unrectifiable, we conclude that $\mathcal{H}^1(E \cap G(\mathbb{R})) = 0$. Since $E \cap G(\mathbb{R}) \supset E_1$, we conclude $\mathcal{H}^1(E_1) = 0$, and therefore (since $f$ is Lipschitz) $\mathcal{L}^1(C) = 0$, a contradiction. $\square$

Now let us show that the condition in Theorem 2.1 that for any compact subset $F' \subset F$ and any local Lipschitz quotient $f\colon F' \to \mathbb{R}$ the image $f(F')$ has Lebesgue measure 0 is essential (i.e., if, for example, $F \subset \mathbb{R}^m$ is compact then in order to conclude that $F$ is purely unrectifiable it is not enough to require that the image $f(F)$ has Lebesgue measure 0 for any local Lipschitz quotient $f\colon F \to \mathbb{R}$).

In Theorem 2.3, we construct a rectifiable compact subset $K$ of the plane, of positive and finite 1-dimensional Hausdorff measure, such that for any continuous function $f\colon K \to \mathbb{R}$, which is local open (that is, for any open $U$ the image $f(U \cap K)$ is equal to an intersection of $f(K)$ with an open set), $f(K)$ is a single point. It is clear that any local Lipschitz quotient mapping is continuous local open. In fact the set $K$ will be a union of countably many closed intervals with finite sum of lengths.

**Lemma 2.2.** *If $K = I \cup K_1$, where $I = [a, b]$ is a closed interval in the plane, $K_1 \subset \mathbb{R}^2$ is compact, $K_1 \cap I = \{a\}$, and $f\colon K \to \mathbb{R}$ is continuous and local* *open, $f(K) = [x, y]$, then $f(b) = x$ or $f(b) = y$.*

PROOF. Assume $f(b)$ lies inside the open interval $(x, y)$. Denote by $I_\varepsilon$ the half-open interval, which is the intersection of $[a, b]$ and the open ball of radius $\varepsilon$ around point $b$. For sufficiently small $\varepsilon$ one has $I_\varepsilon = K \cap B_\varepsilon(b)$. Therefore, for sufficiently small $\varepsilon$ the image $f(I_\varepsilon)$ contains an open interval around $f(b)$. Then there exist $z_1, z_2 \in I_\varepsilon$ such that $f(z_1) > f(b) > f(z_2)$. Then there exists $w_1 \in [z_1, z_2] \subset I_\varepsilon$ such that $f(w_1) = f(b)$. Without loss of generality assume that the order of points is $b$–$z_1$–$w_1$–$z_2$.

Then the continuous mapping $f|_{[b,w_1]}$ takes the same value $f(b)$ at the ends of the interval. Moreover, since $z_1 \in (b, w_1)$, we conclude that $\max_{[b,w_1]} f \geq f(z_1) > f(b)$. Assume $t \in (b, w_1)$ is such that $f(t) = \max_{[b,w_1]} f$. Then $f$ is not open at $t$, since the image of a small neighborhood of $t$ should contain an open interval around $f(t)$.

This contradiction finishes the proof of the lemma. □

**Theorem 2.3.** *Let $K = \cup_{n\geq 1}[0, e^{i/n^2}/n^2]$. Then for any continuous function $f\colon K \to \mathbb{R}$, such that $f$ is local open, $f(K)$ is a single point.*

PROOF. First of all, note that $K$ is closed and connected, and so since $f$ is continuous, $f(K)$ is a closed interval $[x, y]$. Assume $x \neq y$.

If $z_n = e^{i/n^2}/n^2$, then by Lemma 2.2, $f(z_n)$ is equal to either $x$ or $y$ for any $n \geq 1$. Since $z_n \to 0$ as $n \to \infty$, the value $f(0)$ is either $x$ or $y$. Assume $f(0) = x$.

Then there exists $\varepsilon > 0$ such that $f(B(0, \varepsilon) \cap K) \subset [x, \frac{x+y}{2}]$ (and therefore, all values of $f$ in this intersection are strictly less than $y$). If $n > 1/\varepsilon$, then $f(z_n)$ cannot be equal to $y$, so that it is equal to $x$. For such $n$, let $t \in [0, z_n]$ be such that $f(t) = \max_{[0,z_n]} f$. Note that if $f(t) = x$, then $f|_{[0,z_n]} \equiv x$, and therefore $f$ is not local open (the intersection of a small ball around inner point of the interval $[0, z_n]$ with $K$ is a small subinterval of $[0, z_n]$, and its image has to contain a small open neighborhood of $x$). Therefore, $f(t) > x$. But $n$ is such that $f([0, z_n]) \subset [x, x + y/2]$; thus, $x < f(t) < y$. Then the image of the small neighborhood of $t$ has to contain an open neighborhood of $f(t)$. This is impossible, since this image does not contain values greater than $f(t)$. □

## 3 Sets of Hausdorff Dimension Greater Than 1.

In this section we show that unlike in the case of Hausdorff dimension 1, local Lipschitz quotient mappings fail to distinguish planar subsets of higher Hausdorff dimensions. For every $D \in (1, 2]$ we construct a $D$-dimensional compact subset $S$ of the plane with the property that for every local Lipschitz quotient mapping $f\colon S \to \mathbb{R}$ the image $f(S)$ is necessarily finite. We show that for $D < 2$ the set $S$ may be constructed to be purely unrectifiable, and in the case $D = 2$ we construct a totally disconnected compact set $S$.

The next lemma proves an important property of mappings which are local Lipschitz quotient.

**Lemma 3.1.** *Let $X$ and $Y$ be metric spaces, $S \subset X$, $f\colon S \to Y$ be a local Lipschitz quotient mapping, with Lipschitz constant $L$ and co-Lipschitz constant $1$. If $T \subset S$ and the points $t_0 \in T$, $x_1, \dots, x_n \in S$ are such that $(T_{Lr} \setminus T) \cap S = \emptyset$, $d(f(x_1), f(t_0)) < Lr$ and $d(x_i, x_{i+1}) < r$ for every $i = 1, \dots, n-1$, then there exist $t_1, \dots, t_n \in T$ such that $f(t_i) = f(x_i)$ for all $i = 1, \dots, n$.*

PROOF. Let us prove this lemma by induction on $n$. Since $f$ is local co-Lipschitz, for $n = 1$ we have that

$$f(B_{Lr}(t_0) \cap S) \supset B_{Lr}(f(t_0)) \cap f(S) \ni f(x_1),$$

therefore there exists a point $t_1 \in T_{Lr} \cap S = T$ such that $f(t_1) = f(x_1)$.

Suppose now we have already constructed $t_i \in T$ such that $f(t_i) = f(x_i)$, $i \le n-1$. Since $f$ is Lipschitz with constant $L$ we have $d(f(t_{n-1}), f(x_n)) = d(f(x_{n-1}), f(x_n)) < Lr$. Then by what we have just proved there exists $t_n \in T$ such that $f(t_n) = f(x_n)$. $\square$

For the construction of the set $S$ we need the following lemma.

**Lemma 3.2.** *For every positive integer $n$ there exists a function*

$$\gamma^n = (\gamma_1^n, \gamma_2^n)\colon \{1, \dots, 4n^2\} \to \{0, \dots, 2n-1\} \times \{0, \dots, 2n-1\}, \qquad (3)$$

*such that*

1. $|\gamma_1^n(k) - \gamma_1^n(k+1)| + |\gamma_2^n(k) - \gamma_2^n(k+1)| = 1$ *for every* $1 \le k \le 4n^2 - 1$,
2. $\gamma^n$ *is injective,*
3. $\gamma^n(\{1, \dots, 2n^2\}) = \{0, \dots, 2n-1\} \times \{n, \dots, 2n-1\}$ *and* $\gamma^n(\{2n^2+1, \dots, 4n^2\}) = \{0, \dots, 2n-1\} \times \{0, \dots, n-1\}$,
4. $\gamma^n(1) = (0, n)$ *and* $\gamma^n(4n^2) = (0, n-1)$.

PROOF. Let us denote by $[x]$ the integer part of the real number $x$, put $\{x\} = x - [x]$. For each $n \ge 1$ and $v = 1, 2$ we define the functions $\gamma_v^n\colon \{1, \dots, 4n^2\} \to \{0, \dots, 2n-1\}$ by the following formulas. If $1 \le j \le 2n^2$, then $\gamma_1^n(j) = \left[\frac{j-1}{n}\right]$,

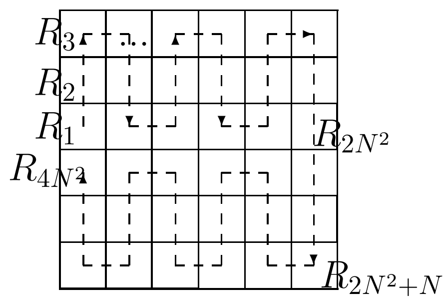



Figure 1: Squares $R_1, \ldots, R_{4N^2}$, $N = 3$.

$$\gamma_2^n(j) = \begin{cases} n + n\{\frac{j-1}{n}\} & \text{if } \left[\frac{j-1}{n}\right] \text{ is even} \\ 2n - n\{\frac{j-1}{n}\} - 1 & \text{if } \left[\frac{j-1}{n}\right] \text{ is odd.} \end{cases}$$

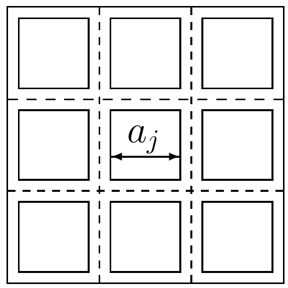
If $2n^2 + 1 \leq j \leq 4n^2$, then we put $\gamma_1^n(j) = \gamma_1^n(4n^2 + 1 - j)$ and $\gamma_2^n(j) = 2n - \gamma_2^n(4n^2 + 1 - j)$. $\square$

The set $S$ is defined as an intersection of the decreasing sequence of sets $S_k$, with each $S_k$ being a finite disjoint union of closed squares $Q_m^k$, $m = 1, \ldots, M_k$, with horizontal and vertical sides.

Let $S_1$ be the unit square and suppose that $S_k$ is already constructed. For a suitable integer $N = N_k$, which will be precisely defined below and any fixed $1 \leq m \leq M_k$, we divide each side of $Q_m^k$ into $2N$ equal intervals in order to obtain $4N^2$ smaller squares. We denote these squares $R_1, \ldots, R_{4N^2}$ in such a way that $R_j$ denotes the square in the intersection of the $\gamma_1^N(j)$th column and $\gamma_2^N(j)$th row (see Figure 1).

For every $1 \leq j \leq 4N^2$ we choose then an appropriate positive integer $n_j$ and some $a_j \in (0, s(Q_m^k)/(2Nn_j))$, where $s(Q_m^k)$ denotes the length of the side of $Q_m^k$. Now let us divide $R_j$ into $n_j^2$ squares $\{R_{j,s}\}_{1 \leq s \leq n_j^2}$ (by dividing the sides of $R_j$ into $n_j$ equal parts) and let $\mathcal{P}(Q_m^k) = \bigcup_{j=1}^{4N^2} \bigcup_{s=1}^{n_j^2} A_{a_j/2}(c_{j,s})$, where $c_{j,s}$ is the center of $R_{j,s}$ (see Figure 2 (a)). Then we define $S_{k+1}$ as a union $\bigcup_{m=1}^{M_k} \mathcal{P}(Q_m^k)$. Figure 2 (b) shows an example of how $S_2$ may be obtained from $S_1$.

We will choose $n_j$ and $a_j$ in such a way that the squares $R_{j+1,s}$ will be much smaller than the squares $R_{j,s}$, and the distance between two neighbor squares among $A_{a_{j+1}/2}(c_{j+1,s})$ (i.e., those lying in adjacent squares $R_{j+1,s}$) will be much smaller than between neighbor squares among $A_{a_j/2}(c_{j,s})$. The



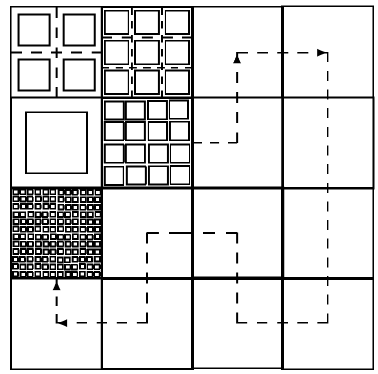



Figure 2:
(a): An example of a square $R_j$ such that $n_j = 3$.
(b): An example of how $S_2$ may be obtained from $S_1$.

freedom in the choice of $n_j$ and $a_j$ makes it possible to construct a set $S$ of Hausdorff dimension arbitrarily close to 2.

The next lemma will be our main tool to prove that the set $S$ we are going to construct has finite image under any local Lipschitz quotient function.

**Lemma 3.3.** *Let $N \geq 40$, $n_1, \ldots, n_{4N^2}$ be positive integers; $L$, $s$, $\{a_j\}$ and $\{d_j\}$ $(j = 1, \ldots, 4N^2)$ be positive real numbers such that*

$$\begin{array}{ll} a_j + d_j = \frac{s}{2Nn_j} & \text{for } 1 \leq j \leq 4N^2, \\ a_j > a_{j+1}, d_j > d_{j+1} & \text{for } 1 \leq j \leq 4N^2 - 1, \\ d_j \geq 4L\sqrt{a_{j+1}^2 + (2a_{j+1} + d_{j+1})^2} & \text{for } 1 \leq j \leq 4N^2 - 1, \\ 1 \leq L \leq \frac{N}{40}. & \end{array}$$

*For each $1 \leq j \leq 4N^2$, $1 \leq l, m \leq n_j$ consider the collection of closed squares in the square $[0, s] \times [0, s]$*

$$Q_{l,m,j} = A_{a_j/2}\Big(s\big(\tfrac{\gamma_1^N(j)}{2N} + \tfrac{l-1/2}{2Nn_j}\big), s\big(\tfrac{\gamma_2^N(j)}{2N} + \tfrac{m-1/2}{2Nn_j}\big)\Big).$$

*If a set $S \subset \mathbb{R}^2$ intersects every square $Q_{l,m,j}$ and is contained in the union of all squares $\bigcup_{j=1}^{4N^2} \bigcup_{l,m=1}^{n_j} Q_{l,m,j}$, then for any local Lipschitz quotient mapping $f\colon S \to \mathbb{R}$ with Lipschitz constant $L$ and co-Lipschitz constant 1, one has $f(S) = f(Q_{1,1,1} \cap S)$.*

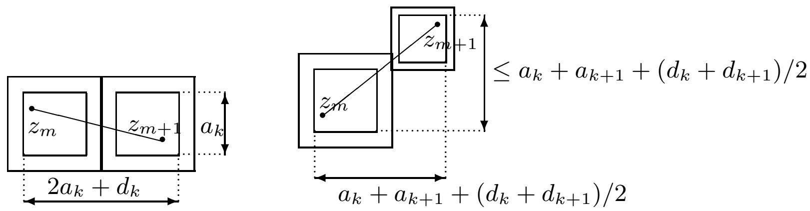



Figure 3: Neighboring squares.

PROOF. For each $1 \leq j \leq 4N^2$, let $Q_j$ denote the union of all $a_j \times a_j$ squares $Q_{\cdot,\cdot,j}$ from the considered family of squares $Q_j = \bigcup_{l,m=1}^{n_j} Q_{l,m,j}$. Note that the distance between two neighbor squares $Q_{l,m,j}$ and $Q_{l',m',j}$ (that is, lying in the adjacent $(s/2Nn_j) \times (s/2Nn_j)$ squares) is equal to $d_j$. More generally, we will say that $Q = Q_{l,m,j}$ and $Q' = Q_{l',m',j'}$ are neighboring, if the "grid" squares $R = A_{s/(4Nn_j)}(c_Q)$ and $R' = A_{s/(4Nn_{j'})}(c_{Q'})$ ($c_Q$ and $c_{Q'}$ denote the centers of $Q$ and $Q'$ respectively) they are contained in, intersect by a nontrivial interval.

Assume $x, y \in S$ do not belong to the same $Q_{l,m,j}$. Then there exist disjoint squares $Q_{l_x,m_x,j_x} \ni x$ and $Q_{l_y,m_y,j_y} \ni y$. If $j_x = j_y = j$, then there is a finite sequence of points $z_i \in Q_j \cap S$ $(1 \leq i \leq I)$, such that $z_1 = x$, $z_I = y$, and $z_i$ and $z_{i+1}$ are in neighbor squares $Q_{\cdot,\cdot,j}$ (see Figure 3). Then for every $i$ the distance between $z_i$ and $z_{i+1}$ is at most $\delta_j = \sqrt{a_j^2 + (2a_j + d_j)^2}$.

Note that if $Q = Q_{l,m,j}$ and $Q' = Q_{l',m',j+1}$ are neighboring, then the distance between any two points $z \in Q$ and $z' \in Q'$ is at most $\sqrt{2}(a_j + a_{j+1} + d_j)$, since $d_j > d_{j+1}$. Since $a_j > a_{j+1}$, this expression is less than $\sqrt{2}(2a_j + d_j) < \sqrt{2}\delta_j$ (see Figure 3 for an illustration).

This means that if $j_x \neq j_y$, then there exists a finite sequence of points from $\bigcup_{n=\min\{j_x,j_y\}}^{\max\{j_x,j_y\}} Q_n \cap S$, which starts at $x$, finishes at $y$, and the distance between each two consequent points is not greater than $\sqrt{2}\delta_{\min\{j_x,j_y\}}$.

Let us show that the diameter of $f(S)$ is not greater than $\frac{10Ls}{N}$. Assume $\mathrm{diam} f(S) > \frac{10Ls}{N}$. Consider any point $x \in Q_{1,1,4N^2} \cap S$. There exists a point $x' \in S$ such that $|f(x) - f(x')| > \frac{5Ls}{N}$. Let $1 \leq j \leq 4N^2$ be such that $x' \in Q_j$. Choose any sequence of points $z_i \in Q_i \cap S$, $i = 1, \ldots, j-1$. Let $z_0 = x$, $z_j = x'$. Since $z_0 \in Q_{1,1,4N^2} \subset Q_{4N^2}$, and $Q_{4N^2}$ is neighbor to $Q_1$, we conclude that

$$\|z_i - z_{i+1}\| \leq \frac{s\sqrt{5}}{2N} \text{ for every } i = 0, \ldots, j-1$$

(the maximal distance between any two points of neighbor squares $\frac{s}{2N} \times \frac{s}{2N}$

is $\frac{s\sqrt{5}}{2N}$).

Since $f$ is $L$-Lipschitz, it follows that $|f(z_i) - f(z_{i+1})| \leq \frac{Ls\sqrt{5}}{2N}$ whenever $0 \leq i \leq j-1$. But we also know that $|f(z_0) - f(z_j)| > \frac{5Ls}{N}$. Therefore there exists $0 \leq i_0 \leq j$ such that

$$\min\{|f(x') - f(z_{i_0})|, |f(z_{i_0}) - f(x)|\} > Ls/N \text{ and}$$
$$f(z_{i_0}) \text{ belongs to the interval with endpoints } f(x) \text{ and } f(x').$$

This immediately implies $1 \leq i_0 \leq j-1$.

On the other hand there is a finite sequence $y_1, \dots, y_I \in S$ such that $y_1 = x$, $y_I = x'$, $\|y_i - y_{i+1}\| \leq \sqrt{2}\delta_j$ for all $1 \leq i \leq I-1$, and $y_i \in \bigcup_{n \geq j} Q_n \cap S$ for all $1 \leq i \leq I$. Since $f(z_{i_0})$ is between $f(x)$ and $f(x')$, there exists $1 \leq i_1 \leq I$ such that $|f(y_{i_1}) - f(z_{i_0})| \leq L\sqrt{2}\delta_j$. Put $t_0 = z_{i_0}$, $x_i = y_{i_1+i-1}$, $r = 2\delta_j$ and $T = Q_{l,m,i_0} \cap S$ where $l, m$ are such that $z_{i_0} \in Q_{l,m,i_0}$. Note that $Lr \leq \frac{d_{j-1}}{2} \leq \frac{d_{i_0}}{2}$, since $i_0 \leq j-1$. Therefore, $T_{Lr} \setminus T$ does not intersect $S$ ($T$ is contained in the $Q_{l,m,i_0}$, which has distance greater than $d_{i_0}/2$ to any other $Q_{\cdot,\cdot,\cdot}$, so that there are no points of $S$ in $T_{Lr}$ other than points from $T$). This means all hypotheses of Lemma 3.1 are fulfilled for $T$, $S$, $t_0$, $\{x_v\}$ and $f$ ($t_0 \in T$, $|f(x_1) - f(t_0)| < Lr$ and $\|x_i - x_{i+1}\| < r$ for all $i$). Then there exists $t \in T$ such that $f(t) = f(x_{I-i_1+1}) = f(y_I) = f(x')$. Therefore, $\frac{Ls}{N} < |f(z_{i_0}) - f(x')| = |f(z_{i_0}) - f(t)| \leq L\sqrt{2}a_{i_0}$ since $z_{i_0}, t \in T \subseteq Q_{l,m,i_0}$. This is a contradiction, since $a_{i_0} \leq \frac{s}{2N} < \frac{s}{\sqrt{2}N}$. Thus $\operatorname{diam} f(S) \leq \frac{10Ls}{N}$.

Consider now the set $S_{\text{half}} = \bigcup_{n=2N^2+1}^{4N^2} Q_n \cap S$. Let $x \in Q_{2N^2+N} \cap S$ be any point in the bottom right corner of the square $[0,s] \times [0,s]$ (see Figure 1). Since $f$ is 1-local co-Lipschitz, $\operatorname{diam} f(S) \leq \frac{10Ls}{N}$ and $1 \leq L \leq \frac{N}{40}$ we get

$$\begin{aligned} f(S) \subset B_{10Ls/N}(f(x)) \cap f(S) &\subset f\big(B_{10Ls/N}(x) \cap S\big) \\ &\subset f\big(B_{s/4}(x) \cap S\big) \subset f(S_{\text{half}}). \end{aligned}$$

This implies $f(S_{\text{half}}) = f(S)$.

Let us show now that in fact $f(S) = f(Q_{1,1,1} \cap S)$. Fix any $\alpha \in f(S)$ and consider arbitrary $x \in Q_{1,1,1}$. Since $\alpha, f(x) \in f(S_{\text{half}})$, there exist $x', y' \in S_{\text{half}}$ such that $f(x') = f(x)$ and $f(y') = \alpha$. There exists a finite sequence of points $x_1, x_2, \dots, x_I \in S_{\text{half}}$, such that $x_1 = x'$, $x_I = y'$ and $\|x_i - x_{i+1}\| \leq \sqrt{2}\delta_{2N^2+1}$ for all $1 \leq i \leq I-1$. If we put now $T = Q_{1,1,1} \cap S$, $t_0 = x$ and $r = 2\delta_{2N^2+1}$, then all the assumptions of Lemma 3.1 hold (since $Lr \leq \frac{d_{2N^2}}{2} < \frac{d_1}{2}$, one has $(T_{Lr} \setminus T) \cap S = \emptyset$). Therefore, there exists $t \in T$ such that $f(t) = f(x_I) = f(y') = \alpha$. This implies $f(S) \subset f(Q_{1,1,1} \cap S)$ or, equivalently, $f(S) = f(Q_{1,1,1} \cap S)$. □

In what follows, the length of a side of a square $Q = A_r(p)$ (which equals $2r$) is denoted by $s(Q)$.

**Lemma 3.4.** *Let $D \in (0,2)$ be fixed. Assume $S_k$ is a decreasing sequence of compact sets in the plane and $N_k$ are positive integers such that the following three conditions hold:*

– *$S_1$ is the unit square $[0,1] \times [0,1]$.*

– *Every $S_k$ is a finite union $\bigcup_{m=1}^{M_k} Q_m^k$ of disjoint closed squares with horizontal and vertical sides.*

– *$S_{k+1}$ is obtained from $S_k$ in the following way. Let $R$ be one of the disjoint squares $Q_m^k$. Consider the partition of $R$ into $4N_k^2$ squares (which are obtained by dividing each side of $R$ into $2N_k$ equal segments) $R_1, \dots, R_{4N_k^2}$. For each $1 \le j \le 4N_k^2$, let $a_j$ and $d_j$ be positive real numbers, $n_j$ be positive integer, such that $2N_k n_j(a_j + d_j) = s(R)$ and $(2N_k n_j)^2 a_j^D = s(R)^D$.*

*For any closed square $A_r(x,y)$, $a < 2r/n$ let*

$$\mathcal{P}_{a,n}(A_r(x,y)) = \bigcup_{1 \le i,i' \le n} A_{a/2}(x + (\tfrac{2i-1}{n} - 1)r, y + (\tfrac{2i'-1}{n} - 1)r).$$

*We then define $\mathcal{P}(R) = \bigcup_{j=1}^{4N_k^2} \mathcal{P}_{a_j,n_j}(R_j)$ and $S_{k+1} = \bigcup_{m=1}^{M_k} \mathcal{P}(Q_m^k)$.*

*Then the $D$-dimensional Hausdorff measure of the intersection $S = \cap_{k=1}^{\infty} S_k$ is nonzero and finite.*

Proof. Note that it is enough to prove that the $D$-dimensional Hausdorff measure of $S$ with respect to $\ell_\infty^2$-norm is a non-zero finite number.

Let us observe that for each $k$, the sum of $D$th powers of sides of all squares which form $S_k$, equals 1. This is because $a_j$ and $d_j$ are so chosen in the lemma, that the sum of $D$th powers of sides of squares in $\mathcal{P}_{a_j,n_j}(R_j)$ equals $s(R)^D/(2N_k)^2$ for each $j$, and therefore, $\sum_{Q \in \mathcal{P}(R)} s(Q)^D = s(R)^D$. Since the squares in $S_k$ become arbitrarily small as $k$ grows, it follows that the Hausdorff measure of $\bigcap_{k \ge 1} S_k$ is at most 1.

The positivity of $\mathcal{H}^D(S)$ is due to the following fact. There exists an absolute constant $c > 0$ such that for an arbitrary closed square $Q$ (with horizontal and vertical sides) intersecting $S$, there is $n_0 = n_0(Q)$ such that for any $n \ge n_0$

$$s(Q)^D \ge c \sum_{\substack{1 \le m \le M_n \\ Q_m^n \cap Q \ne \emptyset}} s(Q_m^n)^D. \tag{4}$$

Indeed, let $\{B_v\}_{v \ge 1}$ be any countable covering of $S$ by open $\ell_\infty^2$-balls. Since $S$ is compact, there exists a finite subcovering, $\{B_{v_l}\}_{1 \le l \le L}$. The closed squares $K_l = \overline{B}_{v_l}$ also form a covering of $S$. Then for any $n \ge \max_{1 \le l \le L} n_0(K_l)$ one has

$$\sum_{1 \le l \le L} s(K_l)^D \ge c \sum_{1 \le m \le M_n} s(Q_m^n)^D = c.$$

Therefore, $\sum_{v\geq 1} s(B_v)^D \geq c$, and since the covering $\{B_v\}$ is arbitrary, this implies $\mathcal{H}^D(S) \geq c$.

It remains to prove inequality (4) for any closed square $Q$. Denote $q = s(Q)$. We may assume $Q \subset S_1 = [0,1]^2$ (otherwise replace $Q$ by a square inside $S_1$ whose side is shorter than $q$ but which contains $Q \cap S_1$). We will consider four cases. In each case, we will show that there exists $n_0$, such that (4) holds for $n = n_0$. Then (4) holds for any $n > n_0$ since, as we showed in the beginning of the proof of this lemma, the sum of $D$th powers of all sides of squares obtained from $Q_m^{n_0}$ at each subsequent step remains equal to $s(Q_m^{n_0})^D$.

Let

$$k = 1 + \max\{n\colon \text{there exists } m, \text{ such that } Q \subset Q_m^n\} \tag{5}$$

(note that $k \geq 2$). Among the squares in $S_{k-1}$, let $R$ be the one which contains $Q$.

Denote by $R_1, \ldots, R_{4N_{k-1}^2}$ the squares which are obtained from $R$ by dividing each side of $R$ into $2N = 2N_{k-1}$ equal segments, let $s = s(R)$.

Case 1. $Q$ contains at least one of the squares $R_1, \ldots, R_{4N^2}$.

Then $m = \left[q/(\frac{s}{2N})\right]$ is at least 1, and $Q$ is covered by at most $(m+2)^2$ squares among the $R_j$. Therefore

$$\sum_{\substack{1\leq i\leq M_k,\\ Q_i^k\cap Q\neq\emptyset}} s(Q_i^k)^D \leq (m+2)^2 \sum_{\substack{1\leq i\leq M_k,\\ Q_i^k\in\mathcal{P}(R_j)}} s(Q_i^k)^D = (m+2)^2\frac{s^D}{4N^2}.$$

Note that $m+2 \leq 3m$ since $m \geq 1$, therefore

$$\sum_{\substack{1\leq i\leq M_k,\\ Q_i^k\cap Q\neq\emptyset}} s(Q_i^k)^D \leq 9m^2\frac{s^D}{4N^2} = 9s^D\Big(\frac{m}{2N}\Big)^2 \leq 9\Big(\frac{sm}{2N}\Big)^D \leq 9q^D.$$

(The inequality $\left(\frac{m}{2N}\right)^2 \leq \left(\frac{m}{2N}\right)^D$ holds since $m \leq 2N$.) Thus, (4) holds with $c = 1/9$ and $n = k$.

Case 2. $Q$ does not contain any of $R_1, \ldots, R_{4N^2}$; $Q \cap S_k$ intersects at least two of the $R_j$'s.

Then $Q$ may intersect up to 6 squares among the $R_j$. If $Q$ intersects at least five $R_i$'s, then $q \geq \frac{s}{2N}$. Then

$$\sum_{\substack{1\leq i\leq M_k,\\ Q_i^k\cap Q\neq\emptyset}} s(Q_i^k)^D \leq 6\frac{s^D}{(2N)^2} \leq 6\Big(\frac{s}{2N}\Big)^D \leq 6q^D,$$

therefore (4) holds with $c = 1/6$ and $n = k$.

Suppose $Q$ intersects at most four squares $R_j$. Let $q_j$ be the bigger side of the rectangle $Q \cap R_j$ for each of those values of $j$. Since $Q \cap S_k$ intersects at least two of the $R_j$'s, for each such $j$ one has $q_j \geq d_j/2$ (because the distance between $R_j \cap S_k$ and the border of $R_j$ is at least $d_j/2$).

Let us prove that for each such $j$,

$$c_0 \sum_{\substack{1 \leq i \leq M_k,\\ Q_i^k \in \mathcal{P}(R_j), Q_i^k \cap Q \neq \emptyset}} s(Q_i^k)^D \leq q_j^D$$

where $c_0 = \min\{(2^{\frac{2-D}{D}} - 1)^D/8, 1/9\}$. Then (4) holds with $c = c_0/4$ and $n = k$.

Consider a square $Q' \supset Q \cap R_j$ such that $s(Q') = q_j$ and $Q' \subset R_j$, let $m = \left[\frac{q_j}{a_j + d_j}\right]$. If $m \geq 1$, then $Q'$ intersects at most $(m+2)^2$ squares among $Q_i^k \in \mathcal{P}(R_j)$. Therefore,

$$\sum_{\substack{1 \leq i \leq M_k,\\ Q_i^k \in \mathcal{P}(R_j), Q_i^k \cap Q \neq \emptyset}} s(Q_i^k)^D \leq (m+2)^2 a_j^D \leq 9m^2 a_j^D.$$

On the other hand,

$$q_j^D \geq m^D (a_j + d_j)^D = m^D a_j^D (2Nn_j)^{2-D} \geq m^D a_j^D m^{2-D} = m^2 a_j^D.$$

Thus (4) holds with $c = 1/9$ and $n = k$.

If $m = 0$, then $Q'$ intersects at most 2 squares among $Q_i^k \in \mathcal{P}(R_j)$. Therefore,

$$\sum_{\substack{1 \leq i \leq M_k,\\ Q_i^k \in \mathcal{P}(R_j), Q_i^k \cap Q \neq \emptyset}} s(Q_i^k)^D \leq 2a_j^D$$

and, since $q_j \geq d_j/2$,

$$\begin{aligned} q_j^D \geq \frac{1}{4} d_j^D = \frac{a_j^D}{4} \Big(\frac{a_j + d_j}{a_j} - 1\Big)^D &= \frac{a_j^D}{4} \Big((2Nn_j)^{\frac{2-D}{D}} - 1\Big)^D \\ \geq a_j^D \frac{(2^{\frac{2-D}{D}} - 1)^D}{4}. \end{aligned}$$

Therefore, in this case (4) holds with $c = (2^{\frac{2-D}{D}} - 1)^D/8$ and $n = k$.

Case 3. $Q$ does not contain any of $R_1, \ldots, R_{4N^2}$; $Q \cap S_k$ is contained in only one of these squares, say $R_j$; $q \geq d_j/2$.

If $m = \left[\frac{q}{a_j+d_j}\right] \geq 1$, then $Q$ intersects at most $(m+2)^2$ squares among $Q_i^k \in \mathcal{P}(R_j)$. Therefore, the same argument as in Case 2 shows that (4) holds with $c = 1/9$ and $n = k$. If $m = 0$; that is, $d_j/2 \leq q < a_j + d_j$, then the same argument as in Case 2 shows that (4) holds with $c = (2^{\frac{2-D}{D}} - 1)^D/8$ and $n = k$.

Case 4. $Q$ does not contain any of $R_1, \dots, R_{4N^2}$; $Q \cap S_k$ is contained in only one of these squares, say $R_j$; $q < d_j/2$.

In this case $Q$ intersects only one square among $Q_i^k \in \mathcal{P}(R_j)$. Denote this square by $R^*$. There exists a square $Q_1$, which lies inside $R^*$ and intersects the boundary of $R^*$, such that $s(Q_1) \leq s(Q)$ and $Q_1 \supset Q \cap S$. Then $\max\{n\colon \text{there exists } m, \text{ such that } Q_1 \subset Q_m^n\} = k$ (see (5)). Now divide $R^*$ into $4N_{k+1}^2$ squares $R_1^*, \dots, R_{4N_{k+1}^2}^*$ and consider $n = k+1$ and the square $Q_1$ instead of $Q$. Since $Q_1$ intersects the boundary of $R^*$, $Q_1$ intersects at least one $R_j^*$ for which $s(Q_1) \geq d_j^*/2$. Therefore, $Q_1$ will fit into one of the first three cases, and we will get a positive constant $c_1$ such that

$$q^D \geq s(Q_1)^D \geq c_1 \sum_{\substack{1 \leq i \leq M_{k+1}, \\ Q_i^{k+1} \cap Q_1 \neq \emptyset}} s(Q_i^{k+1})^D. \qquad \square$$

**Theorem 3.5.** *For every $0 < D < 2$ there is a compact purely unrectifiable set $S$ in the plane, such that its Hausdorff dimension is equal to $D$ and for any local Lipschitz quotient map $f\colon S \to \mathbb{R}$ the image $f(S)$ is finite.*

Proof. Let $S_k$ be a decreasing sequence of compact sets, where every $S_k$ is the union of finitely many disjoint closed squares as in Lemma 3.3 and Lemma 3.4. We put $S_1 = [0,1] \times [0,1]$. Assume $S_k = \bigcup_{1 \leq i \leq M_k} Q_i^k$ is already constructed. Let $N_k = 40k$. Let us show that for each $1 \leq i \leq M_k$, there exist sequences of positive real numbers $a_j = a_j(i), d_j = d_j(i)$ and positive integers $n_j = n_j(i)$ $(1 \leq j \leq 4N_k^2)$ such that

(A) $4N_k^2 n_j^2 a_j^D = \big(2N_k n_j(a_j + d_j)\big)^D$,

(B) $2N_k n_j(a_j + d_j) = s(Q_i^k)$,

(C) $a_{j+1} < a_j$, $d_{j+1} < d_j$,

(D) $\frac{d_j}{4} \geq k\delta_{j+1} = k\sqrt{(a_{j+1})^2 + (2a_{j+1} + d_{j+1})^2}$.

Note that (A) implies ${a_j}^D = (2N_k n_j)^{-2}\big(2N_k n_j(a_j + d_j)\big)^D$. Therefore from (B) we can find $a_j = s(2N_k n_j)^{-2/D}$, where $s = s(Q_i^k)$. Thus $d_j = \frac{s}{2N_k n_j} -$

$\frac{s}{(2N_k n_j)^{2/D}}$. If the sequence $n_j$ increases, then $a_j$ decrease; if $\frac{1}{2N_k n_1} < (\frac{D}{2})^{\frac{D}{2-D}}$, then $d_j$ also decrease, since $(x - x^{2/D})' = 1 - (2/D)x^{\frac{2-D}{D}} > 0$ if $x < (\frac{D}{2})^{\frac{D}{2-D}}$. If $n_j$ is already constructed, we always can find $n_{j+1} > n_j$ such that (D) hold, since the right-hand side of (D) tends to 0 as $n_{j+1} \to \infty$. Then we put $S_{k+1} = \bigcup_{1 \le i \le M_k} \mathcal{P}(Q_i^k)$. Lemma 3.4 guarantees that $S = \bigcap_{k \ge 1} S_k$ has Hausdorff dimension $D$.

Assume now that $f\colon S \to \mathbb{R}$ is a local Lipschitz quotient mapping. Assume also that $k \ge 1$ is such that $L_1 = L/c$, the ratio of the Lipschitz and co-Lipschitz constants of $f$ is less than $N_k/40$. Since we may rescale $f$ ($\widetilde{f}(x) = f(x/c)$), we can assume without loss of generality that $f$ is 1-local co-Lipschitz and is $L_1$-Lipschitz.

Let us show that $f(Q_i^k \cap S)$ is a single point for every $1 \le i \le M_k$. Indeed, fix $1 \le i \le M_k$ and let $T_1 = Q_i^k \cap S$. By Lemma 3.3 there exists a square $Q_{i_1}^{k+1}$ among the finite family of squares $\mathcal{P}(Q_i^k)$ such that $f(T_1 \cap Q_{i_1}^{k+1}) = f(T_1)$. The same lemma may be applied to the set $T_2 = T_1 \cap Q_{i_1}^{k+1}$, and so there exists a square $Q_{i_2}^{k+2} \in \mathcal{P}(Q_{i_1}^{k+1})$ such that $f(T_2 \cap Q_{i_2}^{k+2}) = f(T_2)$. Note that since $T_1 \cap Q_{i_2}^{k+2} = T_2 \cap Q_{i_2}^{k+2}$, this implies $f(T_1 \cap Q_{i_2}^{k+2}) = f(T_1)$. If we proceed further, we find a sequence of squares $\{Q_{i_n}^{k+n}\}_{n \ge 1}$ such that $Q_{i_n}^{k+n} \supset Q_{i_{n+1}}^{k+n+1}$ and $f(T_1) = f(T_1 \cap Q_{i_n}^{k+n})$ for every $n \ge 1$. Since the diameter of $Q_{i_n}^{k+n}$ tends to 0, and $f$ is continuous, this implies that $f(T_1)$ is a single point. Therefore $f(S)$ is finite, since there are finitely many squares $Q_i^k$.

Finally let us note that $S$ is purely unrectifiable. For any $s \in S$ and any $\varepsilon > 0$ there exists a positive $\rho < \varepsilon$ such that $(B(s,\rho) \setminus B(s,\frac{\rho}{2})) \cap S = \emptyset$; hence $S$ is purely unrectifiable. □

**Theorem 3.6.** *There is a compact, totally disconnected set $S$ in the plane of positive 2-dimensional Lebesgue measure such that for any local Lipschitz quotient $f\colon S \to \mathbb{R}$ the image $f(S)$ is finite.*

PROOF. The proof of this theorem is similar to the proof of Theorem 3.5. We construct sets $S_k$ in the same way, but when defining $S_{k+1}$ instead of condition (A), we impose the following restriction on positive real numbers $a_j, d_j$ $(1 \le j \le 4N_k^2)$.

(A$_1$) $\frac{a_j}{a_j + d_j} = 1 - \frac{1}{(k+1)^2}$.

Then $\frac{\lambda(S_k)}{\lambda(S_{k+1})} = (1 - \frac{1}{(k+1)^2})^2$, so that $\lambda(S) = \frac{1}{4}$. The other properties of $S$ can be verified in the same way as in Theorem 3.5. □

**Remark 3.7.** The rather technical construction in Section 3 of purely unrectifiable (or totally disconnected in case $D = 2$) planar sets $S$ of Hausdorff dimension $D \in (1,2]$ such that any local Lipschitz quotient map $f\colon S \to \mathbb{R}$ has finite image, can be generalized to the case of arbitrary dimension. There exist sets $S \subset \mathbb{R}^m$ of Hausdorff dimension $D \in (1,m]$ with the same property.

Below we state without proof two important changes one has to make in Lemmas 3.2–3.4 and Theorems 3.5 and 3.6 in order to construct such sets.

First, one has to find a higher dimensional analogue of "Peano curve" (3) from Lemma 3.2.

$$\gamma^{n,(m)}\colon \{1,\dots,(2n)^m\} \to \underbrace{\{0,\dots,2n-1\} \times \cdots \times \{0,\dots,2n-1\}}_{m \text{ times}},$$

such that

1. $\sum_{i=1}^m |\gamma_i^{n,(m)}(k) - \gamma_i^{n,(m)}(k+1)| = 1$ for every $1 \le k \le (2n)^m - 1$,
2. $\gamma^{n,(m)}$ is injective,
3. the image of $\{1,\dots,\frac{(2n)^m}{2}\}$ under $\gamma^{n,(m)}$ is one half of $\{0,\dots,2n-1\}^m$ in the same sense as it was for $\gamma^n$ in Lemma 3.2,
4. $\sum_{i=1}^m |\gamma_i^{n,(m)}((2n)^m) - \gamma_i^{n,(m)}(1)| = 1$.

Here by $\gamma_i^{n,(m)}(k)$ we denote the $i$th coordinate of $\gamma^{n,(m)}(k)$.

Such maps $\gamma^{n,(m)}$ are easily constructed by induction. Let $\gamma^{n,(2)} = \gamma^n$ be as in (3). If $\gamma^{n,(m)}$ is already constructed, then for $0 \le s \le 2n-1$, $1 \le k \le N = \frac{(2n)^m}{2}$, $s' = 2n-1-s$ and $k' = \frac{(2n)^m}{2} + 1 - k$ define $\gamma^{n,(m+1)}$ by

$$\gamma^{n,(m+1)}(sN+k) = \begin{cases} (\gamma^{n,(m)}(k), s) & \text{if } s \text{ is even} \\ (\gamma^{n,(m)}(k'), s) & \text{if } s \text{ is odd} \end{cases}$$

$$\gamma^{n,(m+1)}((2n)N+sN+k) = \begin{cases} (\gamma^{n,(m)}(N+k'), s') & \text{if } s \text{ is even} \\ (\gamma^{n,(m)}(N+k), s') & \text{if } s \text{ is odd} \end{cases}$$

The other important change should be made in conditions (A)–(D) (proof of Theorem 3.5). We replace them by $N_k = 10(m+3)k$,

(A′) $(2N_k n_j)^m a_j^D = \big(2N_k n_j(a_j+d_j)\big)^D$,

(B′) $s(Q_i^k) = 2N_k n_j(a_j+d_j)$,

(C′) $a_{j+1} < a_j$, $d_{j+1} < d_j$,

(D′) $\frac{d_j}{2\sqrt{m}} \ge k\delta_{j+1} = k\sqrt{(m-1)(a_{j+1})^2 + (2a_{j+1}+d_{j+1})^2}$.

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