A27251 No Calculator

UNIVERSITY OF BIRMINGHAM

SCHOOL OF MATHEMATICS

Programmes in the School of Mathematics — Final Examination

Programmes including Mathematics — Final Examination

06 22791

MSM 4P21 Linear Analysis

Summer Examinations 2011

Time allowed: 3 hours

Full marks may be obtained with complete answers to FOUR questions out of SIX. Only the best FOUR answers will be credited. Please use separate answer booklets for Section A and Section B.
An indication of the number of marks allocated to parts of questions is shown in square brackets.
No Calculator is permitted in this examination.

Section A

1. Let $(X, ||\cdot||_X)$ be a normed vector space over $\mathbb{R}$.

   (a) Let α and β be real numbers. Prove that if $x_n, y_n, x, y \in X$ are such that the sequence of vectors x_n converges to x and the sequence of vectors y_n converges to y, then the sequence of vectors $\alpha x_n + \beta y_n$ converges to $\alpha x + \beta y$.

   [5]

   (b) Assume $(X, ||\cdot||_X)$ is a Banach space and Y is a linear subspace of X, such that $(Y, ||\cdot||_X)$ is a Banach space too. Prove that Y is closed; in other words, prove $\overline{Y}^{||\cdot||_X} = Y$.

   [5]

   (c) Let $X = \ell_1$ be the space of sequences of real numbers $(x_1, x_2, ...)$ such that

   $$||(x_1, x_2, ...)||_1 = \sum_{n=1}^{\infty} |x_n| < \infty.$$

   Let $\bar{v} = (v_n)_{n=1}^{\infty}$ be a sequence of real numbers, such that $0 < v_n \leq 1$ for every $n \geq 1$. For each $x = (x_1, x_2, ...) \in \ell_1$, define

   $$N_{\bar{v}}(x) = \sum_{n=1}^{\infty} v_n |x_n| .$$

   Prove that $N = N_{\bar{v}}$ is a well-defined function on ℓ_1 and that N is a norm on ℓ_1.

   [5]

   (d) In each of the questions (i) and (ii), give an example of a sequence $\bar{v} = (v_n)_{n=1}^{\infty}$ satisfying $0 < v_n \leq 1$ for every $n \geq 1$, such that the norms $N_{\bar{v}}$ and $||\cdot||_1$ are

   (i) equivalent; [4]

   (ii) not equivalent. [6]

   Justify your answer.

2. Let X be a vector space over $\mathbb{R}$ and let $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ be an inner product on X.

   (a) Assume $x, y \in X$ are such that $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in X$. Prove that $x = y$.

   [4]

   (b) Give a definition for the norm induced on X by the inner product $\langle \cdot, \cdot \rangle$ and state the Cauchy-Schwarz inequality.

   [4]

   (c) Using (b) or otherwise, prove the triangle inequality for the induced norm.

   [5]

   (d) Let now $X = C[0,1]$ be the space of continuous functions $f : [0,1] \to \mathbb{R}$. For $f, g \in X$, let

   $$\langle f, g \rangle = \int_0^1 t f(t) g(t) dt.$$

   Prove that this formula defines an inner product on X.

   [6]

   (e) Let $X = C[0,1]$ be the space of continuous functions $f : [0,1] \to \mathbb{R}$, with the inner product defined as in (d). Let $f_0 \in X$ be the function $f_0(t) = t$. Find a function $g \in X$, not identically 0, such that g is orthogonal to f_0. Justify your answer.

   [6]

3. Let $(X, ||\cdot||)$ be a normed vector space over $\mathbb{R}$.

(a) Give the definition of elements of its dual space X^* and the definition of the norm $||\cdot||_{X^*}$. You do not need to prove that $||\cdot||_{X^*}$ is a norm.

[4]

(b) Prove that $(X^*, ||\cdot||_{X^*})$ is a Banach space.

[5]

(c) Assume Y is a linear subspace of X, such that Y is dense in X, and $\phi \in X^*$ is such that $\phi(y) = 0$ for every $y \in Y$. Prove $\phi(x) = 0$ for every $x \in X$.

[4]

(d) Let now $X = C[0,1]$ be the space of continuous functions $f : [0,1] \to \mathbb{R}$ with the supremum norm $||\cdot||_\infty$. For each $f \in X$, define

$$Uf = g,$$

where

$$g(x) = \cos(x) \cdot f(x^2), \qquad x \in [0,1].$$

Prove that U is a bounded linear operator from $(C[0,1], ||\cdot||_\infty)$ to $(C[0,1], ||\cdot||_\infty)$.

[6]

(e) Find the norm $||U|| = \sup\{||Uf||_\infty : ||f||_\infty \leq 1\}$ of the bounded linear operator U defined in (d).

[6]

Section B

4. (a) Define the outer Lebesgue measure λ^* of subsets of $\mathbb{R}$.

[4]

(b) For each closed interval $[a,b] \subseteq \mathbb{R}$ of finite length, prove $\lambda^*([a,b]) = b - a$.

[5]

(c) Prove that any set $E \subseteq \mathbb{R}$ with $\lambda^*(E) = 0$ is Lebesgue measurable.

[5]

(d) Is the set of irrational numbers $x \in [0,1]$

(i) a Borel set?

(ii) Lebesgue measurable set?

In each case justify your answer.

[5]

(e) Let

$$f(x) = \begin{cases} e^{\sin\frac{1}{x}} & \text{if } x \neq 0, \\ 5 & \text{if } x = 0 \end{cases}$$

be defined on the whole real line. Is f Borel measurable? Justify your answer.

[6]

5. Let $(\Omega, \sum, \mu)$ be a measure space.

(a) Assume $f : \Omega \to \overline{\mathbb{R}}$ is a measurable function. Prove that the formula

$$v(E) = \mu(f^{-1}(E))$$

defines a measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the σ- algebra of Borel subsets of $\mathbb{R}$.

[6]

(b) Assume $f_n : \Omega \to \overline{\mathbb{R}}$ is a sequence of measurable functions. Prove that the function $f(x) = \inf_{n\geq 1} f_n(x)$ is measurable.

[3]

(c) State carefully the Monotone Convergence Theorem for nonnegative functions.

[4]

(d) Assume $g_n : \Omega \to \overline{\mathbb{R}}$ is a sequence of nonnegative measurable functions satisfying

$$\int g_n d\mu < \frac{1}{n^2}$$

for each $n \geq 1$. Using (c) or otherwise prove that

$$\sum_{n=1}^{\infty} g_n(x) < +\infty$$

μ-almost everywhere.

[7]

(e) Let $\mu = \lambda$ be the Lebesgue measure on $\Omega = [0, 1]$; define the sequence of functions

$$h_n(x) = n^2 \chi_{(0,\frac{1}{n})}(x),$$

where $\chi_{(0,\frac{1}{n})}$ is the characteristic function of $(0, \frac{1}{n})$.
Find $h(x) = \lim_{n\to\infty} h_n(x)$ and verify that

$$\lim_{n\to\infty} \int h_n d\lambda \neq \int h d\lambda.$$

Explain why this example does not contradict the Monotone Convergence Theorem.

[5]

6. Let $(\Omega, \sum, \mu)$ be a measure space, and for each $1 \leq p \leq +\infty$ let $\mathcal{L}_p(\Omega)$ be the collection of measurable functions $f : \Omega \to \overline{\mathbb{R}}$ such that $||f||_p < +\infty$.

   (a) Suppose $1 \leq p, q \leq +\infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. State carefully Hölder's inequality for two functions $f \in \mathcal{L}_p(\Omega)$ and $g \in \mathcal{L}_q(\Omega)$. [5]

   (b) For a sequence $f_n \in L_p(\Omega)$ define what it means that f_n converges to f in $L_p(\Omega)$. Prove that if $1 \leq p < +\infty$ and $E_n \in \sum$ is a sequence of sets such that $\mu(E_n) \to 0$, then the characteristic functions χ_{E_n} converge to the zero function in $L_p(\Omega)$. [5]

   (c) Suppose $h \in \mathcal{L}_\infty(\Omega)$. Prove that for any $\epsilon > 0$ there exists $\delta > 0$ such that if $E \subseteq \Omega$ is any measurable set of measure less than δ, then
$$\int_E |h| \, d\mu < \epsilon.$$
[5]

   (d) For this part only, suppose also that $\mu(\Omega) < +\infty$. Using (a) or otherwise, prove that
$$\mathcal{L}_3(\Omega) \subseteq \mathcal{L}_1(\Omega).$$
[5]

   (e) Give an example of a measure space $(\Omega, \sum, \mu)$ with $\mu(\Omega) = +\infty$ and $\mathcal{L}_3(\Omega) \not\subseteq \mathcal{L}_1(\Omega)$. Justify your example. [5]