METRIC DERIVED NUMBERS AND CONTINUOUS METRIC DIFFERENTIABILITY VIA HOMEOMORPHISMS

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Abstract. We define the notions of unilateral metric derivatives and "metric derived numbers" in analogy with Dini derivatives (also referred to as "derived numbers") and establish their basic properties. We also prove that the set of points where a path with values in a metric space with continuous metric derivative is not "metrically differentiable" (in a certain strong sense) is σ-symmetrically porous and provide an example of a path for which this set is uncountable. In the second part of this paper, we study the continuous metric differentiability via a homeomorphic change of variable.

1. Introduction

The main aim of this paper is to study analogues of the usual notion of differentiability which work for mappings with values in metric spaces. Let (X,ρ) be a metric space and $f:[a,b]\to X$ be any mapping. As every metric space isometrically embeds in some Banach space (see e.g. [BL, Lemma 1.1]), we can suppose that the distance in X is in fact generated by a complete norm $\|\cdot\|$. Define

$$md_{\pm}(f,x)=\lim_{t\to 0+}\frac{\|f(x\pm t)-f(x)\|}{t}$$

to be the *unilateral right* (*resp. left*) *metric derivatives* of the mapping f at x. If $md_+(f,x)$ and $md_-(f,x)$ exist, and are equal, then we call $md(f,x):=md_+(f,x)$ the metric derivative of f at the point x.

We say that f *is metrically differentiable at* x provided $md(f,x)$ exists and

$$\|f(y)-f(z)\|-md(f,x)|y-z|=o(|y-x|+|z-x|),\ \text{when } (y,z)\to(x,x). \tag{1.1}$$

Note that in this terminology, the existence of the "metric derivative" $md(f,x)$ of f at x does not necessarily imply that f is metrically differentiable at x! The basic example of such mapping would be $f(t)=|t|:\mathbb{R}\to\mathbb{R}$ and $x=0$.

Metric derivatives were introduced by Kirchheim in [Kh] (see also [A, DP, KS]), and were studied by several authors (see e.g. [AKh, D1, D2, DZ]). In [AKh], the authors work with a slightly weaker version of metric differentiability.

We start section 3 by noting that the set of points where $md_{\pm}(f,x)$ exist, but $md_+(f,x)\neq md_-(f,x)$, is countable; see Theorem 3.1. This is analogous to a similar theorem for unilateral derivatives of real-valued functions.

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There is a well established theory of derived numbers (or Dini derivatives) of real-valued functions $f : \mathbb{R} \to \mathbb{R}$ (see e.g. [Br]). In section 3, we generalize theorems about relationships among the Dini derivatives to the context of metric derived numbers $mD^{\pm}, mD_{\pm}$.

In Theorem 3.2, we prove that the set of "angular" points of each $f : \mathbb{R} \to X$, i.e. points $x \in \mathbb{R}$ where either $mD_{-}(f,x) > mD^{+}(f,x)$ or $mD_{+}(f,x) > mD^{-}(f,x)$, is countable. Theorem 3.3 (resp. Theorem 3.4 if f is pointwise-Lipschitz) shows that the sets of points $x \in \mathbb{R}$ where $mD^{+}(f,x) \neq mD^{-}(f,x)$ (resp. $mD_{+}(f,x) \neq mD_{-}(f,x)$) is σ-porous. Theorem 3.5 (see also Corollary 3.6) is a metric analogue of the so-called Denjoy-Young-Saks theorem about Dini derivatives (see e.g. [Br, Theorem 4.4]).

In section 4, we show that if $md(f,\cdot)$ is a continuous function, then the set of points x, where f is not metrically differentiable, is σ-symmetrically porous (Theorem 4.7). In Theorem 4.9, we show that this set is not necessarily countable. This means that the properties of metric derivatives are different from the properties of standard ones; in the latter case, the set considered in section 4, would necessarily be countable (if say $md(f,\cdot) \equiv 1$ for a real-valued f then the standard unilateral derivatives of f are equal to ± 1 at all points).

In section 5, we discuss sufficient conditions for a mapping to be metrically differentiable at a point. This is closely related to the notion of bilateral metric regularity.

In a recent paper [DZ], L. Zajíček together with the first author characterized those mappings $f : [a,b] \to X$ that allow a metrically differentiable (resp. boundedly metrically differentiable) parameterization. In section 6, we study the situation when f allows a continuously metrically differentiable parameterization (by this we mean that for a suitable homeomorphism h, the composition $f \circ h$ is metrically differentiable and its metric derivative is continuous), or just a parameterization with continuous metric derivative; see Theorems 6.2 and 6.1 for more details.

2. Preliminaries

By λ we denote the 1-dimensional Lebesgue measure on $\mathbb{R}$, and by $\mathcal{H}^1$ the 1-dimensional Hausdorff measure. In the following, X is always a real Banach space.

The following is a version of the Sard's theorem. For a proof see e.g. [DZ, Lemma 2.2].

Lemma 2.1. *Let $f : [a,b] \to X$ be arbitrary. Then $\mathcal{H}^1(f(\{x \in [a,b] : md(f,x) = 0\})) = 0$.*

By $B(x,r)$, we denote the open ball in X with center $x \in X$ and radius $r > 0$. Let $M \subset \mathbb{R}$, $x \in M$, and $R > 0$. Then we define $\gamma(x,R,M)$ to be the supremum of all $r > 0$ for which there exists $z \in \mathbb{R}$ such that $B(z,r) \subset B(x,R) \setminus M$. Also, we define $S\gamma(x,R,M)$ to be the supremum of all $r > 0$ for which there exists $z \in \mathbb{R}$ such that $B(z,r) \cup B(2x-z,r) \subset B(x,R) \setminus M$. Further, we define the *upper porosity* of M at x as

$$\overline{p}(M,x) := 2 \limsup_{R \to 0+} \frac{\gamma(x,R,M)}{R},$$

and the *symmetric upper porosity* of M at x as

$$S\overline{p}(M,x) := 2 \limsup_{R \to 0+} \frac{S\gamma(x,R,M)}{R}.$$

We say that M is *porous*[1] (resp. *symmetrically porous*) provided $\overline{p}(M,x) > 0$ for all $x \in M$ (resp. $S\overline{p}(M,x) > 0$ for all $x \in M$). We say that $N \subset \mathbb{R}$ is *σ-porous* (resp. *σ-symmetrically*

[1] In the terminology of [Z], this corresponds to M being "an upper-porous set".

porous) provided it is a countable union of porous (resp. symmetrically porous) sets. For more information about porous sets, see a recent survey [Z].

Let $f : [a, b] \to X$. Then we say that f *has finite variation* or that f *is BV*, provided $\bigvee_a^b f < \infty$. (Recall that $\bigvee_a^b f = \sup_D \sum_{i=0}^{n(D)-1} \|f(x_i) - f(x_{i+1})\|$, where the supremum is taken over all partitions $D = \{a = x_0 < x_1 < \cdots < x_n = b\}$, of $[a, b]$ and $n(D) = \#D - 1$.) We define $\bigvee_v^u f := -\bigvee_u^v f$ for $a \le u < v \le b$. We will denote $v_f(x) := \bigvee_a^x f$ for $x \in [a, b]$.

We say that $f : \mathbb{R} \to X$ is *pointwise-Lipschitz* if $\limsup_{y \to x} \frac{\|f(x)-f(y)\|}{|x-y|} < \infty$ for every $x \in \mathbb{R}$.

A considerable part of the present article is devoted to metric analogues of derived numbers (Dini derivatives). Now, we give a definition of metric derived numbers. Let $f : \mathbb{R} \to X$. Define

$$mD^{\pm}(f, x) = \limsup_{t \to 0+} \frac{\|f(x \pm t) - f(x)\|}{t},$$

and

$$mD_{\pm}(f, x) = \liminf_{t \to 0+} \frac{\|f(x \pm t) - f(x)\|}{t},$$

to be the *unilateral upper* (*resp. lower*) *metric derived numbers* (we also allow the value $+\infty$).

Note that if all four metric derived numbers of a mapping $f : \mathbb{R} \to X$ agree at a point x, then $md(f, x)$ exists, but still f is not necessarily metrically differentiable at x.

3. Unilateral metric derivatives

It is well known that the set where the standard unilateral derivatives of a real function of a real variable exist but are not equal is countable (see e.g. [J, Theorem 7.2]). The following theorem shows that it is also true for unilateral metric derivatives.

Theorem 3.1. *Let $f : \mathbb{R} \to X$. Then the set of points $x \in \mathbb{R}$ where $md_+(f, x)$, $md_-(f, x)$ exist but $md_+(f, x) \neq md_-(f, x)$, is countable.*

Proof. The proof is similar to the proof of [J, Theorem 7.2] and thus we omit it. □

It is well known that for a real function of a real variable the set of angular points (i.e. points where $D_- f > D^+ f$ or $D_+ f > D^- f$; $D^{\pm} f$, $D_{\pm} f$ are the standard derived numbers) is countable; see e.g. [J, Theorem 7.2]. The following theorem shows what happens for metric derived numbers.

Theorem 3.2. *Let $f : \mathbb{R} \to X$. Then the set of points $x \in \mathbb{R}$ where either $mD_-(f, x) > mD^+(f, x)$ or $mD_+(f, x) > mD^-(f, x)$ is countable.*

Proof. By symmetry, it is enough to prove that the set $E = \{x \in \mathbb{R} : mD_-(f, x) > mD^+(f, x)\}$ is countable. Let $h < k$ be two positive rational numbers. For a positive integer n let E_{hkn} be the set of points $x \in E$ for which $\frac{\|f(\xi)-f(x)\|}{|\xi-x|} < h$ and $\frac{\|f(\xi')-f(x)\|}{|\xi'-x|} > k$ whenever $0 < \xi - x < 1/n$ and $0 < x - \xi' < 1/n$. Then $E_{hkn} \cap (x - 1/n, x + 1/n) = \{x\}$. Suppose that is not true, and there is a point $x_1 \in E_{hkn} \cap (x - 1/n, x + 1/n)$ such that $x_1 \neq x$. Then assuming $x > x_1$, say, we get $\frac{\|f(x_1)-f(x)\|}{|x_1-x|} < h$ and $\frac{\|f(x)-f(x_1)\|}{|x-x_1|} > k$, a contradiction. Thus all points of E_{hkn} are isolated, and E_{hkn} is countable. Because $E \subset \bigcup_{h,k,n} E_{hkn}$, we obtain the conclusion of the theorem. □

We have the following two theorems concerning the points where unilateral lower and upper metric derivatives differ. In the proofs, we use similar ideas as in [EH, Theorem 1].

Theorem 3.3. *Let X be a Banach space, and $f : \mathbb{R} \to X$ be arbitrary. Then the set*

$$\{x \in \mathbb{R} : mD^+(f,x) \neq mD^-(f,x)\}$$

is σ-porous.

Proof. We will only prove that the set

$$A = A_f = \{x \in \mathbb{R} : mD^-(f,x) < mD^+(f,x)\},$$

is σ-porous (and notice that $\{x \in \mathbb{R} : mD^-(f,x) > mD^+(f,x)\}$ is σ-porous as it is equal to $A_{f(-\cdot)}$). To that end, it is enough to establish that

$$A_{rs} = \{x \in A : mD^-(f,x) < r < s < mD^+(f,x)\},$$

is σ-porous for all $r < s$ pairs of positive rational numbers. Define

$$A_{rsn} = \left\{ x \in A_{rs} : \frac{\|f(x) - f(y)\|}{|x-y|} < r \text{ for } y \in (x - 1/n, x) \right\}.$$

We easily see that $A_{rs} = \bigcup_n A_{rsn}$. We will prove that A_{rsn} is $\frac{\delta-1}{\delta}$-porous, where $\delta = \min(2, (s+r)/2r)$. Let $x \in A_{rsn}$. Then there exist $x_k \to x+$ such that $\frac{\|f(x)-f(x_k)\|}{|x-x_k|} > s$. Choose k large enough such that $|x - x_k| < 1/n$. Define $w_k = x + \delta\,(x_k - x)$, and let $y \in [x_k, w_k] \cap A_{rsn}$. Then

$$\begin{aligned}
\|f(x) - f(y)\| &\geq \|f(x) - f(x_k)\| - \|f(x_k) - f(y)\| \\
&\geq s\,|x - x_k| - r\,|x_k - y| \geq s\,|x - x_k| - r\,|x_k - w_k| \\
&= s\,|x - x_k| - r\,(\delta - 1)\,|x_k - x| \\
&= |x - x_k|\,(s - r\,(\delta - 1)) = |w_k - x|\,\frac{(s - r\,(\delta-1))}{\delta} \\
&\geq r\,|x - y|,
\end{aligned}$$

by the choice of δ (we used that $w_k - x = \delta\,(x_k - x)$, and $w_k - x_k = (\delta - 1)\,(x_k - x)$). Thus $y \notin A_{rsn}$, and $[x_k, w_k] \cap A_{rsn} = \emptyset$. Finally, note that $\frac{w_k - x_k}{w_k - x} = \frac{\delta-1}{\delta} > 0$. □

Theorem 3.4. *Let X be a Banach space, and $f : \mathbb{R} \to X$ be pointwise-Lipschitz. Then the set*

$$\{x \in \mathbb{R} : mD_+(f,x) \neq mD_-(f,x)\}$$

is σ-porous.

Proof. We will only prove that the set

$$B = B_f = \{x \in \mathbb{R} : mD_-(f,x) < mD_+(f,x)\},$$

is σ-porous, and notice that $\{x \in \mathbb{R} : mD_-(f,x) > mD_+(f,x)\}$ is σ-porous as it is equal to $B_{f(-\cdot)}$. We will prove that B_f is σ-porous for f that is pointwise-Lipschitz. To that end, it is enough to establish that $B_{rs} = \{x \in B : mD_-(f,x) < r < s < mD_+(f,x)\}$, is σ-porous for all $r < s$ pairs of positive rational numbers. For $n \in \mathbb{N}$, define

$$\begin{aligned}
B_{rsn} = \Big\{ x \in B_{rs} :& \frac{\|f(x) - f(y)\|}{|x-y|} > s \text{ for } y \in (x, x + 1/n), \\
&\text{and } \frac{\|f(x) - f(z)\|}{|x-z|} < n \text{ whenever } 0 < |z - x| < 1/n \Big\}.
\end{aligned}$$

Since f is pointwise-Lipschitz, we easily see that $B_{rs} = \bigcup_n B_{rsn}$. We will prove that B_{rsn} is $\frac{\delta-1}{\delta}$-porous, where $\delta = \min(\frac{s-r}{n} + 1, 2)$. Let $x \in B_{rsn}$. Then there exist $x_k \to x-$ such that

$\frac{\|f(x)-f(x_k)\|}{|x-x_k|} < r$. Choose k large enough such that $|x - x_k| < 1/n$. Define $w_k = x - \delta(x - x_k)$, and let $y \in [w_k, x_k] \cap B_{rsn}$. Then

$$\begin{aligned}\|f(x) - f(y)\| &\le \|f(x) - f(x_k)\| + \|f(x_k) - f(y)\| \\ &\le r\,|x - x_k| + n\,|x_k - y| \le r\,|x - x_k| + n\,|x_k - w_k| \\ &= r\,|x - x_k| + n\,(\delta - 1)\,|x_k - x| = |x - x_k|\,(r + n\,(\delta - 1)) \\ &\le s\,|x - y|,\end{aligned}$$

by the choice of δ (we used that $x-w_k = \delta\,(x-x_k)$, $x_k-w_k = (\delta-1)\,(x-x_k)$, and $|x_k-w_k| < 1/n$). Thus $y \notin B_{rsn}$, and $[x_k, w_k] \cap B_{rsn} = \emptyset$. Finally, note that $\frac{x_k-w_k}{x-w_k} = \frac{\delta-1}{\delta} > 0$. □

The following theorem asserts that outside of a set of measure 0, the fact that $mD^+(f,x) < \infty$ already implies that $md(f,x)$ exists.

Theorem 3.5. *Let $f : \mathbb{R} \to X$ be arbitrary. Then there exists a set N with Lebesgue measure zero such that*

> *if $x \in \mathbb{R} \setminus N$ and $mD^+(f,x) < \infty$, then $md(f,x)$ exists, and $md(f,x) = mD^+(f,x)$.*

Proof. Let N_1 be the set of points $x \in \mathbb{R}$ where $mD^-(f,x) \neq mD^+(f,x)$. Then, by Theorem 3.3 N_1 is σ-porous. Therefore, by the Lebesgue density theorem, its Lebesgue measure $\lambda(N_1)$ is zero. Let

$$A_n = \{x \in \mathbb{R} : \|f(x+h) - f(x)\| \le nh \text{ for } 0 < h < 1/n\}.$$

Let A be the set of points x such that $mD^+(f,x) < \infty$. Then $A = \bigcup_n A_n$. Let $A_{n,j}$ be subsets of A_n, such that $A_n = \bigcup A_{n,j}$, and $\operatorname{diam}(A_{n,j}) < 1/n$. Then $f|_{A_{n,j}}$ is n-Lipschitz, and thus, by Kirszbraun theorem, see [Kb], it can be extended to an n-Lipschitz function $f_{n,j}$ defined on the whole real line. By [D2, Theorem 2.7], we obtain that $f_{n,j}$ is metrically differentiable at all $x \in D_{n,j}$, where $\lambda(\mathbb{R} \setminus D_{n,j}) = 0$. Let $E_{n,j} \subset D_{n,j} \cap A_{n,j}$ be the set of points of density of $D_{n,j} \cap A_{n,j}$. By the Lebesgue density theorem we have that $\lambda(D_{n,j} \cap A_{n,j} \setminus E_{n,j}) = 0$. We shall prove that $md(f,x)$ exists and is equal to $mD^+(f,x)$ at all points $x \in E_{n,j}$ for all $n, j \in \mathbb{N}$. This will conclude the proof, as the set $N = \bigcup_{n,j}(A_{n,j} \setminus E_{n,j})$ has Lebesgue measure 0.

To finish the proof, let $x \in E_{n,j}$. Fix $\varepsilon > 0$. Find $\delta > 0$ such that $\lambda(E_{n,j} \cap (x, x+t)) \ge \left(1 - \frac{\varepsilon}{4n}\right)t$ for $0 < t < \delta$, and $\left|\frac{\|f_{n,j}(x+t)-f_{n,j}(x)\|}{|t|} - md(f_{n,j}, x)\right| \le \varepsilon$, whenever $0 < |t| < \delta$. Thus for each $0 < h < \delta$ there exists $y \in E_{n,j} \cap (x, x+h)$ such that $|y - (x+h)| \le \frac{\varepsilon h}{2n}$. Now,

$$\begin{aligned}\|f(x+h) - f(x)\| &\le \|f(y) - f(x)\| + \|f(x+h) - f(y)\| \\ &\le (md(f_{n,j}, x) + \varepsilon)(y - x) + \varepsilon h \\ &\le (md(f_{n,j}, x) + \varepsilon)h,\end{aligned}$$

since x and y belong to $E_{n,j} \subset A_n$ and $y > x$. On the other hand,

$$\begin{aligned}\|f(x+h) - f(x)\| &\ge \|f(y) - f(x)\| - \|f(x+h) - f(y)\| \\ &\ge (md(f_{n,j}, x) - \varepsilon)(y - x) - \varepsilon h \\ &\ge ((md(f_{n,j}, x) - \varepsilon)(1 - \varepsilon h \cdot (2n)^{-1} - \varepsilon)h.\end{aligned}$$

Thus $md_+(f,x) = md(f_{n,j}, x) = mD^+(f,x)$.

A similar argument shows that $md_-(f,x) = md(f_{n,j},x) = mD^+(f,x)$ for $x \in E_{n,j}$, and thus $md(f,x)$ exists for all $x \in A \setminus N$. $\square$

Theorem 3.5 has the following corollary.

Corollary 3.6. *Let $f : \mathbb{R} \to X$ be arbitrary. Then there exists a set $N \subset \mathbb{R}$ with $\lambda(N) = 0$, such that if $x \in \mathbb{R} \setminus N$, and $\min(mD^-(f,x), mD^+(f,x)) < \infty$, then $md(f,x)$ exists.*

Corollary 3.6 together with [D2, Theorem 2.6] imply the following:

Corollary 3.7. *Let $f : \mathbb{R} \to X$ be arbitrary. Then there exists a set $M \subset \mathbb{R}$ with $\lambda(M) = 0$, such that if $x \in \mathbb{R} \setminus M$, and $\min(mD^-(f,x), mD^+(f,x)) < \infty$, then f is metrically differentiable at x.*

4. Points of metric non-differentiability

We will use following lemma proved in [DZ, Lemma 2.4].

Lemma 4.1. *Let $f : [c,d] \to X$, $x \in [c,d]$. Then the following hold.*

(i) *If $md(f,x) = 0$, then f is metrically differentiable at x.*
(ii) *If $h : [a,b] \to [c,d]$ is differentiable at $w \in [a,b]$, $h(w) = x$, and f is metrically differentiable at x, then $f \circ h$ is metrically differentiable at w, and $md(f \circ h, w) = md(f,x) \cdot |h'(w)|$.*

Lemma 4.2. *Let X be a Banach space, and let $f : [a,b] \to X$. If $md(f,\cdot)$ is continuous at $x \in [a,b]$, then there exists $\delta > 0$ such that*

$$\bigvee_s^t f = \int_s^t md(f,y)\,dy \qquad \text{for all } s < t, s,t \in [x-\delta, x+\delta] \cap [a,b].$$

Proof. Let $\delta > 0$ be chosen such that for all $s \in [x-\delta, x+\delta] \cap [a,b]$ we have that $md(f,s)$ exists and $|md(f,x) - md(f,s)| \le 1$. It follows from [F, §2.2.7] that $f|_{[x-\delta,x+\delta]\cap[a,b]}$ is Lipschitz. We obtain that

$$\int_s^t md(f,y)\,dy = \int_{f([s,t])} N(f|_{[s,t]}, y)\,d\mathcal{H}^1(y) = \bigvee_s^t f,$$

for all $s < t$, $s,t \in [x-\delta, x+\delta] \cap [a,b]$ (here, $N(f|_{[s,t]}, y)$ is the multiplicity with which the function $f|_{[s,t]}$ assumes a value y). The first equality follows from [Kh, Theorem 7], the second equality follows from [F, Theorem 2.10.13]. $\square$

Let $f : [a,b] \to X$, $I = [a,b]$. We say that $x \in I$ is *metrically regular point of the function f*, provided

$$\lim_{\substack{t\to 0\\ x+t\in I}} \frac{\|f(x+t) - f(x)\|}{|\bigvee_x^{x+t} f|} = 1.$$

Lemma 4.3. *Let X be a Banach space, $g : [a,b] \to X$, $x \in [a,b]$, $md(g,x) > 0$, and $md(g,\cdot)$ is continuous at x. Then x is metrically regular point of the function g.*

Proof. Let $\varepsilon > 0$. Find $\delta_0 > 0$ such that $(1-\varepsilon)\, md(g,x)|t| \le \|g(x+t)-g(x)\|$ and $md(g,x+t) < (1+\varepsilon)\cdot md(g,x)$, whenever $|t| < \delta_0$ and $x+t \in [a,b]$. Using Lemma 4.2, we can find $0 < \delta < \delta_0$ such that for all $|t| < \delta$ we have

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)\left|\bigvee_x^{x+t} g\right| = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)\left|\int_x^{x+t} md(g,s)\,ds\right|$$

$$\le (1-\varepsilon)\cdot md(g,x)\,|t| \le \|g(x+t)-g(x)\| \le \left|\bigvee_x^{x+t} g\right|.$$

If $t \neq 0$, by dividing by $\left|\bigvee_x^{x+t} g\right|$ (which is strictly positive), we obtain $\frac{1-\varepsilon}{1+\varepsilon} \le \frac{\|g(x+t)-g(x)\|}{|\bigvee_x^{x+t} g|} \le 1$, and thus x is metrically regular point of f. $\square$

The following lemma shows that the condition (1.1) is satisfied "unilaterally" at a point x provided $md(f,\cdot)$ is continuous at x.

Lemma 4.4. *Let X be a Banach space, and let $f : [a,b] \to X$. If $md(f,\cdot)$ is continuous at $x \in [a,b]$, then*

$$\|f(y)-f(z)\| - md(f,x)|y-z| = o(|x-z|+|x-y|), \tag{4.1}$$

whenever $(y,z) \to (x,x)$, and $\operatorname{sign}(z-x) = \operatorname{sign}(y-x)$.

Proof. If $md(f,x) = 0$, then the conclusion follows from Lemma 4.1, so we can assume that $md(f,x) > 0$. Lemma 4.3 implies that x is metrically regular point of f. Now we will prove that f satisfies (4.1) at x. Let $0 < \varepsilon < 1$. Using Lemma 4.2, find $\delta > 0$ such that for all t with $x+t \in [a,b] \cap [x-\delta, x+\delta]$ we have that $(1-\varepsilon)\left|\bigvee_x^{x+t} f\right| \le \|f(x+t)-f(x)\|$,

$$(1-\varepsilon)\, md(f,x) \le md(f,x+t) \le (1+\varepsilon)\, md(f,x),$$

and $\bigvee_y^z f = \int_y^z md(f,s)\,ds$ for all $y,z \in [a,b]\cap[x-\delta,x+\delta]$. Let $y,z \in [a,b]\cap[x-\delta,x+\delta]$ with $\operatorname{sign}(z-x) = \operatorname{sign}(y-x)$. Without any loss of generality, we can assume that $z > x$, and $|z-x| \ge |y-z|$. We obtain that

$$\begin{aligned}
\|f(y)-f(z)\| &\ge \|f(z)-f(x)\| - \|f(y)-f(x)\| \\
&\ge (1-\varepsilon)\bigvee_x^z f - \|f(y)-f(x)\| \\
&\ge (1-\varepsilon)\int_x^z md(f,t)\,dt - \bigvee_x^y f \\
&\ge (1-\varepsilon)\int_x^z md(f,t)\,dt - \left|\int_x^y md(f,t)\,dt\right| \\
&\ge (1-\varepsilon)^2\, md(f,x)(z-x) - (1+\varepsilon)\, md(f,x)|y-x| \\
&= md(f,x)|z-y| - \varepsilon\cdot \underbrace{((2-\varepsilon)(z-x)+|y-x|)\cdot md(f,x)}_{\eta(\varepsilon,y,z)}.
\end{aligned}$$

It is easy to see that $\frac{\eta(\varepsilon,y,z)}{|z-x|+|y-x|}$ is bounded from above by $2\cdot md(f,x)$ for all $\varepsilon \in (0,1)$. For the other inequality, note that

$$\|f(y)-f(z)\| \le \bigvee_y^z f = \int_y^z md(f,s)\,ds \le (1+\varepsilon)\, md(f,x)|z-y|, \tag{4.2}$$

and the conclusion easily follows. □

We now show that if the metric derivative of f exists at each point and is continuous, then the mapping is metrically differentiable on a large set of points. We prove this in several steps.

Proposition 4.5. *Let X be a Banach space, $f : [a, b] \to X$ be such that $md(f, x) = 1$ for each $x \in [a, b]$. Then the set of points $x \in [a, b]$ such that f is not metrically differentiable at x, is σ-symmetrically porous.*

Proof. Let A be the set of points $x \in (a, b)$ such that f is not metrically differentiable at x. By Lemma 4.4, we see that the condition (1.1) is satisfied unilaterally at each $x \in [a, b]$.

Suppose that $x \in A$. We claim that there exist $\delta_j = \delta_j(x) \to 0+$ such that

$$\liminf_{j\to\infty} \frac{\|f(x + \delta_j) - f(x - \delta_j)\|}{2\delta_j} < 1. \tag{4.3}$$

To see this, note that because $x \in A$, there exist $(y_j)_j$, $(z_j)_j$ such that $y_j < x < z_j$ (because (1.1) is satisfied unilaterally at x), $\lim_j y_j = \lim_j z_j = x$, and $\liminf_{j\to\infty} \frac{\|f(z_j)-f(y_j)\|}{z_j-y_j} < 1-\varepsilon$, for some $\varepsilon > 0$. Without any loss of generality, we can assume that $z_j - x \le x - y_j$. Let $\tilde{y}_j = 2x - y_j$, and note that $z_j \le \tilde{y}_j$. If $\tilde{y}_j = z_j$, take $\delta_j = z_j - x$, otherwise note that for $j \in \mathbb{N}$ large enough we have

$$\begin{aligned}\|f(y_j) - f(\tilde{y}_j)\| &\le \|f(y_j) - f(z_j)\| + \|f(z_j) - f(\tilde{y}_j)\| \\ &\le (1 - \varepsilon)(z_j - y_j) + (\tilde{y}_j - z_j).\end{aligned}$$

Now, as $\tilde{y}_j - z_j \le z_j - y_j$, we obtain $(\tilde{y}_j - z_j) - \frac{\varepsilon}{2}(z_j - y_j) \le \big(1 - \frac{\varepsilon}{2}\big)(\tilde{y}_j - z_j)$, and thus $\|f(y_j) - f(\tilde{y}_j)\| \le \big(1 - \frac{\varepsilon}{2}\big)(\tilde{y}_j - y_j)$. Now define $\delta_j = \tilde{y}_j - x = x - y_j$, and (4.3) follows.

Let A_{nm} be the set of all $x \in A$ such that

- there exist a sequence $(\delta_j)_j$, such that $\delta_j \to 0+$, and $\|f(x-\delta_j)-f(x+\delta_j)\| \le \big(1-\frac{1}{m}\big)\, 2\,\delta_j$,
- for each $t \in [0, 1]$ with $0 < |x - t| < 1/n$ we have $\big(1 - \frac{1}{2m}\big)|x - t| < \|f(t) - f(x)\|$.

By the above argument, it is easy to see that $A = \bigcup_{n,m} A_{nm}$.

Fix $n, m \in \mathbb{N}$. Let $x \in A_{nm}$. There exists $j_0 \in \mathbb{N}$ such that for all $j \ge j_0$ we have $0 < \delta_j < (4n)^{-1}$. Let $z_j := x + \delta_j$, and $y_j := x - \delta_j$. Fix $j \ge j_0$ and suppose that $w \in [z_j, z_j + 2\delta_j]$. Then $|w - y_j| < 1/n$, and we have that

$$\begin{aligned}\|f(y_j) - f(w)\| &\le \|f(y_j) - f(z_j)\| + \|f(z_j) - f(w)\| \\ &\le \left(1 - \frac{1}{m}\right) 2\delta_j + |w - z_j|.\end{aligned}$$

By the choice of w we have $-\frac{2\delta_j}{2m} + (w - z_j) \le \big(1 - \frac{1}{2m}\big)(w - z_j)$, and thus $\|f(y_j) - f(w)\| \le \big(1 - \frac{1}{2m}\big)(w - y_j)$. This implies that $w \notin A_{nm}$. We obtained that $[z_j, z_j + 2\delta_j] \cap A_{nm} = \emptyset$. Similarly, $[y_j - 2\delta_j, y_j] \cap A_{nm} = \emptyset$, and the symmetric porosity of A_{nm} follows. □

We will need the following auxiliary lemma.

Lemma 4.6. *Let $B \subset [a, b]$ be symmetrically porous and $h : [a, b] \to \mathbb{R}$ be continuously differentiable and bilipschitz. Then $h(B)$ is symmetrically porous.*

Proof. Let $L > 0$ be such that $L^{-1}|x-y| \le |h(x)-h(y)| \le L|x-y|$ for all $x, y \in [a,b]$. Let $x \in B$. Let $\delta_n, \alpha_n > 0$ be such that $B(x-\delta_n, \alpha_n) \cup B(x+\delta_n, \alpha_n) \subset \mathbb{R} \setminus B$, $\alpha_n \to 0$, and $c\delta_n \le \alpha_n$. First, we will show that

$$B(h(x \pm \delta_n), \alpha_n/(2L)) \cap h(B) = \emptyset. \tag{4.4}$$

Let $z \in B(h(x \pm \delta_n), \alpha_n/(2L))$. If $y \in [a,b]$ is such that $h(y) = z$, then

$$|x \pm \delta_n - y| \le L|h(x \pm \delta_n) - h(y)| \le \alpha_n/2,$$

and thus $z \notin h(B)$ (since h is one-to-one), and (4.4) holds.

Note that since $h'(x) \neq 0$,

$$\left|1 - \frac{h(x+\delta_n) - h(x)}{h(x) - h(x-\delta_n)}\right| = \left|1 - \frac{h'(x)\delta_n + o(\delta_n)}{h'(x)\delta_n + o(\delta_n)}\right| \to 0,$$

as $n \to \infty$, and thus

$$|h(x+\delta_n) - h(x) - (h(x) - h(x-\delta_n))| \le c/(4L^2)|h(x-\delta_n) - h(x)| \le c\delta_n/(4L) \le \alpha_n/(4L) \tag{4.5}$$

for n large enough. Now, we will show that $2h(x) - z \in B(h(x+\delta_n), \alpha_n/(2L))$ whenever $z \in B(h(x-\delta_n), \alpha_n/(4L))$, and n is large enough. Together with (4.4), this easily implies that $S\overline{p}(h(B), h(x)) > 0$.

Assume that $z \in B(h(x-\delta_n), \alpha_n/(4L))$, and that (4.5) holds. Then

$$\begin{aligned} |2h(x) - z - h(x+\delta_n)| &\le |h(x+\delta_n) - h(x) + (h(x-\delta_n) - h(x))| + |h(x-\delta_n) - z| \\ &\le \alpha_n/(4L) + \alpha_n/(4L) = \alpha_n/(2L), \end{aligned}$$

and thus $2h(x) - z \in B(h(x+\delta_n), \alpha_n/(2L))$, and the conclusion follows. □

We have the following:

Theorem 4.7. *Let $f : [a,b] \to X$ be such that $md(f,\cdot)$ is continuous on $[a,b]$. Then the set of points, where f is not metrically differentiable, is σ-symmetrically porous.*

Proof. Let $A \subset [a,b]$ be the set where f is not metrically differentiable. Lemma 4.1 implies that if $x \in A$, then $md(f,x) > 0$. Let $A = \bigcup_n A_n$, where $A_n = \{x \in A : md(f,x) > 1/n\}$. It is enough to show that each A_n is σ-symmetrically porous. Because $md(f,\cdot)$ is continuous, we have that each A_n is open. Let (c,d) be an open component of A_n, let $g = f|_{[c,d]}$, $G = g \circ v_g^{-1}$ (see Section 2 for the definition of v_g). Using Lemma 4.2, it is easy to see that $md(G,x) = 1$ for all $x \in v_g((c,d))$. Then Proposition 4.5 implies that G is metrically differentiable outside a σ-symmetrically porous set B. Because v_g is continuously differentiable and bilipschitz, by Lemmas 4.1 and 4.6, we obtain that $g = G \circ v_g$ is metrically differentiable outside a σ-symmetrically porous set $v_g^{-1}(B)$. □

Remark 4.8. It is easy to see that if f is a real-valued function and $md(f,\cdot)$ is continuous on $[a,b]$, then the set of points where f is not metrically differentiable is at most countable. However, in Theorem 4.9 below, we show that already in a 2-dimensional situation such a set may be uncountable. Thus, Theorem 4.9 shows that Theorem 4.7 cannot be strengthened to make the exceptional set countable.

Theorem 4.9. *For any norm $\|\cdot\|$ in the 2-dimensional plane, there exists a curve $\gamma : [0, \ell] \to (\mathbb{R}^2, \|\cdot\|)$ with $md(\gamma, x) = 1$ for all $x \in [0, \ell]$, but such that the set of points where γ is not metrically differentiable is uncountable.*

We will give a detailed proof of this theorem for $\|\cdot\|$ being the Euclidean norm. In Remark 4.14, we explain how this case reflects the most general situation. Note however, that if one uses a "polygonal" norm (for example, the ℓ_1-norm), then much simpler constructions are possible. We explain this in Remark 4.15.

Before we start the proof of Theorem 4.9, let us establish the following property of logarithmic spirals, which will be used in the proof of Lemma 4.11.

Lemma 4.10. *Assume $S_{a,b}$ is a planar curve defined in polar coordinates (r, ϕ) by the equation $r = ae^{b\phi}$ with $a > 0$, $b \neq 0$ (logarithmic spiral). Then the length of the arc of $S_{a,b}$ between the origin and the point with modulus r_0 and argument ϕ_0 is equal to $\frac{\sqrt{b^2+1}}{|b|} r_0$.*

In other words, if $S_{a,b} : [0, +\infty) \to \mathbb{C}$ is the arc-length parameterization of this logarithmic spiral such that $S_{a,b}(0) = 0$, then

$$\frac{|S_{a,b}(t)|}{t} = \frac{|b|}{\sqrt{b^2+1}} \tag{4.6}$$

for all $t > 0$.

Proof. A routine computation of the length of the logarithmic spiral with the given equation in polar coordinates proves the lemma. □

Lemma 4.11. *For any angle $\alpha \in (0, \pi/2)$ and a constant $q \in (0, 1)$ there is a piecewise smooth planar curve such that its arc-length parameterization $g = g_{q,\alpha} : \mathbb{R} \to \mathbb{C}$ has the following properties:*

(a) *$g([0, 1])$ is a horizontal interval and there exists $L_{q,\alpha} > 0$ such that $g([1 + L_{q,\alpha}, +\infty))$ and $g((-\infty, -L_{q,\alpha}])$ are horizontal rays;*
(b) *there exists $t_{q,\alpha} > 1/2$ such that the arguments of $z_\pm = g(1/2 \pm t_{q,\alpha}) - g(1/2)$ are equal to $(-\alpha)$ and $(\pi + \alpha)$ resp.;*
(c) *$|g(t) - g(s)|/|t - s| > q$ for all $s \in [0, 1]$ and $t \neq s$.*

Proof. Let $B > 0$ be large enough so as to ensure that

$$\frac{B}{\sqrt{B^2+1}} > q \quad \text{and} \quad -B\sin\alpha + \cos\alpha < 0. \tag{4.7}$$

In (4.13), we will impose another condition on B which also bounds B from below. Fix $b > B$ and denote $k = \frac{b}{\sqrt{b^2+1}}$.

We first construct a piecewise smooth planar curve $f = f_{q,\alpha} : [0, +\infty) \to \mathbb{C}$ such that

$$g(t) = \begin{cases} f(t), & \text{if } t \geq 0, \\ 1 - \overline{f(1-t)}, & \text{if } t < 0, \end{cases} \tag{4.8}$$

has the desired properties.

For $t \in [0, 1]$ we set $f(t) = t + 0i$. Now let $S_{1,-b} : [0, +\infty) \to \mathbb{C}$ be the arc-length parameterization of the logarithmic spiral from Lemma 4.10. Identity (4.6) implies that the point $S_{1,-b}(k^{-1})$ has modulus 1, therefore, it coincides with $f(1)$.

For $t \in [1, 1 + k^{-1}(e^{b\alpha} - 1)]$ we put $f(t) = S_{1,-b}(t + k^{-1} - 1)$. Then for every $s \geq 0$ one has:

$$\frac{1+s}{|f(1+s) - f(0)|} < \frac{k^{-1}+s}{|f(1+s) - f(0)|} = \frac{k^{-1}+s}{|S_{1,-b}(k^{-1}+s)|} = k^{-1}. \tag{4.9}$$

Let $s_0 = k^{-1}(e^{b\alpha} - 1)$. Then the point $f(1 + s_0) = S_{1,-b}(k^{-1}e^{b\alpha})$ has modulus $e^{b\alpha}$ and argument $-\alpha$.

Now let $S_{e^{2b\alpha},b} : [0, +\infty) \to \mathbb{C}$ be another logarithmic spiral parametrized by the arc-length. For $t \in [1 + s_0, 1 + s_0 + s_1]$ (where s_1 is defined below), let $f(t) = S_{e^{2b\alpha},b}(t + k^{-1} - 1)$. Again, note that $S_{1,-b}(t + k^{-1} - 1)$ and $S_{e^{2b\alpha},b}(t + k^{-1} - 1)$ are equal at $t = 1 + s_0$, since by (4.6) the lengths of the arcs of both logarithmic spirals between the origin and the point with modulus $e^{b\alpha}$ and argument $-\alpha$ are equal to $k^{-1}e^{b\alpha} = k^{-1} + s_0$. Furthermore, for every $s \geq 0$ one has:

$$\frac{1+s_0+s}{|f(1+s_0+s) - f(0)|} < \frac{k^{-1}+s_0+s}{|S_{e^{2b\alpha},,b}(k^{-1}+s_0+s)|} = k^{-1}. \tag{4.10}$$

Let us find the slope of the tangent to the logarithmic spiral $S_{e^{2b\alpha},b}$ at the point with modulus $e^{b\alpha}$ and argument $-\alpha$. If we denote by $z(\phi) = e^{2b\alpha}e^{b\phi}e^{i\phi}$ the polar parameterization of $S_{e^{2b\alpha},b}$, then $\operatorname{Im} \frac{dz}{d\phi}(-\alpha)$ is equal to $e^{b\alpha}(-b\sin\alpha + \cos\alpha) < 0$. Therefore, the y-coordinate of $f(t)$ continues to decrease as ϕ increases from $-\alpha$ to some $-\beta \in (-\alpha, 0)$ such that $-b\sin\beta + \cos\beta = 0$ (i.e., $\tan\beta = 1/b$). Let s_1 be such that $f(1+s_0+s_1) = S_{e^{2b\alpha},b}(e^{2b\alpha}e^{-b\beta})$ is the point with modulus $e^{2b\alpha - b\beta}$ and argument $-\beta$, i.e., $s_1 = k^{-1}e^{b\alpha}(e^{b(\alpha-\beta)} - 1)$.

For $t \geq 1 + s_0 + s_1$, define $f(t)$ as $f(1 + s_0 + s_1) + (t - 1 - s_0 - s_1)$. Then one easily checks that since $\cos\beta = k$, the law of cosines for the triangle with vertices in $f(0)$, $f(1 + s_0 + s_1)$ and $f(1 + s_0 + s_1 + s)$ guarantees that the inequality

$$\frac{(1+s_0+s_1+s)^2}{|f(1+s_0+s_1+s) - f(0)|^2} \leq k^{-2} \tag{4.11}$$

holds for all $s \geq 0$.

Inequalities (4.9), (4.10), (4.11) imply that

$$\frac{t}{|f(t) - f(0)|} \leq k^{-1} \tag{4.12}$$

for all $t > 0$.

Note that if we now define g as in (4.8), then property (a) in the Lemma holds for $L_{q,\alpha} = s_0 + s_1$.

The argument of $f(1 + s_0) - f(0)$ is equal to $(-\alpha)$. Then the arguments of $g(1/2 \pm (1/2 + s_0)) - g(1/2)$ are equal to $(-\alpha')$ and $(\pi + \alpha')$ respectively, where $\alpha' > \alpha$. Since the argument of $g(1/2 + t) - g(1/2)$ is continuous in t, there is a value $t_{q,\alpha}$ between $1/2$ and $1/2 + s_0$ such that property (b) in the present lemma holds for $t_{q,\alpha}$.

We have already proved, see (4.12), that property (c) in the present lemma holds for $s = 0$ and all $t > 0$ (as $g(t) = f(t)$ for $t \geq 0$). If $s \in [0, 1]$ and $1 \leq t \leq 1 + s_0 + s_1$, then $g(s) = s$ and

$$\begin{aligned} t - s &= (t-1) + (1-s) = (k^{-1}|g(t)| - k^{-1}) + (1-s) \\ &= k^{-1}(|g(t)| - s) - (k^{-1} - 1)(1-s) < k^{-1}|g(t) - g(s)|. \end{aligned}$$

If $t \geq 1 + s_0 + s_1$, then $|g(t) - g(0)|/t \geq k$, and therefore,

$$\frac{|g(t) - g(s)|}{t-s} \geq \frac{|g(t)| - s}{t-s} \geq \frac{k-x}{1-x},$$

where $x = \frac{s}{t} \le \frac{1}{1+s_0+s_1}$. Then $\frac{k-x}{1-x} \ge k - \frac{1-k}{s_0+s_1} \ge k - \frac{k(1-k)}{e^{b\alpha}-1}$. Note that the latter expression is an increasing function of b (as k is a function of b), which tends to 1 as b tends to infinity. Therefore, if in addition to (4.7) we require that

$$\frac{B}{\sqrt{B^2+1}} - \frac{\frac{B}{\sqrt{B^2+1}}(1 - \frac{B}{\sqrt{B^2+1}})}{e^{B\alpha}-1} > q, \tag{4.13}$$

then property (c) in the Lemma holds for all $s \in [0,1]$ and $t \ge 1$. It remains to note that this property trivially holds for $s,t \in [0,1]$ and that by symmetry, the case $s \in [0,1]$, $t < 0$ is analogous to $1-s \in [0,1]$, $1-t > 1$.

Thus, conditions (a)–(c) hold for g with $t_{q,\alpha} \in (1/2, 1/2+s_0)$ and $L_{q,\alpha} = s_0 + s_1$. □

Remark 4.12. In addition to properties (a)–(c) of Lemma 4.11 we may assume that the curve $g_{q,\alpha}$ is a graph of Lipschitz piecewise smooth function $F_{q,\alpha} : \mathbb{R} \to \mathbb{R}$.

Proof. Let us analyze the tangent vector to $g_{q,\alpha}$ when the argument t changes from 1 to $1+s_0$ and from $1+s_0$ to $1+s_0+s_1$ (see the proof of Lemma 4.11).

The arc of $g_{q,\alpha}$ between $g_{q,\alpha}(1)$ and $g_{q,\alpha}(1+s_0)$ has the polar parameterization $z(\phi) = e^{b\phi}e^{-i\phi}$, ϕ increases from 0 to α. Then the x-coordinate $\operatorname{Re}\frac{dz}{d\phi}(\phi)$ of the tangent vector is equal to $e^{b\phi}(b\cos\phi - \sin\phi)$. This is positive provided $\tan\phi < b$. Thus, we impose the following additional restriction on B:

$$\tan\alpha < B. \tag{4.14}$$

Since $\operatorname{Re}\frac{dz}{d\phi}(\phi)$ is continuous, we conclude that its minimum on $\phi \in [0,\alpha]$ is positive. The y-coordinate of the tangent vector is continuous in t, therefore is bounded for $t \in [1, 1+s_0]$. Thus, the slope of the tangent vector is bounded. Hence $g_{q,\alpha}|_{[1,1+s_0]}$ is a graph of Lipschitz function.

The arc of $g_{q,\alpha}$ between $g_{q,\alpha}(1+s_0)$ and $g_{q,\alpha}(1+s_0+s_1)$ has the polar parameterization $z(\phi) = e^{2b\alpha+b\phi}e^{i\phi}$, ϕ increases from $-\alpha$ to $-\beta$. Then $\operatorname{Re}\frac{dz}{d\phi}(\phi) = e^{2b\alpha+b\phi}(b\cos\phi - \sin\phi) > 0$ since $\cos\phi > 0$ and $\sin\phi < 0$ for $\phi \in (-\alpha, -\beta)$. In the same way this implies that $g_{q,\alpha}|_{[1+s_0, 1+s_0+s_1]}$ is a graph of Lipschitz function. □

Proof of Theorem 4.9 in the Euclidean case. Let $\alpha_n \to \pi/2$ and $q_n \to 1$, $n \ge 1$ be two increasing sequences of positive reals. For every pair (α_n, q_n) consider a Lipschitz function $F_n(x) = F_{q_n,\alpha_n}(x + 1/2)$, where F_{q_n,α_n} is a Lipschitz piecewise smooth function described in Remark 4.12 (whose graph is the curve g_{q_n,α_n} from Lemma 4.11). The function F_n is even. Note that $F_n(x)$ is constant for $|x| \ge x_n = \operatorname{Re}(g_{q_n,\alpha_n}(1 + L_{q_n,\alpha_n}))$. Denote by $\Gamma_{F_n}(x) = x + iF_n(x)$ the graph of F_n and for each $n \ge 1$ choose $L_n > x_n$ such that

$$L_n - x_n > n\mathcal{H}^1(\Gamma_{F_n}[-x_n, x_n]). \tag{4.15}$$

Now let $G_n(x) = \frac{F_n(L_n x) - F_n(L_n)}{L_n}$. The function G_n has the following properties:

- G_n is a nonnegative even piecewise smooth Lipschitz function on $\mathbb{R}$,
- G_n is zero on $(-\infty, -1] \cup [1, \infty)$,
- $G_n(x)$ attains its maximum at $x = 0$, $G_n(0) < 1$ and $G_n^{-1}(G_n(0)) = [-1/L_n, 1/L_n]$ ($L_n > |F_n(x_n)|$ from (4.15)),
- If $\gamma_n = \Gamma_{G_n}$ is the graph of G_n, then $\mathcal{H}^1(\gamma_n[-a_n, a_n]) < 1/n$, where $a_n = \sup\{x \colon G_n(x) > 0\}$,
- There exists $t_n \in (0, a_n)$ such that the argument of $\gamma_n(t_n) - \gamma_n(0)$ is equal to $(-\alpha_n)$,

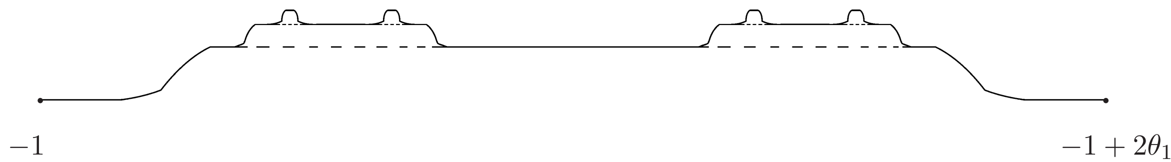

FIGURE 1. A graph of $S_3(x)$

- The ratio $\frac{|\gamma_n(x)-\gamma_n(y)|}{\mathcal{H}^1(\gamma_n[x,y])}$ is bounded from below by q_n for all pairs of $x \neq y$ such that $|x| \leq 1/L_n$.

Denote by p_n the length of $\gamma_n[-a_n, a_n]$. Let $\theta_n \searrow 0$ $(n \geq 1)$ be such that

$$\theta_{n+1}p_{n+1} < (\theta_n p_n)/4, \quad 2\theta_{n+1} < \theta_n/L_n, \quad \theta_1 < 1/2, \text{ and } \sum_{n\geq 1}\theta_n < 1.$$

The first property of θ_n guarantees that for every $n \geq 1$

$$\sum_{j\geq 1} 2^{j-1}\theta_{n+j}p_{n+j} < 2\theta_{n+1}p_{n+1}.$$

Note that as $G_n(x)$ is a hat-like function on $[-1,1]$, the graph of $G_n^{(\rho,\theta)}(x) = \theta G_n(\theta^{-1}(x-\rho))$ is the rescaled "hat" on $[\rho-\theta, \rho+\theta]$. For any closed interval $I = [\rho-\theta, \rho+\theta]$ denote by $I^{(L)}$ the interval $[\rho-\theta/L, \rho+\theta/L]$.

For $x \in [-1,1]$, let

$$h_1(x) = S_1(x) = \sum_{\rho\in\{-1+\theta_1, 1-\theta_1\}} G_1^{(\rho,\theta_1)}(x).$$

Let $\mathcal{G}_1 = \{[-1, -1+2\theta_1], [1-2\theta_1, 1]\}$ (since $\theta_1 < 1/2$, these intervals are disjoint). Now we define inductively two sequences of families of intervals as follows:

$$\mathcal{F}_n = \{I^{(L_n)} \text{ such that } I \in \mathcal{G}_n\};$$
$$\mathcal{G}_{n+1} = \{[a, a+2\theta_{n+1}], [b-2\theta_{n+1}, b] \text{ such that } [a,b] \in \mathcal{F}_n\}.$$

For every $n \geq 1$, $x \in [-1,1]$, let

$$h_{n+1}(x) = \sum_{[a,b]\in\mathcal{F}_n} \sum_{\rho\in\{a+\theta_{n+1}, b-\theta_{n+1}\}} G_{n+1}^{(\rho,\theta_{n+1})}(x); \tag{4.16}$$
$$S_{n+1}(x) = S_n(x) + h_{n+1}(x)$$

(Figure 1 shows a possible graph of $S_3(x)$). Note that the definition of h_1 agrees with (4.16) if we let $\mathcal{F}_0 = \{[-1,1]\}$. For all n, $\mathcal{F}_n$ consists of 2^n disjoint closed intervals of the same length $2\theta_n/L_n$, whose union is equal to the preimage $S_n^{-1}(\max_x S_n(x))$. Since $4\theta_{n+1} \leq 2\theta_n/L_n$, intervals in $\mathcal{G}_{n+1}$ are disjoint.

For $x \in [-1,1]$, define $G(x) = \lim_n S_n(x)$. Note that each S_n is continuous and $|G - S_n| = |\sum_{k\geq n+1} h_k| \leq \sum_{k\geq n+1}\theta_k$ which tends to zero as $n \to \infty$. Therefore, G is continuous. Since the length ℓ of the graph of G is finite (it is bounded from above by $1 + \sum_{n\geq 1} 2^n\theta_n p_n < 1 + 4\theta_1 p_1 < 5$), we conclude that the graph of G has an arc-length parameterization.

Let $\gamma = \Gamma_G \colon [-1,1] \to \mathbb{C}$ be the graph of G. The curve γ consists of points of two types: points in $A_1 = \bigcup \Gamma_{S_n}[-1,1] \cap \gamma[-1,1]$ and points in $A_2 = \gamma[-1,1] \setminus A_1$. The set A_2 is a Cantor-like set which will be described below.

For any $t \in \gamma^{-1}(A_1)$, the metric derivative of the normal parameterization of γ at t is clearly equal to 1, since the functions S_n are piecewise smooth. Consider $c \in C = \gamma^{-1}(A_2)$. Since $\gamma(c)$ does not belong to $\Gamma_{S_n}[-1,1]$ for any n, there is a sequence of intervals $I_n \in \mathcal{G}_n$ such that $c = \bigcap_{n\geq 1} I_n$. Then $\gamma(c)$ corresponds to a certain infinite sequence $\varepsilon \in \{0,1\}^\infty$: depending whether I_n has center at $\rho = a + \theta_n$ or at $\rho = b - \theta_n$ (see (4.16)), we let ε_n be equal to 0 or 1. Therefore, C is a Cantor set, and thus it is uncountable. We show that for any $c \in C$, the metric derivative of the normal parameterization of γ at c is equal to 1, but the normal parameterization of γ is not metrically differentiable at c.

For any point $c \in C$ there is a pair of sequences of points $y_n, z_n \to c$, $y_n < c < z_n$ such that $G(y_n) = S_n(y_n)$, $G(z_n) = S_n(z_n)$ and the points $\gamma(y_n)$, $\gamma(z_n)$ and $\Gamma_{S_n}(\frac{y_n+z_n}{2})$ form an isosceles triangle with vertex angle $\pi - 2\alpha_n$. This means that not only the ratio between the distance $|\gamma(y_n) - \gamma(z_n)|$ divided by the length of $\gamma[y_n, z_n]$ does not tend to 1, but moreover, it tends to 0. Therefore, the normal parameterization of γ is not metrically differentiable at c.

It remains to show that for any point $c \in C$ the metric derivative of the normal parameterization of γ at c is equal to 1. We will show that the ratio

$$\mathcal{H}^1(\gamma[c, c+t])/|\gamma(c+t) - \gamma(c)| \tag{4.17}$$

tends to 1 as $t \to 0$.

Assume $t > 0$ is small. Let $\varepsilon \in \{0,1\}^\infty$ be a sequence corresponding to $\gamma(c)$. Without any loss of generality we may assume $c + t \in \bigcup_{I \in \mathcal{G}_1} I$. Let $\delta \in \{0,1\}^\infty$ be a sequence corresponding to $\gamma(c+t)$. If $\gamma(c+t) \in A_1$, then δ is a finite sequence; otherwise, δ is infinite.

Since t is small, we may assume that $\delta_1 = \varepsilon_1$. Let $n \geq 1$ be such that $(\varepsilon_1, \dots, \varepsilon_n) = (\delta_1, \dots, \delta_n)$ and $\varepsilon_{n+1} \neq \delta_{n+1}$ (if such n does not exist, that is, if the sequence δ constitutes the beginning of the infinite sequence ε, we let n be equal to the length of δ). Note that when c is fixed and t tends to 0, then n tends to ∞.

In order to find an upper bound for (4.17), we will use the following estimate:

$$\begin{aligned}\frac{\mathcal{H}^1(\gamma[c,c+t])}{|\gamma(c+t)-\gamma(c)|} &\leq \frac{\mathcal{H}^1(\gamma[x,c+t]) + \mathcal{H}^1(\gamma[c,x])}{|\gamma(c+t)-\gamma(x)| - |\gamma(x)-\gamma(c)|} \\ &\leq \Big(\frac{\mathcal{H}^1(\gamma[x,c+t])}{|\gamma(c+t)-\gamma(x)|} + y\Big)/(1-y),\end{aligned} \tag{4.18}$$

for any $x \in (c, c+t)$, such that the expression $y = \frac{\mathcal{H}^1(\gamma[c,x])}{|\gamma(c+t)-\gamma(x)|}$ is strictly less than 1.

Consider first the case when δ coincides with $(\varepsilon_1, \dots, \varepsilon_n)$. In this case, $G(c+t) = S_n(c+t)$ and there is an interval $I_n \in \mathcal{G}_n$ of length $2\theta_n$ containing both c and $c+t$. Let $J_1, J_2 \subset I_n$ be disjoint intervals in $\mathcal{G}_{n+1}$ such that $c \in J_1 \cup J_2$. Since δ has length n, we get $c + t \notin J_1 \cup J_2$. Also note that $c \in J_1^{(L_{n+1})} \cup J_2^{(L_{n+1})}$ since $G(c) \neq S_{n+1}(c)$.

If $c \in J_i^{(L_{n+1})}$, then let $x = \sup\{z \in J_i \colon S_{n+1}(z) > S_n(z)\}$. Since $S_{n-1}|_{I_n}$ is constant and $G(x) = S_n(x)$, $G(c+t) = S_n(c+t)$, we may deduce that by the property of G_{n+1}, the expression $\frac{\mathcal{H}^1(\gamma[x,c+t])}{|\gamma(c+t)-\gamma(x)|}$ does not exceed q_{n+1}^{-1}. Now we want to find an upper estimate for $y = \frac{\mathcal{H}^1(\gamma[c,x])}{|\gamma(c+t)-\gamma(x)|}$. The numerator is not greater than $\sum_{j\geq 1} 2^{j-1}\theta_{n+j}p_{n+j} < 2\theta_{n+1}p_{n+1}$, and the denominator is at least $\theta_{n+1}(n+1)p_{n+1}$ (this follows from the property of L_n, see (4.15)). Therefore, $y \leq 2/(n+1)$. Thus, the quantity (4.17) is at most $\psi_{n+1}(q_{n+1}^{-1})$, where $\psi_k(t) = (t + 2/k)/(1 - 2/k)$.

Now consider the case when δ has length at least $n+1$. In the above notation this implies that $c \in J_1^{(L_{n+1})}$ and $c+t \in J_2$. Choose $x = \sup\{z \in J_1 \colon S_{n+1}(z) > S_n(z)\}$ as before. If $c+t \in J_2^{(L_{n+1})}$, then the same proof as in the previous paragraph shows that $\frac{\mathcal{H}^1(\gamma[x,c+t])}{|\gamma(c+t)-\gamma(x)|} \le \psi_{n+1}(1)$ (in this case γ connects $\gamma(x)$ and $\gamma(x')$, where $x' = \inf\{z \in J_2 \colon S_{n+1}(z) > S_n(z)\}$, by a straight line interval). Thus, the quantity (4.17) is at most $\psi_{n+1}(\psi_{n+1}(1))$.

If $c+t \in J_2 \setminus J_2^{(L_{n+1})}$, then $\mathcal{H}^1(\gamma[x,c+t]) \le |\gamma(c+t)-\gamma(x)| + \sum_{j\ge 1} 2^{j-1}\theta_{n+j}p_{n+j} < |\gamma(c+t)-\gamma(x)| + 2\theta_{n+1}p_{n+1}$, so together with $|\gamma(c+t)-\gamma(x)| > \theta_{n+1}(n+1)p_{n+1}$ we get that the quantity (4.17) is at most $\psi_{n+1}(1+2/(n+1))$.

It remains to observe that the length n of the initial part of sequences ε and δ tends to ∞ as $t \to 0$ and to note that $\psi_{n+1}(q_{n+1}^{-1})$, $\psi_{n+1}(\psi_{n+1}(1))$ and $\psi_{n+1}(1+2/(n+1))$ tend to 1 as n tends to infinity. □

Remark 4.13. Note that in fact we proved that the curve γ constructed above has the following property: for every $c \in C$ there exist $y_n < c < z_n$ such that $(y_n, z_n) \to (c,c)$ and

$$\frac{|\gamma(y_n)-\gamma(z_n)|}{\mathcal{H}^1(\gamma[y_n,z_n])} \to 0.$$

This means that this curve has uncountably many "spikes".

Remark 4.14. For a general norm $\|\cdot\|$ on the 2-dimensional plane, one can produce an analogue of the curve constructed in Lemma 4.11 in the following way.

We may assume the $\|\cdot\|$-norm of the point 1 on the complex plane is equal to 1. Define $g([0,1])$ to be a horizontal interval as in (4.8) ($f(t) = t$ for $t \in [0,1]$), then $\|g(t)\| = t$ for $0 \le t \le 1$. Next find a small $\varepsilon > 0$, such that if we define $g(t)|_{t>1}$ to be a ray with slope $-\varepsilon$, then the condition (c) in Lemma 4.11 with the norm $\|\cdot\|$ instead of Euclidean norm $|\cdot|$ holds for all $t > 1$. Next thing would be to note that the ratio $\|g(t)-g(s)\|/|t-s|$ tends to 1 as s remains in $[0,1]$ and t tends to infinity (we define $g(t) = 1 + (t-1)z_{-\varepsilon}$, where $\|z_{-\varepsilon}\| = 1$ and $\tan\arg z_{-\varepsilon} = -\varepsilon$). So we may choose a sufficiently large T_1 such that if we redefine $g(t)|_{t>T_1}$ to be a ray with slope -2ε, then we again have condition (c) in Lemma 4.11 still valid for $\|\cdot\|$. If we continue this way, the curve g would consist of straight intervals such that each new interval "turns" by less than $-\varepsilon$ with respect to the previous interval, and in the end point of each interval the ratio from condition (c) is very close to 1 (much closer to 1 than q is). Since $N\varepsilon \to \infty$, the angle between the horizontal axis and the subsequent intervals which form the curve g tends to $\pi/2$. So there will be a moment when this angle becomes bigger than α. At this moment, we stop the process, and start "rotating" intervals towards horizontal axis (making the slope less negative) in order to obtain a broken line satisfying the conditions (a)–(c) from Lemma 4.11.

One can check that since the arc-length parameterization of the boundary of a unit ball of arbitrary norm is uniformly continuous, the algorithm explained above can be implemented for every 2-dimensional norm (of course, ε would depend on the norm).

The curve g constructed above will in fact be an approximation of two logarithmic spirals (such as those used in the proof of Lemma 4.11). Then we prove Theorem 4.9 in the same way, each time putting two rescaled "hats" on top of the previous "hat". The curve obtained in this way will not be metrically differentiable at the points of the Cantor set, since if we consider a sequence of isosceles triangles $A_nB_nC_n$ with vertex angle $\angle B_n$ tending to 0, the ratio between $\|A_nC_n\|$ and $\|A_nB_n\| + \|B_nC_n\|$ will tend to zero as $n \to \infty$, for any norm $\|\cdot\|$.

Remark 4.15. If we work with the ℓ_1-norm, then for a fixed $\alpha \in (0, \pi/2)$ let $h = \frac{1}{2}\tan\alpha$ and

$$g(t) = \begin{cases} (t+h) - hi, & \text{if } t < -h, \\ ti, & \text{if } t \in [-h, 0], \\ t, & \text{if } t \in [0, 1], \\ 1 - (t-1)i, & \text{if } t \in [1, 1+h], \\ (t-h) - hi, & \text{if } t > 1+h. \end{cases}$$

The curve g satisfies conditions of Lemma 4.11 with any $q < 1$ (for the ℓ_1 norm), and although it cannot be made into a graph of a function in the usual sense, one can easily see that putting together such "boxes" (rescaling as necessary and taking $\alpha_n \to \pi/2$), we obtain the example of a planar curve with metric derivative 1 at every point, but with uncountable set of points where it is not metrically differentiable.

5. Metric regularity and metric differentiability

This section contains mainly auxiliary results. Let $f : [a, b] \to X$, $I = [a, b]$. We say that $x \in I$ is *bilaterally metrically regular point of the function* f, provided

$$\lim_{\substack{(y,z)\to(x,x) \\ a\le y\le x\le z\le b}} \frac{\|f(y) - f(z)\|}{\bigvee_y^z f} = 1.$$

See the beginning of section 4 for the definition of a metrically regular point. Note that every bilaterally metrically regular point of a function is also its metrically regular point.

Lemma 5.1. *Let X be a Banach space, $g : [a, b] \to X$, $x \in [a, b]$, g is metrically differentiable at x with $md(g, x) > 0$, and $md(g, \cdot)$ is continuous at x. Then x is bilaterally metrically regular point of the function g.*

Proof. Lemma 4.3 implies that x is a metrically regular point of g. Let $\varepsilon > 0$. By metric differentiability of g at x, by Lemma 4.2, and by continuity of $md(g, \cdot)$ at x find $\delta > 0$ such that $(1 - \varepsilon)\, md(g, x)|z - y| \le \|g(z) - g(y)\|$, $\bigvee_y^z g = \int_y^z md(g, s)\, ds$, for $x - \delta < y < x < z < x + \delta$, and $md(g, x + t) < (1 + \varepsilon) \cdot md(g, x)$ for $|t| < \delta$ with $x + t \in [a, b]$. Thus, for y, z with $x - \delta < y \le x \le z < x + \delta$ we have

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right) \bigvee_y^z g = \left(\frac{1-\varepsilon}{1+\varepsilon}\right) \int_y^z md(g, s)\, ds \le (1 - \varepsilon) \cdot md(g, x)\, |z - y|$$
$$\le \|g(z) - g(y)\| \le \bigvee_y^z g.$$

If $y \ne z$, then by dividing by $\bigvee_y^z g$, we obtain $\frac{1-\varepsilon}{1+\varepsilon} \le \frac{\|g(y)-g(z)\|}{\bigvee_y^z g} \le 1$, and thus x is bilaterally metrically regular point of g. ∎

Lemma 5.2. *Let X be a Banach space, and let $f : [a, b] \to X$. If $md(f, \cdot)$ is continuous at $x \in [a, b]$, and x is a bilaterally metrically regular point of f, then f is metrically differentiable at x.*

Proof. If $md(f,x)=0$, then the conclusion follows from Lemma 4.1(i), and thus we can assume that $md(f,x)>0$. Lemma 4.4 implies that the condition (1.1) holds provided $\operatorname{sign}(z-x)=\operatorname{sign}(y-x)$. Thus, we only need to treat the case $\operatorname{sign}(z-x)=-\operatorname{sign}(y-x)$ since the cases when either $y=x$ or $z=x$ follow easily from the existence of $md(f,x)$. Let $\varepsilon>0$. Find $\delta>0$ such that for all $x-\delta<y\le x\le z<x+\delta$ with $(y,z)\ne(x,x)$ we have that $\|f(y)-f(z)\|\ge(1-\varepsilon)\bigvee_y^z f$, $\bigvee_y^z f=\int_y^z md(f,t)\,dt$, and $(1-\varepsilon)\,md(f,t)\le md(f,x)\le(1+\varepsilon)\,md(f,t)$ for $|x-t|<\delta$ with $t\in[a,b]$. Let $x-\delta<y\le x\le z<x+\delta$. Then

$$\begin{aligned}\|f(y)-f(z)\| &\ge (1-\varepsilon)\bigvee_y^z f=(1-\varepsilon)\int_y^z md(f,t)\,dt\\ &\ge (1-\varepsilon)^2\,md(f,x)(z-y).\end{aligned}$$

The other inequality follows from the same reasoning as in (4.2). □

Lemma 5.3. *Let X be a Banach space, let $f:[a,b]\to X$ be continuous, BV, and such that it is not constant on any subinterval of $[a,b]$. Let $x\in(a,b)$, $y=v_f(x)$, and $g=f\circ v_f^{-1}$. Then*

(i) *if x is a metrically regular point of f, then $md(g,y)=1$,*

(ii) *if x is a bilaterally metrically regular point of f, and there exists a neighbourhood U of x such that all $z\in U$ are metrically regular points of f, then g is metrically differentiable at y.*

Proof. To prove (i), note that

$$\begin{aligned}1&=\lim_{z\to x}\frac{\|f(z)-f(x)\|}{\left|\bigvee_x^z f\right|}=\lim_{z\to x}\frac{\|f(z)-f(x)\|}{|v_f(z)-v_f(x)|}\\ &=\lim_{w\to y}\frac{\|f\circ v_f^{-1}(w)-f\circ v_f^{-1}(y)\|}{|w-y|}=md(g,y).\end{aligned}$$

For (ii), first note that $md(g,y)=1$ by part (i). Let U be the neighbourhood of x such that all $z\in U$ are metrically regular points of f. Then part (i) implies that $md(g,w)=1$ for all $w=v_f(z)$, where $z\in U$. To apply Lemma 5.2, it is enough to show that y is a bilaterally metrically regular point of g, but

$$\begin{aligned}\lim_{\substack{(s,t)\to(y,y)\\ 0\le s\le y\le t\le v_f(b)}}\frac{\|g(t)-g(s)\|}{\bigvee_s^t g}&=\lim_{\substack{(s,t)\to(y,y)\\ 0\le s\le y\le t\le v_f(b)}}\frac{\|f\circ v_f^{-1}(t)-f\circ v_f^{-1}(s)\|}{t-s}\\ &=\lim_{\substack{(u,v)\to(x,x)\\ a\le u\le x\le v\le b}}\frac{\|f(v)-f(u)\|}{v_f(v)-v_f(u)}=1,\end{aligned}$$

where the last equality follows from the fact that x is a bilaterally metrically regular point of f, and $v_f(v)-v_f(u)=\bigvee_u^v f$ for any $u,v\in U$, $u<v$ by Lemma 4.2. Now, application of Lemma 5.2 yields the conclusion. □

We will also need the following simple lemma.

Lemma 5.4. *Let $f:[a,b]\to X$, $x\in[a,b]$, be such that $md(f,x)$ exists, but f is not metrically differentiable at x. Then if h is a homeomorphism of $[a,b]$ onto itself such that $f\circ h$ is metrically differentiable at $h^{-1}(x)$, then $md(f\circ h,h^{-1}(x))=0$.*

Proof. Lemma 4.1 shows that $md(f,x) > 0$ (otherwise we have a contradiction with the fact that f is not metrically differentiable at x). Suppose that h is an (increasing) homeomorphism such that $f \circ h$ is metrically differentiable at $y = h^{-1}(x)$. For a contradiction, suppose that $md(f \circ h, y) > 0$. Note that

$$\frac{|h(y+t) - h(y)|}{|t|} = \frac{|h(y+t) - h(y)|}{\|f(h(y+t)) - f(h(y))\|} \cdot \frac{\|f(h(y+t)) - f(h(y))\|}{|t|},$$

and it follows that $h'(y) = \frac{md(f \circ h, y)}{md(f,x)} > 0$. Thus $h'(y)$ exists and is non-zero. This implies that $(h^{-1})'(x)$ exists. Because $f = (f \circ h) \circ h^{-1}$, Lemma 4.1 implies that f is metrically differentiable at x, a contradiction. We conclude that $md(f \circ h, y) = 0$. $\square$

6. Continuous metric differentiability via homeomorphisms

Let $f : [a,b] \to X$. Let M_f be the set of all points $x \in [a,b]$ with the following property: there is no neighbourhood $U = (x - \delta, x + \delta)$ of x such that either $f|_U$ is constant or all points of U are metrically regular points of the function f. Obviously, M_f is closed, and $a, b \in M_f$.

Theorem 6.1. *Let X be Banach space, and let $f : [a,b] \to X$. Then the following are equivalent.*

- (i) *There exists a homeomorphism k of $[a,b]$ onto itself such that $md(f \circ k, \cdot)$ is continuous on $[a,b]$.*
- (ii) *f is continuous, BV, and $\mathcal{H}^1(f(M_f)) = 0$.*

Proof. To prove that (i) $\Longrightarrow$ (ii), note that the existence of continuous metric derivative implies continuity and boundedness of variation of the function, and these properties are preserved when the function is composed with a homeomorphism. Thus, it is enough to prove that $\mathcal{H}^1(f(M_f)) = 0$. Note that $M_f = k(M_{f \circ k})$, and thus it is enough to prove that $\mathcal{H}^1((f \circ k)(M_{f \circ k})) = 0$. Let $g = f \circ k$. We claim that

$$M_g \subset \{x \in [a,b] : md(g,x) = 0\}. \tag{6.1}$$

Indeed, Lemma 4.3 implies that every point $x \in (a,b)$, such that $md(g,x) > 0$, is metrically regular point of g. By continuity of $md(g, \cdot)$, there exists a neighbourhood U of x such that $md(g,y) > 0$ at all $y \in U$, and thus all points of U are metrically regular points of g. So we get (6.1), and then by Lemma 2.1, we see that $\mathcal{H}^1(g(M_g)) = 0$.

To prove that (ii) $\Longrightarrow$ (i), let $(U_i)_i$ be the collection of all maximal open intervals inside $[a,b]$ such that $f|_{U_i}$ is constant, and put $U = \bigcup_i U_i$. Define $\varphi(t) = v_f(t) + \lambda(U \cap [a,t])$ for $t \in [a,b]$. Let (a_j, b_j) be the maximal open components of $[a,b]$ such that all points of (a_j, b_j) are metrically regular points of f. Let $\alpha_j = \varphi(a_j)$, $\beta_j = \varphi(b_j)$. Then $\varphi(b_j) - \varphi(a_j) = \bigvee_{a_j}^{b_j} f$. Note that

$$\varphi(b) = \lambda(U) + \bigvee_a^b f = \lambda(U) + \sum_j \bigvee_{a_j}^{b_j} f = \lambda(U) + \sum_j (\beta_j - \alpha_j) = \lambda(\varphi([a,b] \setminus M_f)), \tag{6.2}$$

by [DZ, Lemma 2.7], and thus $\lambda(\varphi(M_f)) = \lambda(M_{f \circ \varphi^{-1}}) = 0$ (the left-hand side of (6.2), $\varphi(b)$, is equal to $\lambda(\varphi[a,b])$, and φ is increasing). Let $g = f \circ \varphi^{-1}$. It is easy to see that g is Lipschitz (because φ is a homeomorphism). By Zahorski's lemma (see e.g. [GNW, p. 27]) there exists a continuously differentiable homeomorphism h of $[0, \varphi(b)]$ onto itself such that $h'(x) = 0$ if and only if $x \in h^{-1}(M_g)$. Now, by the equality

$$\frac{g(h(x+t))-g(h(x))}{t} = \frac{g(h(x+t))-g(h(x))}{h(x+t)-h(x)} \cdot \frac{h(x+t)-h(x)}{t}, \tag{6.3}$$

and by Lemma 5.3, we obtain that $md(g\circ h, x)$ exists and is continuous at all $x \in \varphi(U) \cup \bigcup_j (\alpha_j, \beta_j)$. By (6.3), by the choice of h and the fact that g is Lipschitz, we easily obtain that $md(g \circ h, x) = 0$ for all $x \in h^{-1}(M_g)$, and that $md(g\circ h, \cdot) = md(f \circ k, \cdot)$ is continuous at all such points (where $k = \varphi^{-1} \circ h$). □

Let M_f^{b} be the set of all points $x \in [a,b]$ with the following property: there is no neighbourhood $U = (x-\delta, x+\delta)$ of x such that either $f|_U$ is constant or all points of U are bilaterally metrically regular points of the function f. Obviously, M_f^{b} is closed and $a, b \in M_f^{\mathrm{b}}$.

Theorem 6.2. *Let X be Banach space, and let $f : [a,b] \to X$. Then the following are equivalent.*

(i) *There exists a homeomorphism h of $[a,b]$ onto itself such that $f \circ h$ is metrically differentiable at every point of $[a,b]$, and $md(f\circ h, \cdot)$ is continuous.*
(ii) *f is continuous, BV, and $\mathcal{H}^1(f(M_f^{\mathrm{b}})) = 0$.*

Proof. The proof is similar to the proof of Theorem 6.1, and thus we omit it. It uses Lemmas 5.1 and 5.3(ii). □

The following example shows that the scopes of Theorems 6.1 and 6.2 are different (see also Remark 6.4).

Example 6.3. There exists 1-Lipschitz mapping $f : [0,1] \to \ell_2$ such that $md(f,x) = 1$ for all $x \in [0,1]$, but f is not metrically differentiable at a dense subset S of $[0,1]$.

Proof. Choose $t_n > 0$ with $\sum_n t_n^2 = 1$, and $q_n \in (0,1)$ such that $S = \{q_n : n \in \mathbb{N}\}$ is dense in $[0,1]$. Let $f_n : [0,1] \to \mathbb{R}^2$ be defined as

$$f_n(t) = \begin{cases} (t, 0) & \text{for } 0 \le t \le q_n, \\ \frac{(t-q_n)}{\sqrt{2}} \cdot (1,1) + (q_n, 0) & \text{for } q_n < t \le 1. \end{cases}$$

It is easy to see that $f_n(0) = 0$ and f_n is 1-Lipschitz for each $n \in \mathbb{N}$. Define $f : [0,1] \to \ell_2 = \sum \oplus_{\ell_2} \ell_2^2$ as $f(t) = (t_n \cdot f_n(t))_n$. It is easy to see that f is well defined, and 1-Lipschitz. First, we will show that $md(f,x) = 1$ for all $x \in [0,1]$. Choose $x \in [0,1]$ and $\varepsilon > 0$. Find $n_0 \in \mathbb{N}$ such that $\sum_{n \ge n_0} t_n^2 < \varepsilon^2$. Find $\delta > 0$ such that $(x-\delta, x+\delta) \cap \{q_j : j \le n_0\} \subset \{x\}$. Let $y \in (x-\delta, x+\delta)$ and notice that

$$\begin{aligned} |y-x| &\ge \|f(y)-f(x)\| \\ &= \Big(\sum_{n \le n_0} t_n^2 \, \|f_n(y)-f_n(x)\|^2_{\ell_2^2} + \sum_{n > n_0} t_n^2 \, \|f_n(y)-f_n(x)\|^2_{\ell_2^2} \Big)^{1/2} \\ &\ge \Big(\Big(\sum_{n \le n_0} t_n^2 \Big)^{1/2} - \varepsilon \Big) |y-x| \ge (1-2\varepsilon)|y-x|. \end{aligned}$$

Conclude by sending ε to 0.

Now we will show that f is not metrically differentiable at any $x \in S$. Fix $x = q_m \in S$ for some m, and let $\delta > 0$ be such that $0 \le x - \delta < x + \delta \le 1$. Then

$$\begin{aligned}\frac{\|f(x-\delta) - f(x+\delta)\|}{2\delta} &= \frac{1}{2\delta}\Big(t_m^2 \|f_m(x-\delta) - f_m(x+\delta)\|_{\ell_2^2}^2 \\ &\qquad + \sum_{n\neq m} t_n^2 \|f_n(x+\delta) - f_n(x-\delta)\|_{\ell_2^2}^2\Big)^{1/2} \\ &\le \frac{1}{2\delta}\Big(t_m^2\delta^2(2+\sqrt{2}) + \sum_{n\neq m} 4\delta^2 t_n^2\Big)^{1/2} \\ &= \Big(\frac{2+\sqrt{2}}{4} t_m^2 + \sum_{n\neq m} t_n^2\Big)^{1/2} = C_m < 1,\end{aligned}$$

and thus f is not metrically differentiable at x, as the condition (1.1) is violated. □

Remark 6.4. Lemma 5.4 implies that if h is a homeomorphism of $[0,1]$ onto itself such that $f \circ h$ is metrically differentiable at all $x \in [0,1]$, then $md(f \circ h, y) = 0$ for all $y \in h^{-1}(S)$, which is a dense subset of $[0,1]$. If h could be chosen to further make $md(f \circ h, \cdot)$ continuous, then f would have to be constant. Thus, there exists no homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is metrically differentiable at all points of $[0,1]$ while $md(f \circ h, \cdot)$ is continuous.

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