UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics — Final Examination

Programmes including Mathematics — Final Examination

06 22791 MSM4P21 Level M

Linear Analysis

Summer Examinations 2012

Three Hours

Full marks will be obtained with complete answers to FOUR out of SIX questions. If more than FOUR questions are attempted then only the best FOUR will count towards the final mark. An indication of the number of marks allocated to parts of questions is shown in square brackets. No calculator is permitted in this examination.

1. Let p,q be real numbers satisfying the *strict* inequalities

$$1 < p,q < \infty$$

and the equation

$$1 = \frac{1}{p} + \frac{1}{q}.$$

(a) State and prove Young's inequality for real numbers $A,B \geq 0$. [5]

(b) Let ℓ_p be the space of complex-valued sequences $x = (x_n)_{n\in\mathbb{N}}$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$ and let

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

Show, using part (a) or otherwise, that if $x \in \ell_p$ and $y \in \ell_q$ then the pointwise product xy belongs to ℓ_1 and

$$\|xy\|_1 \leq \|x\|_p \|y\|_q.$$ [5]

(c) Show that if $1 \leq s,t < \infty$ and $w \in \ell_s$ then $w' \in \ell_t$, where

$$w'_n = |w_n|^{s/t}$$

for $n \in \mathbb{N}$. How does the norm $\|w'\|_t$ relate to the norm $\|w\|_s$? [5]

(d) Suppose now that $1 \leq p,q,r < \infty$ with

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Show that if $x \in \ell_p$ and $y \in \ell_q$ then the pointwise product xy belongs to ℓ_r and

$$\|xy\|_r \leq \|x\|_p \|y\|_q.$$ [5]

(e) Deduce from part (d), or otherwise, that if $x_1, \ldots, x_n \geq 0$ and $1 \leq r < p < \infty$, then we have

$$\left(\frac{x_1^r + x_2^r + \ldots + x_n^r}{n}\right)^{1/r} \leq \left(\frac{x_1^p + x_2^p + \ldots + x_n^p}{n}\right)^{1/p}.$$ [5]

2. Suppose X is a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and that $||\cdot||$ and $||\cdot||'$ are two norms on X.

(a) Show that if $(X, ||\cdot||)$ is complete and there exist $A, B > 0$ with

$$B||x|| \leq ||x||' \leq A||x||$$

for all $x \in X$, then $(X, ||\cdot||')$ is also complete. [5]

(b) Suppose now that we only know that there exists $A > 0$ with

$$||x||' \leq A||x||$$

for all $x \in X$. Let $i\colon (X, ||\cdot||) \to (X, ||\cdot||')$ be given by

$$i(x) = x$$

for all $x \in X$. Prove that i is a bounded linear operator, whose operator norm $||i||_{\text{op}}$ satisfies

$$||i||_{\text{op}} \leq A.$$ [5]

(c) State, without proof, the open mapping theorem and the inverse mapping theorem. [5]

(d) Suppose now that we know $(X, ||\cdot||)$ and $(X, ||\cdot||')$ are both complete and that, as in part (b), there exists $A > 0$ such that

$$||x||' \leq A||x||$$

for all $x \in X$. Show that there exists $B > 0$ such that

$$B||x|| \leq ||x||'$$

for all $x \in X$. [10]

3. Suppose that H is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and that $P\colon H \to H$ is linear, $P^2 = P$ and $\langle Px,y\rangle = \langle x,Py\rangle$ for all $x,y \in H$.

(a) State Pythagoras's theorem in H. [4]

(b) Prove that

$$||x||^2 = ||Px||^2 + ||x - Px||^2,$$

for all $x \in H$, where $||\cdot||$ is the induced norm on H. [6]

(c) Deduce that P is a bounded linear operator with operator norm $||P||_{\text{op}} \leq 1$. [5]

(d) Prove that P is the orthogonal projection onto a closed linear subspace of H. [10]

4. In this question we consider the outer Lebesgue measure λ^* defined on subsets of the real line $\mathbb{R}$.

(a) Prove that if $E \subseteq \mathbb{R}$ is countable then $\lambda^*(E) = 0$. [5]

(b) List (without proof) properties of the outer Lebesgue measure λ^* that are needed to imply that it is an outer measure on $\mathbb{R}$. [5]

(c) For each of the following statements decide whether it is true or false. If the statement is true, give a proof; if the statement is false give a short counterexample without proof.

(i) Every subset E of $\mathbb{R}$ is Lebesgue measurable; [2]

(ii) Every subset E of $\mathbb{R}$ with $\lambda^*(E) = 0$ is Lebesgue measurable. [2]

(d) Let $\mathscr{C}_1$ be the σ-algebra generated by the collection

$$\{[a,b] : -\infty < a < b < +\infty\}$$

of all closed intervals, and $\mathscr{C}_2$ be the σ-algebra generated by the collection

$$\{(a,b) : -\infty < a < b < +\infty\}$$

of all open intervals. Prove that $\mathscr{C}_1 = \mathscr{C}_2$. Give (without proof) the name of this σ-algebra. [5]

(e) Consider a collection $\mathscr{F}$ of all subsets of the real line $\mathbb{R}$ that either contain an interval of length at least 1 or their complement contains an interval of length at least 1. Is $\mathscr{F}$ a σ-algebra? Justify your answer. [6]

5. Let (Ω,Σ) be a measurable space.

(a) Assume μ_1,μ_2 are two measures on (Ω,Σ) and a_1,a_2 are two nonnegative real numbers. Prove that the formula

$$\nu = a_1\mu_1 + a_2\mu_2$$

defines a measure on (Ω,Σ). [4]

(b) Let f, g be two measurable real-valued functions on Ω. Prove that the set

$$\{x\in\Omega : f(x) > g(x)\}$$

is measurable. [6]

(c) State the Monotone Convergence Theorem for nonnegative functions. [3]

(d) Under additional assumptions that measurable functions f,g take only nonnegative values and using (c) prove that

$$\int (f+g)d\mu = \int f d\mu + \int g d\mu$$

for any measure μ on (Ω,Σ). [6]

(e) Let λ be the Lebesgue measure on the real line $\mathbb{R}$ and let a measurable set $S\subseteq\mathbb{R}^2$ be such that for λ-almost all $a\in\mathbb{R}$, the Lebesgue measure of

$$S_a = \{y\in\mathbb{R} :\ (a,y)\in S\}$$

is zero. Prove that for λ-almost all $b\in\mathbb{R}$, the set

$$S^b = \{x\in\mathbb{R} :\ (x,b)\in S\}$$

has Lebesgue measure zero. State any theorems, lemmas or propositions from the course that you use. [6]

6. Let (Ω, Σ, μ) be a measure space. For a measurable function $f : \Omega \to \mathbb{R}$ and any real number $t > 0$, denote

$$C_f(t) = \mu\{x \in \Omega : |f(x)| \geq t\}.$$

Recall that for $1 \leq p < \infty$, a measurable function $f : \Omega \to \mathbb{R}$ belongs to $L_p(\Omega)$ if $\int |f|^p \, d\mu < \infty$ and that in this case we define $\|f\|_p$ to be equal to $\left(\int |f|^p \, d\mu \right)^{1/p}$.

(a) Let f, g be two measurable real-valued functions. Prove that

$$C_{f+g}(t) \leq C_f(t/2) + C_g(t/2).$$ [3]

(b) For any integers $n, m \geq 1$ prove that

$$C_f(t) \leq \frac{1}{t^n + t^m} \int (|f|^n + |f|^m) d\mu.$$ [4]

Now let $\Omega = \mathbb{R}$ and $\mu = \lambda$ be the Lebesgue measure.

(c) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\|f - \chi_{[0,1]}\|_2 < 1/10$, where $\chi_{[0,1]}$ is the characteristic function of $[0, 1]$. [3]

(d) Assume $1 \leq p < \infty$ and $f \in L_p(\mathbb{R})$; define $f_n = f\chi_{[-n,n]}$ for each $n \geq 1$. Prove that the sequence of functions (f_n) converges to f in $L_p(\mathbb{R})$. [7]

(e) Show that $L_2(\mathbb{R}) \not\subseteq L_1(\mathbb{R})$ and $L_1(\mathbb{R}) \not\subseteq L_2(\mathbb{R})$. [8]