

UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics

Final Examination

Programmes including Mathematics

Final Examination

06 22788 MSM3P21 Level H

Linear Analysis

06 22791 MSM4P21 Level M

Linear Analysis

Summer Examinations 2014

3 hours

Full marks will be obtained with complete answers to FOUR out of SIX questions. If more than FOUR questions are attempted then only the best FOUR will count towards the final mark. An indication of the number of marks allocated to parts of questions is shown in square brackets. No calculator is permitted in this examination.

1. (a) For $1 \leq p < \infty$ define the real vector space ℓ^p . Give a definition of $\|x\|_p$ where $x \in \ell^p$. [5]

(b) Give an example of $x = (a_n)_{n \geq 1}$ such that $a_n \rightarrow 0$ and $x \notin \ell^3$. [5]

(c) Assume that $y = (b_n)_{n \geq 1} \in \ell^3$. For each $n \geq 1$, define the vector

$$Y_n = (b_{n+1}, b_{n+2}, \dots) = (b_{n+k})_{k \geq 1}.$$

Prove that the sequence of vectors $(Y_n)_{n=1}^{\infty}$ converges to 0 in $(\ell^3, \|\cdot\|_3)$. [5]

(d) For each $z = (c_n)_{n \geq 1} \in \ell^3$, let

$$N(z) = \left(\sum_{n=1}^{\infty} \left| \frac{c_n}{n} \right|^3 \right)^{1/3}.$$

Prove that N is a norm on ℓ^3 . You may use without proof standard facts proved in the course lectures. [5]

(e) Are the norms N and $\|\cdot\|_3$, both defined on ℓ^3 , equivalent? Justify your answer. [5]

2. (a) Let X be a real inner product space and $u, v, w \in X$ be vectors of norm 1 such that $u + v + w = 0$. Prove that $\langle u, v \rangle = -\frac{1}{2}$. Give a geometric interpretation of this fact when $X = \mathbb{R}^2$ equipped with the Euclidean inner product given by

$$\langle x, y \rangle = a_1b_1 + a_2b_2$$

for each $x = (a_1, a_2)$ and $y = (b_1, b_2)$ in \mathbb{R}^2 . [5]

- (b) Give an example of an inner product on the space $Y = C[0, 1]$ of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$. Verify that the formula you give indeed defines an inner product. [5]

- (c) In $C[0, 1]$, let $f(t) = t$. In terms of the inner product defined in (b), find $g \in f^\perp$ such that $g(0) = 5$. [5]

- (d) Let $H = \ell^2$. Give an example of a linear subspace L of H and a vector $x \in H$ such that there is no $y \in L$ with $\|x - y\| = \text{dist}(x, L)$. Explain which property of L allows for such a vector x to exist in your example. [5]

- (e) Let $H = \ell^2$ and $e_n \in H$ be the vector with all coordinates equal to 0 except for the n th coordinate which is equal to 1. Let M be the 2-dimensional plane spanned by vectors $e_1 + e_2$ and $e_3 + e_4$. Let $P : H \rightarrow M$ be the orthogonal projection. Find a formula for Px in terms of the coordinates of $x = (a_n)_{n \geq 1} \in H$. You can use, without proof, any results from the course. [5]

3. (a) Let $(X, \|\cdot\|)$ be a real normed space. Define a bounded linear functional $f : X \rightarrow \mathbb{R}$. [4]

(b) Let $(X, \|\cdot\|) = (\ell^1, \|\cdot\|_1)$ and let $f \in X^*$. Prove that there exists a bounded sequence $(v_1, v_2, v_3, \dots, v_n, \dots)$ of real numbers such that

$$f(x) = f(a_1, a_2, a_3, \dots, a_n, \dots) = \sum_{n=1}^{\infty} a_n v_n$$

for all $x = (a_1, a_2, a_3, \dots, a_n, \dots) \in \ell^1$. [6]

(c) For a bounded linear functional f as in (b) give, without proof, a formula for the norm of f in terms of v_1, v_2, \dots . [3]

(d) Prove that the formula

$$T(b_1, b_2, b_3, \dots, b_n, \dots) = (b_1, \frac{b_2}{2}, \frac{b_3}{3}, \dots, \frac{b_n}{n}, \dots)$$

defines a bounded linear operator $T : (\ell^\infty, \|\cdot\|_\infty) \rightarrow (\ell^\infty, \|\cdot\|_\infty)$. [6]

(e) Show that $w = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \dots, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n+1}}, \dots)$ does not belong to $T(\ell^\infty)$, the image of the operator T defined in (d), but is one of the accumulation points of the image of T . [6]

4. (a) Let $[a, b]$ be a segment in \mathbb{R} . Give a definition of a partition P of $[a, b]$. Define the norm of a partition P . [4]

- (b) Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function.

Prove that if $P_n = (x_i^{(n)})_{i=0}^n$ is a partition of $[a, b]$ and $\|P_n\| \xrightarrow{n \rightarrow \infty} 0$ then any sequence

$$\left(\sum_{i=1}^n f(c_i) |\Delta_i^{(n)}| \right)_{n \geq 1}$$

of Riemann sums, where $c_i \in \Delta_i^{(n)} = [x_{i-1}^{(n)}, x_i^{(n)}]$ ($1 \leq i \leq n$), converges to $\int_a^b f$.

You may use without proof that f is Riemann integrable and for each $\varepsilon > 0$ there exists $\delta > 0$ such that $U(P, f) - L(P, f) < \varepsilon$ whenever $\|P\| < \delta$. [6]

- (c) Using (b) or otherwise find $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^2 e^{k/n}}{n^3}$. [6]

- (d) Give a definition of the outer Lebesgue measure λ^* on the real line \mathbb{R} . [4]

- (e) Recall that the Cantor set C is a subset of $[0, 1]$ obtained by repeatedly deleting the open middle thirds of a set of line segments, starting from $[0, 1]$, then $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, then

$$\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right],$$

and so on. Calculate the Lebesgue measure of C .

[5]

5. (a) Decide whether the following statements are true or false. In each case, justify your assertion by giving a proof if the answer is true, and by a counterexample without proof if the answer is false.

(i) The union of any two σ -algebras of subsets of \mathbb{R} is a σ -algebra. [3]

(ii) The intersection of any two σ -algebras of subsets of \mathbb{R} is a σ -algebra. [3]

(iii) For any set $A \subseteq \mathbb{R}$ there exists a σ -algebra Σ of subsets of \mathbb{R} such that $A \notin \Sigma$. [4]

- (b) Let $A, B \subseteq \mathbb{R}$ be disjoint infinite sets. For any $E \subseteq \mathbb{R}$ define

$$m^*(E) = \begin{cases} 2, & \text{if } E \text{ intersects both } A \text{ and } B, \\ 1, & \text{if } E \text{ intersects exactly one of the sets } A \text{ or } B, \\ 0, & \text{if } E \cap A = E \cap B = \emptyset. \end{cases}$$

Verify that m^* is an outer measure on \mathbb{R} . [5]

- (c) For m^* defined in (d), find an interval $[x, y]$ which is not m^* -measurable. [5]

- (d) Let (X, Σ) be a measurable space and $f, g : X \rightarrow \mathbb{R}$ be measurable functions. Prove that $\{x \in X : f(x) = g(x) + 2\} \in \Sigma$. You may use, without proof, the algebra of measurable functions. [5]

6. (a) Let (X, Σ, μ) be a measure space and $f : X \rightarrow \mathbb{R}$ be a non-negative measurable function. Define $\int f d\mu$. [5]
- (b) Assuming $f : X \rightarrow \mathbb{R}$ is a non-negative measurable function and $\int f d\mu = 0$, prove that $f(x) = 0$ at μ -almost every $x \in X$. [5]
- (c) Let $\lambda_2 = \lambda \otimes \lambda$ be the product Lebesgue measure in \mathbb{R}^2 and D be the closed disk of radius 1 centered at $(0, 0)$. Is D λ_2 -measurable? Justify your answer. [5]
- (d) Give an example of a sequence of non-negative Lebesgue measurable functions $f_n : [0, 1] \rightarrow \mathbb{R}$ in the form

$$f_n = c_n \chi(F_n), \quad \text{where } F_n \subseteq [0, 1],$$

such that

$$\|f_n\|_2 \rightarrow \infty \quad \text{and} \quad \|\sqrt{f_n}\|_2 \leq 1.$$

Justify your example. Here $\|h\|_2$ stands for the L^2 -norm of function h with respect to the Lebesgue measure on $[0, 1]$. [5]

- (e) Show that for *any* sequence of non-negative Lebesgue measurable functions $g_n : [0, 1] \rightarrow \mathbb{R}$ satisfying $\|\sqrt{g_n}\|_2 \leq 1$, the sequence $\{\|g_n\|_1\}$ is bounded. Here $\|h\|_1$ stands for the L^1 -norm of function h with respect to the Lebesgue measure on $[0, 1]$. [5]