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UNIVERSITY OF BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics Final Examination

# 06 22788

MSM 3P21: Linear Analysis

Summer Examinations 2010

Time allowed: 3 hours

Full marks may be obtained with complete answers to FOUR questions out of SIX. Only the FOUR best answers will be credited.

An indication of the number of marks allocated to parts of questions is shown in square brackets.

No calculator is permitted in this examination.

**1.** Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$.

(a) Prove that if $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in $(X, \|\cdot\|)$, then the sequence of vectors $(x_n/n)_{n\in\mathbb{N}}$ converges to 0. [5]

(b) Assume that $|||\cdot||| : X \to \mathbb{R}$ is also a norm on $X$. Let $p(x) = \|x\| + |||x|||$ for every $x \in X$. Show that $p : X \to \mathbb{R}$ is a norm on the vector space $X$. [5]

(c) Give the definition of the vector space $\ell_p$ and the norm $\|\cdot\|_p$ on it, where $p \in [1, +\infty)$. [3]

(d) Assume that the sequence of vectors $(x^{(n)})_{n\in\mathbb{N}} \subset \ell_p$ converges in $(\ell_p, \|\cdot\|_p)$ to a vector $x \in \ell_p$; let $x = (x_k)_{k\in\mathbb{N}}$ and $x^{(n)} = (x_k^{(n)})_{k\in\mathbb{N}}$ for every $n \in \mathbb{N}$. Prove that for every $k \geq 1$,

$$x_k^{(n)} \underset{n\to\infty}{\longrightarrow} x_k.$$ [6]

It is known that $(\ell_p, \|\cdot\|_p)$ is a Banach space for every $p \in [1, +\infty)$. You do not have to prove this.

(e) Give an example of a linear subspace $Y$ of $\ell_1$ such that $(Y, \|\cdot\|_1)$ is not a Banach space. Justify your answer. State any theorems, lemmas or propositions from the course that you use. [6]

**2.** Let $X$ be a vector space over $\mathbb{R}$, and let $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}$ be an inner product on $X$. Let $\|\cdot\| : X \to [0,\infty)$ be the induced norm on $X$ given by $\|x\| = \langle x,x\rangle^{1/2}$.

(a) State and prove the Parallelogram Law. [5]

(b) Prove that vectors $x, y \in X$ are orthogonal if and only if $\|x+y\| = \|x-y\|$. [6]

(c) Give a geometric interpretation of the statement in (b) when $X = \mathbb{R}^2$ equipped with the Euclidean inner product given by

$$\langle x,y\rangle = x_1y_1 + x_2y_2$$

for each $x = (x_1,x_2)$ and $y = (y_1,y_2)$ in $\mathbb{R}^2$. [4]

(d) Give without proof an example of an inner product on the space of continuous functions $f : [0,1] \to \mathbb{R}$. [5]

(e) Using (a) or otherwise, prove that there cannot exist an inner product which turns the space of continuous functions on $[0,1]$ with the supremum norm into an inner product space; that is, prove that there cannot exist an inner product $\langle\cdot,\cdot\rangle : C[0,1] \times C[0,1] \to \mathbb{R}$ such that $\langle f,f\rangle = \|f\|_\infty^2$ for each continuous function $f : [0,1] \to \mathbb{R}$. [5]

**3.** Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$.

(a) Define what it means that $\phi : X \to \mathbb{R}$ is a bounded linear functional on $X$. Prove that a bounded linear functional on $X$ is continuous. [7]

(b) State an equivalent condition for a linear functional $\phi : X \to \mathbb{R}$ to be continuous in terms of its kernel. You do not need to prove the statement. [3]

(c) Prove or disprove: If $\phi : X \to \mathbb{R}$ is a linear functional, then the image $\phi(X)$ of $\phi$ is either $\{0\}$ or $\mathbb{R}$. [5]

(d) Assume that $(X, \|\cdot\|) = (\ell_1, \|\cdot\|_1)$ is the space of all sequences of real numbers with a convergent sum of absolute values; let $y = (y_k)_{k\in\mathbb{N}}$ be an element of $\ell_\infty$. Show that the formula

$$\phi(x) = \sum_{k=1}^{\infty} x_k y_k,$$

where $x = (x_k)_{k\in\mathbb{N}} \in \ell_1$, defines a bounded linear functional $\phi : \ell_1 \to \mathbb{R}$. [6]

(e) Find the norm of the bounded linear functional from (d). [4]

**4.** (a) Define the term *outer measure.* If $\mu^*$ is an outer measure on $\Omega$, what subsets of $\Omega$ are called $\mu^*$-measurable? [5]

(b) Define the outer Lebesgue measure on the real line $\mathbb{R}$ as

$$\lambda^*(E) = \inf\Big\{\sum_{i=1}^{\infty} |I_j| : I_j \text{ are open intervals and } \bigcup_{j=1}^{\infty} I_j \supset E\Big\},$$

where $|I|$ denotes the length of an interval $I$. Prove that $\lambda^*$ is an outer measure. [6]

(c) Give an example of a set of positive outer Lebesgue measure that does not contain any nonempty open intervals. Justify your answer. You may use without proof the fact that the outer Lebesgue measure of an interval is equal to its length. [5]

(d) For every $E \subset \mathbb{R}$ let

$$\mu^* E = \begin{cases} 0, & \text{if } E \cap [0,2] = \varnothing, \\ 1, & \text{if } E \cap [0,2] \neq \varnothing. \end{cases}$$

Prove that $\mu^*$ is an outer measure on $\mathbb{R}$. [5]

(e) In the notation of part (d), decide whether the set $[0,1]$ is $\mu^*$-measurable. Justify your answer. [4]

**5.** Let $(\Omega, \Sigma, \mu)$ be a measure space.

(a) Assume that $f, g : \Omega \to \mathbb{R}$ are measurable functions; prove that $\max(f, g)$ is also a measurable function. [4]

(b) Prove that if the integral $\int f d\mu$ of a non-negative function $f$ is equal to zero, then the function $f$ equals zero $\mu$-almost everywhere. [5]

(c) State carefully Fatou's Lemma and the Dominated Convergence Theorem. [5]

(d) Prove the Dominated Convergence Theorem. [6]

(e) Let $\mu = \lambda$ be the Lebesgue measure on $\Omega = \mathbb{R}$; define the sequence of functions

$$f_n(x) = (1/n)\chi_{[0,n]}(x) : \mathbb{R} \to \mathbb{R},$$

where $\chi_{[0,n]}$ is the characteristic function of $[0, n]$.

Show that $f_n(x) \underset{n\to\infty}{\longrightarrow} 0$ for every $x \in \mathbb{R}$ and that $\int f_n d\lambda = 1$ for every $n \geq 1$.

Explain why this example does not contradict the Dominated Convergence Theorem. [5]

**6.** Let $(\Omega, \Sigma, \mu)$ be a measure space and suppose that $1 \leq p < \infty$. Let $\mathscr{L}_p(\mu)$ be the collection of all measurable functions $f : \Omega \to \overline{\mathbb{R}}$ such that the function $|f(x)|^p$ is integrable.

(a) Explain how $\mathscr{L}_p(\mu)$ can be made into a normed space $L_p(\mu)$. Give the definition of the norm on $L_p(\mu)$. [5]

(b) Assume that $f \in L_p(\mu)$ is such that $|f(x)| \leq 1$ for all $x \in \Omega$. Show that $f \in L_{p'}(\mu)$ for every $p' \in (p, +\infty)$. [5]

(c) Assume that $f, g \in L_{10}(\mu)$. Prove that their product $fg$ belongs to $L_5(\mu)$. State any theorems, lemmas or propositions from the course that you use. [5]

(d) Assume that $\Omega = \mathbb{R}$, $\mu = \lambda$ is the Lebesgue measure, and $\Sigma$ is the $\sigma$-algebra of Lebesgue measurable sets. Let $f = \chi_{[0,1]} : \mathbb{R} \to \mathbb{R}$ be the characteristic function of $[0,1]$. Construct a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n$ converges to $f$ in $L_1(\lambda)$. Justify your construction. [5]

(e) For each $n > 1$, let

$$g_n(x) = \frac{1}{x}\chi_{(1,n)}(x) : \mathbb{R} \to \mathbb{R},$$

where $\chi_{(1,n)}$ is the characteristic function of $(1,n)$.

Show that $g_n \in L_1(\lambda)$ for every $n > 1$, and that for every $n > 1$ there exists $m > n$ such that $\|g_n - g_m\|_1 > 1$. Here $\lambda$ is the Lebesgue measure on $\mathbb{R}$ as in (d). [5]



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