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UNIVERSITY OF
BIRMINGHAM

School of Mathematics

Programmes in the School of Mathematics Final Examination

Programmes including Mathematics Final Examination

06 16764

MSM 4P13: Topics in Analysis LM

Summer 2008
Time allowed: 3 Hours

Full marks may be obtained with complete answers to FOUR questions out of SIX. Credit will be given for the best FOUR answers only.
An indication of marks allocated to parts of questions is shown in square brackets.
No calculator is permitted in this examination.

1. Let X be a vector space over $\mathbb{C}$ and let $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{C}$ be an inner product on X.

(i) State and prove the Cauchy-Schwarz inequality on X. [9]

(ii) Let $\|\cdot\| : X \to [0,\infty)$ be the function on X given by $\|x\| := \langle x,x\rangle^{1/2}$. Using (a), or otherwise, show that for all $x,y \in X$,

$$\|x+y\| \leq \|x\| + \|y\|$$

and deduce that $(X, \|\cdot\|)$ is a normed vector space. [7]

(iii) Prove that for each $x,y \in X$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$ [4]

(iv) Let $d \in \{2,3,\ldots\}$ and for each $x = (x_1,x_2,\ldots,x_d) \in \mathbb{C}^d$ define

$$\|x\|_\infty := \max\{|x_j| : j = 1,2,\ldots,d\}.$$

Show that there does not exist an inner product $\langle\cdot,\cdot\rangle : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}$ such that, for all $x \in \mathbb{C}^d$,

$$\|x\|_\infty = \langle x,x\rangle^{1/2}.$$ [5]

2. Let X be the complex Hilbert space $L^2([-\pi,\pi])$ equipped with the inner product $\langle\cdot,\cdot\rangle$ given by

$$\langle x,y\rangle := \int_{-\pi}^{\pi} x(t)\overline{y(t)}\,dt.$$

(a) Suppose that $d \in \mathbb{N}$ and assume that $\{e_1, e_2, \ldots, e_d\}$ is an orthonormal set in X. Show that, for each $x \in X$ and each $(\alpha_1,\ldots,\alpha_d) \in \mathbb{C}^d$,

$$\left\| x - \sum_{n=1}^{d} \alpha_n e_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{d} |\langle x, e_n\rangle|^2 + \sum_{n=1}^{d} |\alpha_n - \langle x, e_n\rangle|^2.$$ [6]

(b) Deduce that, for each $x \in X$,

$$\inf\left\{ \left\| x - \sum_{n=1}^{d} \alpha_n e_n \right\|^2 : (\alpha_1,\alpha_2,\ldots,\alpha_d) \in \mathbb{C}^d \right\} = \|x\|^2 - \sum_{n=1}^{d} |\langle x, e_n\rangle|^2$$

and that this infimum is attained when $\alpha_n = \langle x, e_n\rangle$ for each $n = 1, 2, \ldots, d$. [3]

(c) Prove that if $e_n \in X$ is given by

$$e_n(t) := \frac{1}{\sqrt{\pi}} \sin(nt)$$

then $\{e_1,\ldots,e_d\}$ is an orthonormal set in X. [5]

(d) Show that

$$\inf\left\{ \int_{-\pi}^{\pi} \left| t + t^6 - \sum_{n=1}^{d} \alpha_n \sin(nt) \right|^2 dt : (\alpha_1,\alpha_2,\ldots,\alpha_d) \in \mathbb{C}^d \right\}$$

equals

$$\frac{2}{3}\pi^3 + \frac{2}{13}\pi^{13} - 4\pi \sum_{n=1}^{d} \frac{1}{n^2}.$$ [9]

(e) Deduce that

$$\inf\left\{ \int_{-\pi}^{\pi} \left| t + t^6 - \sum_{n=1}^{\infty} \alpha_n \sin(nt) \right|^2 dt : \alpha_n \in \mathbb{C} \text{ and } \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty \right\} = \frac{2}{13}\pi^{13}.$$

(You may use the fact that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.) [2]

3. (a) State the Riesz-Fréchet theorem without proof. [3]

(b) Let X,Y be complex Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$ respectively. For any $A\in\mathscr{L}(X,Y)$, show that there exists a unique operator $A^*\in\mathscr{L}(Y,X)$ such that

$$\langle Ax,y\rangle_Y=\langle x,A^*y\rangle_X,\quad \text{for all } x\in X \text{ and all } y\in Y,$$

and $\|A^*\|=\|A\|$. [12]

(c) Let $a,b\in\mathbb{R}$ and let X be the complex Hilbert space $L^2([a,b])$ equipped with the inner product $\langle\cdot,\cdot\rangle$ given by

$$\langle x,y\rangle:=\int_a^b x(t)\overline{y(t)}\,dt.$$

For a real-valued function $y\in C([a,b])$, find the spectrum of the multiplication operator $A:X\to X$ defined by

$$(Ax)(t)=y(t)x(t),\quad x\in X.$$

[You may assume the fact that for a symmetric operator $S\in\mathscr{L}(X)$, where X is a Hilbert space, the inequality $\|(S-\lambda)x\|\geq c\|x\|$ for all $x\in X$ with a positive constant c implies that $\lambda\in\rho(S)$, the resolvent set of S.] [10]

4. Every integrable function $f\colon [0,1] \to \mathbb{C}$ has a Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi inx}$, where

$$\hat{f}(n) = \int_0^1 f(x)\, e^{-2\pi inx}\, dx.$$

Further, f is zero (almost everywhere) if and only if $\hat{f}\colon \mathbb{Z} \to \mathbb{C}$ is zero.

The partial sum $D_N(f)$ is defined by

$$D_N(f)(x) = \sum_{|n|\le N} \hat{f}(n)\, e^{2\pi inx} \qquad \forall x \in [0,1].$$

The map D_N is linear on $L^1([0,1])$, and $D_N(f)$ is a smooth bounded function on $[0,1]$.

Fix p in $[1,\infty]$, and denote by $L^p([0,1])$ the usual Lebesgue space on $[0,1]$. A sequence $(m_n)_{n\in\mathbb{Z}}$ is called a Fourier L^p multiplier if $\sum_{n\in\mathbb{Z}} m_n \hat{f}(n)\, e^{2\pi inx}$ is the Fourier series of an $L^p([0,1])$ function $T_m f$ whenever $\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi inx}$ is the Fourier series of an $L^p([0,1])$ function f. The mapping $T_m\colon f \mapsto T_m f$ is linear.

(a) Show that

$$|\hat{f}(n)| \le \|f\|_p \qquad \forall n \in \mathbb{Z} \quad \forall f \in L^p([0,1]).$$ [5]

(b) Deduce that

$$\|D_N(f)\|_p \le (2N+1)\, \|f\|_p \qquad \forall N \in \mathbb{N} \quad \forall f \in L^p([0,1]).$$ [5]

(c) Suppose that $D_N(f) \to f$ in L^p-norm as $N \to \infty$ for all $f \in L^p([0,1])$. Use the uniform boundedness principle to show that there exists a constant C_p such that

$$\|D_N(f)\|_p \le C_p\, \|f\|_p \qquad \forall N \in \mathbb{N} \quad \forall f \in L^p([0,1]).$$ [5]

(d) Suppose that $(f_k)_{k\in\mathbb{N}}$ is a sequence of $L^p([0,1])$ functions such that $f_k \to f$ in $L^p([0,1])$ norm as $k \to \infty$. Show that $\hat{f}_k(n) \to \hat{f}(n)$ as $k \to \infty$ for all $n \in \mathbb{Z}$. [5]

(e) Use the closed graph theorem to show that the mapping T_m is bounded on $L^p([0,1])$. [5]

5. Suppose that A is a complex Banach algebra with identity e and norm $\|\cdot\|$. Denote the circle $\{z \in \mathbb{C} : |z| = R\}$ by Γ_R. Fix $a \in A$ and define the linear map $T\colon H(\overline{B}(0,R)) \to A$ by the formula

$$T\colon f \mapsto \frac{1}{2\pi i}\int_{\Gamma_R} f(z)\,(ze-a)^{-1}\,dz,$$

where $R > \|a\|$; here $H(\overline{B}(0,R))$ denotes the algebra of (complex) functions f that are holomorphic in some neighbourhood of $\overline{B}(0,R)$, which may depend on the function f.

(a) Suppose that $a \in A$. Show that, if $z \in \mathbb{C}$ and $|z| > \|a\|$, then

$$\sum_{n=0}^{\infty} \|z^{-n-1}a^n\| \leq \frac{1}{|z| - \|a\|},$$

and hence that

$$(ze-a)^{-1} = \sum_{n=0}^{\infty} z^{-n-1}a^n,$$

where the sum converges in norm in A. [5]

(b) Show that, for all $f \in H(\overline{B}(0,R))$,

$$\|Tf\| \leq \frac{R}{R - \|a\|}\max\{|f(z)| : |z| = R\}. \qquad [5]$$

(c) Show that, for all complex polynomials p,

$$Tp = p(a). \qquad [5]$$

(d) Show that, given f and g in $H(\overline{B}(0,R))$, there exist sequences of polynomials p_N and q_N such that $p_N \to f$ and $q_N \to g$ as $N \to \infty$, uniformly in $\overline{B}(0,R)$. [5]

(e) Deduce that the linear map T satisfies $T(fg) = T(f)\,T(g)$, for all $f, g \in H(\overline{B}(0,R))$. [5]

6. In this question, take all functions to be real-valued, and equip the real Hilbert space $L^2([0,1])$ with the inner product $\langle \cdot,\cdot \rangle$ given by

$$\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx.$$

Suppose that $k\colon [0,1]\times[0,1]\to\mathbb{R}$ is continuous. For $f\in L^2([0,1])$, define the function $Kf\colon [0,1]\to\mathbb{R}$ by

$$Kf(x) = \int_0^1 k(x,y)f(y)\,dy \qquad \forall x\in[0,1].$$

(a) Show that

$$\|Kf\|_2 \le \max\{|k(x,y)| : (x,y)\in[0,1]\times[0,1]\}\,\|f\|_2 \qquad \forall f\in L^2([0,1]).$$ [4]

(b) It is possible to approximate k uniformly on $[0,1]\times[0,1]$ by polynomials. Give the name of a theoretical justification for this fact. [1]

(c) Show that the linear operator $K\colon L^2([0,1])\to L^2([0,1])$ is compact. [6]

Now define the continuous function $k\colon [0,1]\times[0,1]\to\mathbb{R}$ by

$$k(x,y) = \begin{cases} x(1-y) & \text{if } 0\le x\le y\le 1 \\ y(1-x) & \text{if } 0\le y< x\le 1. \end{cases}$$

(d) Show that $\langle Kf,g\rangle = \langle Kg,f\rangle$ for all $f,g\in L^2([0,1])$. [1]

(e) On the basis of the results of parts (c) and (d), what can you say about the eigenvalues and eigenfunctions of K? [3]

(f) Show that if $f\neq 0$, $\lambda\neq 0$ and $Kf=\lambda f$, then $f(0)=f(1)=1$, and

$$f'' = \frac{-1}{\lambda}f.$$ [5]

(g) On the basis of the results of part (f), what can you say about the eigenvalues and eigenfunctions of K? [5]