UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics — Final Examination

Programmes involving Mathematics — Final Examination

MSM3P21 06 22788 Level H

Linear Analysis

MSM4P21 06 22791 Level M

Linear Analysis

Summer examinations 2012-13

Three Hours

Full marks will be obtained with complete answers to FOUR out of SIX questions. If more than FOUR questions are attempted then only the best FOUR will count towards the final mark.

An indication of the number of marks allocated to parts of questions is shown in square brackets. No calculator is permitted in this examination.

1. Let $X = C[0,1]$ be the vector space of all real-valued continuous functions $f : [0,1] \to \mathbb{R}$.

(a) Give a definition of the *supremum norm* $\|\cdot\|_\infty$ on the space X. [4]

(b) Prove that if the sequence $(f_n)_{n\geq 1} \subseteq X$ converges to $f \in X$ in the supremum norm, then for each $t \in [0,1]$ one necessarily has $\lim_{n\to\infty} f_n(t) = f(t)$. [4]

(c) Let $V \subseteq X$ be the following linear subspace: $V = \{f \in X \mid f(0) = f(1) = 0\}$. Using (b), or otherwise, prove that $(V, \|\cdot\|_\infty)$ is a Banach space.

You may use results from the course without proof, provided that you clearly state them. [6]

(d) Let $L \subseteq X$ be the linear subspace of X consisting of all continuously differentiable functions. For each $g \in L$, let $N(g) = \|g\|_\infty + \sup_{t\in[0,1]} |g'(t)|$.

Prove that N is a norm on L. [5]

(e) Are the norms $\|\cdot\|_\infty$ and N equivalent on L? Justify your answer. [6]

2. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space.

(a) Prove that if $x, y \in X$ and $x \perp y$, then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2,$$

where $\|\cdot\|$ is the induced norm with respect to the inner product $\langle\cdot,\cdot\rangle$. [5]

(b) Show that if $X \neq \{0\}$ then there exist $u, v \in X$ such that $\|u+v\|^2 \neq \|u\|^2 + \|v\|^2$. [5]

(c) For a subset $S \subseteq X$ give a definition of its orthogonal complement $S^\perp$. [4]

Now let $X = C[0,1]$ be the vector space of all real-valued continuous functions $f : [0,1] \to \mathbb{R}$.

(d) For $f, g \in X$ let

$$\langle f, g\rangle = \int_0^1 t^2 f(t) g(t)\, dt.$$

Verify that this formula defines an inner product on X. [6]

(e) Let $f_0 \in X$ be the function defined by $f_0(t) = t$ for all $t \in [0,1]$. Let $L \subseteq X$ be the linear subspace of X defined by

$$L = \{\lambda f_0 \mid \lambda \in \mathbb{R}\}.$$

Prove or disprove: $L^\perp = \{0\}$.

Here the orthogonal complement is taken with respect to the inner product $\langle\cdot,\cdot\rangle$ defined in (d). [5]

3. Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{R}$.

(a) Define what it means for $\phi : X \to \mathbb{R}$ to be a bounded linear functional. [4]

(b) Let $\phi : X \to \mathbb{R}$ be a linear functional that is continuous. Prove that ϕ is bounded. [6]

(c) Let $(X, \|\cdot\|) = (\ell_2, \|\cdot\|_2)$ be the space of all sequences of real numbers such that the series $\sum_{n=1}^{\infty} |a_n|^2$ converges. Let e_n be the vector in ℓ_2 with nth coordinate equal to 1 and the rest of the coordinates equal to 0.

For a bounded linear functional ϕ on X, let $b_n = \phi(e_n)$ for each $n \geq 1$. Find a value of $p \geq 1$ such that the series $\sum_{n=1}^{\infty} |b_n|^p$ necessarily converges. Justify your answer. [5]

(d) Let $Y = C[0,1]$ be the space of real-valued continuous functions equipped with the supremum norm $\|\cdot\|_\infty$, and $Z = \ell_\infty$ be the space of bounded sequences of real numbers equipped with its usual norm

$$\|(c_k)_{k\geq 1}\|_\infty = \sup_{k\geq 1} |c_k|.$$

Prove that the formula

$$Af = (f(2^{-k}))_{k\geq 1}$$

defines a bounded linear operator $A : Y \to Z$. Find $\|A\|$. [6]

(e) Give an example of an element $z = (a_k)_{k\geq 1} \in Z$ such that $z \notin A(Y)$. Justify your answer. [4]

4. (a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Define the notions of lower and upper sums and what it means for f to be Riemann integrable. [6]

(b) Give an example of a bounded function $f : [a,b] \to \mathbb{R}$ which is not Riemann integrable but is Lebesgue integrable. Prove that the function in your example is not Riemann integrable. State without proof the value of $\int_{[a,b]} f \, d\lambda$. Here λ is the Lebesgue measure. [6]

(c) Let $A \subseteq [0,1]$ be a countable set. Show that the outer Lebesgue measure of its complement in this segment, $\lambda^*([0,1] \setminus A)$, is equal to 1. [6]

(d) Let $\mu_1^*, \mu_2^*, \mu_3^*, \ldots$ be a sequence of outer measures on $\mathbb{R}$, such that $\mu_n^*(E) \leq \mu_{n+1}^*(E)$ for every $E \subseteq \mathbb{R}$. For every $E \subseteq \mathbb{R}$, define

$$\mu^*(E) = \lim_{n \to \infty} \mu_n^*(E).$$

Prove that μ^* is an outer measure on $\mathbb{R}$. [7]

5. (a) Suppose that (X,Σ,μ) is an arbitrary measure space, and $f : X \to \mathbb{R}$ is a measurable function such that $\int |f| d\mu < \infty$. Show that $\left|\int f d\mu\right| \leq \int |f| d\mu$. [5]

(b) Give a definition of the Borel sigma-algebra $\mathscr{B}$ of subsets of $\mathbb{R}$. Is the set $(0,4]$ Borel? Justify your answer. [5]

(c) Prove that every continuous function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable. [5]

Now let $X = \mathbb{R}$ and $\Sigma = \mathscr{B}$ be the sigma-algebra of Borel subsets of $\mathbb{R}$. Let us call a measure μ on $(\mathbb{R},\mathscr{B})$ *strong* if every Borel measurable function $f : X \to \mathbb{R}$ such that $\int f d\mu = 0$ satisfies $f(x) = 0$ for μ-almost every $x \in X$.

(d) Show that the Lebesgue measure λ is not strong. [5]

(e) Find an example of a strong measure and justify your answer. [5]

6. (a) Let $A \subseteq [0,1]^2$ be a Borel measurable set. For each $x \in [0,1]$ denote by A^x the set:

$$A^x = \{y \in [0,1] : (x,y) \in A\}.$$

It is given that $\lambda(A^x) < 1/4$ for each $x \in \mathbb{Q} \cap [0,1]$, and $\lambda(A^x) > 3/4$ for each $x \in [0,1] \setminus \mathbb{Q}$. Which one of the following statements is true? Justify your answer:

(i) $\lambda_2(A) \leq 1/4$,

(ii) $1/4 < \lambda_2(A) < 3/4$,

(iii) $\lambda_2(A) \geq 3/4$.

Here λ is the Lebesgue measure on $\mathbb{R}$, and and $\lambda_2 = \lambda \otimes \lambda$ is the product measure defined on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. [5]

Let (X, Σ, μ) be a measure space and $1 < p, q < \infty$ be conjugate indices, i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Recall that we write $f \in \mathscr{L}_p$ whenever f is a measurable function such that $\int |f|^p d\mu < \infty$.

(b) State Hölder's inequality for functions $f \in \mathscr{L}_p$ and $g \in \mathscr{L}_q$. [5]

(c) Recall Young's inequality: For every $s, t \geq 0$

$$st \leq \frac{s^p}{p} + \frac{t^q}{q}.$$

Use Young's inequality to prove Hölder's inequality. [5]

(d) Now let $X = \mathbb{R}$, $\mu = \lambda$ be the Lebesgue measure and functions $f, h : \mathbb{R} \to \mathbb{R}$ be such that $f, h \in \mathscr{L}_3$. Show that if $g = h^2$ then $fg \in \mathscr{L}_1$. [5]

(e) Again let $X = \mathbb{R}$ and $\mu = \lambda$ be the Lebesgue measure. Find an example of two measurable functions $f, h : \mathbb{R} \to \mathbb{R}$ with $h \in \mathscr{L}_1(\mathbb{R})$ and $f \notin \mathscr{L}_1(\mathbb{R})$, such that $f(x) \leq h(x)$ for all $x \in \mathbb{R}$. [5]