A27250 No Calculator

UNIVERSITY OF BIRMINGHAM

SCHOOL OF MATHEMATICS

Programmes in the School of Mathematics Final Examination

Programmes including Mathematics Final Examination

06 22788

MSM 3P21 Linear Analysis

Summer Examinations 2011

Time allowed: 3 hours

Full marks may be obtained with complete answers to FOUR questions out of SIX. Only the best FOUR answers will be credited. Please use separate answer booklets for Section A and Section B.
An indication of the number of marks allocated to parts of questions is shown in square brackets.
No Calculator is permitted in this examination.

Section A

1. Let $(X, ||\cdot||_X)$ be a normed vector space over $\mathbb{R}$.

   (a) Prove that if $x_n, y_n, x, y \in X$ are such that the sequence of vectors x_n converges to x and the sequence of vectors y_n converges to y, then the sequence of vectors $x_n - y_n$ converges to $x - y$.

   [5]

   (b) Assume $(X, ||\cdot||_X)$ is a Banach space and Y is a linear subspace of X, such that $(Y, ||\cdot||_X)$ is a Banach space too. Prove that Y is closed; in other words, prove $\overline{Y}^{||\cdot||_X} = Y$.

   [5]

   (c) Let $X = \ell_1$ be the space of sequences of real numbers $(x_1, x_2, ...)$ such that

   $$||(x_1, x_2, ...)||_1 = \sum_{n=1}^{\infty} |x_n| < \infty.$$

   Let $\bar{v} = (v_n)_{n=1}^{\infty}$ be a sequence of real numbers, such that $0 < v_n \leq 1$ for every $n \geq 1$. For each $x = (x_1, x_2, ...) \in \ell_1$, define

   $$N_{\bar{v}}(x) = \sum_{n=1}^{\infty} v_n |x_n| .$$

   Prove that $N = N_{\bar{v}}$ is a well-defined function on ℓ_1 and that N is a norm on ℓ_1.

   [5]

   (d) In each of the questions (i) and (ii), give an example of a sequence $\bar{v} = (v_n)_{n=1}^{\infty}$ satisfying $0 < v_n \leq 1$ for every $n \geq 1$, such that the norms $N_{\bar{v}}$ and $||\cdot||_1$ are

   (i) equivalent; [4]

   (ii) not equivalent. [6]

   Justify your answer.

2. Let X be a vector space over $\mathbb{R}$ and let $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ be an inner product on X.

(a) Assume $x, y \in X$ are such that $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in X$. Prove that $x = y$.

[4]

(b) Give a definition for the norm induced on X by the inner product $\langle \cdot, \cdot \rangle$ and state the Cauchy-Schwarz inequality.

[4]

(c) Using (b) or otherwise, prove the triangle inequality for the induced norm.

[5]

(d) Let now $X = C[0,1]$ be the space of continuous functions $f : [0,1] \to \mathbb{R}$. For $f, g \in X$, let

$$\langle f, g \rangle = \int_0^1 t f(t) g(t) dt.$$

Prove that this formula defines an inner product on X.

[6]

(e) Let $X = C[0,1]$ be the space of continuous functions $f : [0,1] \to \mathbb{R}$, with the inner product defined as in (d). Let $f_0 \in X$ be the function identically equal to 1, $f_0(t) = 1$ for every $t \in [0,1]$. Find a function $g \in X$, not identically 0, such that g is orthogonal to f_0. Justify your answer.

[6]

3. Let $(X, ||\cdot||)$ be a normed vector space over $\mathbb{R}$.

(a) Give the definition of elements of its dual space X^* and the definition of the norm $||\cdot||_{X^*}$. You do not need to prove that $||\cdot||_{X^*}$ is a norm.

[4]

(b) Prove that $(X^*, ||\cdot||_{X^*})$ is a Banach space.

[5]

(c) Assume Y is a linear subspace of X, such that Y is dense in X, and $\phi \in X^*$ is such that $\phi(y) = 0$ for every $y \in Y$. Prove $\phi(x) = 0$ for every $x \in X$.

[4]

(d) Let now $X = C[0, 1]$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ with the supremum norm $||\cdot||_\infty$. For each $f \in X$, define

$$Uf = g,$$

where

$$g(x) = f(x^2), \qquad x \in [0, 1].$$

Prove that U is a bounded linear operator from $(C[0, 1], ||\cdot||_\infty)$ to $(C[0, 1], ||\cdot||_\infty)$.

[6]

(e) Find the norm $||U|| = \sup\{||Uf||_\infty : ||f||_\infty \leq 1\}$ of the bounded linear operator U defined in (d).

[6]

Section B

4. (a) Define the outer Lebesgue measure λ^* of subsets of $\mathbb{R}$.

[4]

(b) For each closed interval $[a, b] \subseteq \mathbb{R}$ of finite length, prove $\lambda^*([a, b]) = b - a$.

[5]

(c) Prove that any set $E \subseteq \mathbb{R}$ with $\lambda^*(E) = 0$ is Lebesgue measurable.

[5]

(d) Is the set of irrational numbers $x \in [0, 1]$

(i) a Borel set?

(ii) Lebesgue measurable set?

In each case justify your answer.

[5]

(e) Let

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 5 & \text{if } x = 0 \end{cases}$$

be defined on the whole real line. Is f Borel measurable? Justify your answer.

[6]

5. Let $(\Omega, \sum, \mu)$ be a measure space.

(a) Assume $f : \Omega \to \overline{\mathbb{R}}$ is a measurable function. Prove that the formula

$$v(E) = \mu(f^{-1}(E))$$

defines a measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the σ- algebra of Borel subsets of $\mathbb{R}$.

[6]

(b) Assume $f_n : \Omega \to \overline{\mathbb{R}}$ is a sequence of measurable functions. Prove that the function $f(x) = \inf_{n\geq 1} f_n(x)$ is measurable.

[3]

(c) State the Monotone Convergence Theorem for nonnegative functions.

[4]

(d) Assume $g_n : \Omega \to \overline{\mathbb{R}}$ is a sequence of nonnegative measurable functions satisfying

$$\int g_n d\mu < \frac{1}{n^2}$$

for each $n \geq 1$. Using (c) or otherwise prove that

$$\sum_{n=1}^{\infty} g_n(x) < +\infty$$

μ-almost everywhere.

[7]

(e) Let $\mu = \lambda$ be the Lebesgue measure on $\Omega = [0, 1]$; define the sequence of functions

$$h_n(x) = n^2 \chi_{(0,\frac{1}{n})}(x),$$

where $\chi_{(0,\frac{1}{n})}$ is the characteristic function of $(0, \frac{1}{n})$.
Find $h(x) = \lim_{n\to\infty} h_n(x)$ and verify that

$$\int h_n d\lambda \longrightarrow \infty \text{ while } \int h d\lambda \neq \infty.$$

Explain why this example does not contradict the Monotone Convergence Theorem.

[5]

6. Let $(\Omega, \sum, \mu)$ be a measure space, and for each $1 \leq p \leq +\infty$ let $\mathcal{L}_p(\Omega)$ be the collection of measurable functions $f : \Omega \to \overline{\mathbb{R}}$ such that $||f||_p < +\infty$.

   (a) Suppose $1 \leq p, q \leq +\infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. State carefully Hölder's inequality for two functions $f \in \mathcal{L}_p(\Omega)$ and $g \in \mathcal{L}_q(\Omega)$.
   [5]

   (b) For a sequence $f_n \in L_p(\Omega)$ define what it means that f_n converges to f in $L_p(\Omega)$. Prove that if $1 \leq p < +\infty$ and $E_n \in \sum$ is a sequence of sets such that $\mu(E_n) \to 0$, then the characteristic functions χ_{E_n} converge to the zero function in $L_p(\Omega)$.
   [5]

   (c) Suppose $h \in \mathcal{L}_\infty(\Omega)$. Prove that for any $\epsilon > 0$ there exists $\delta > 0$ such that if $E \subseteq \Omega$ is any measurable set of measure less than δ, then
   $$\int_E |h| \, d\mu < \epsilon.$$
   [5]

   (d) For this part only, suppose also that $\mu(\Omega) < +\infty$. Using (a) or otherwise, prove that
   $$\mathcal{L}_3(\Omega) \subseteq \mathcal{L}_1(\Omega).$$
   [5]

   (e) Let $\mu = \lambda$ be the Lebesgue measure on $\Omega = \mathbb{R}$. Explain why in this case
   $$\mathcal{L}_3(\Omega) \nsubseteq \mathcal{L}_1(\Omega).$$
   Justify your answer.
   [5]