Chapter 1. HAHN-BANACH THEOREM

Here we prove a Hahn-Banach Theorem formulated as Theorem 6.8 in the first part of *Topics in Analysis.*

Theorem 1.1 (Hahn-Banach). *Let X be a vector space over $\mathbb{R}$, Y be a linear subspace of X and $q : X \to \mathbb{R}$ be a sublinear functional on X, that is:*

(a) $q(x + x') \le q(x) + q(x')$ *for all* $x, x' \in X$*;*
(b) $q(\lambda x) = \lambda q(x)$ *for all* $x \in X$ *and* $\lambda \ge 0$.

If $f : Y \to \mathbb{R}$ is a linear functional such that $f(x) \le q(x)$ for all $x \in Y$, then there is a linear functional $F : X \to \mathbb{R}$ such that $F|_Y = f$ and $F(x) \le q(x)$ for all $x \in X$.

Corollary 1.2 (Theorem 6.8). *Let X be a normed space, Y be a linear subspace of X and $\psi \in Y^*$. Then there exists $\varphi \in X^*$ such that φ and ψ coincide on Y and $\|\varphi\|_{X^*} = \|\psi\|_{Y^*}$.*

Proof. Apply Theorem 1.1 with $q(x) = \|\psi\|_{Y^*}\|x\|$ and $f = \psi$. Let $\varphi = F$. $\square$

Exercise 1 (Problem Sheet 0). Prove the statement of Corollary 1.2 in case X is a complex normed space.

Lemma 1.3 (Proof of a particular case of Theorem 1.1). *In the hypothesis of Theorem 1.1 assume, in addition, that the dimension of X/Y is equal to* 1. *Then the conclusion of Theorem 1.1 is valid.*

Proof. Fix $x_0 \in X \setminus Y$ so that $X = \mathbb{R}x_0 + Y = \{tx_0 + y \colon t \in \mathbb{R}, y \in Y\}$.

Fix some number a_0; let $F : X \to \mathbb{R}$ be a linear functional such that $F(x_0) = a_0$ and $F|_Y = f$. Then $F(tx_0 + y) = ta_0 + f(y)$ for any $t \in \mathbb{R}$ and $y \in Y$. When $t > 0$, the restriction

$$ta_0 + f(y) \le q(tx_0 + y) \tag{1.1}$$

is equivalent to

$$a_0 \le \frac{-f(y) + q(tx_0 + y)}{t} = -f(y/t) + q(x_0 + y/t) = -f(y') + q(x_0 + y')$$

for every $y' \in Y$. In the same way when $t < 0$, the restriction (1.1) is equivalent to

$$a_0 \ge f(z') - q(-x_0 + z')$$

for all $z' \in Y$.

We now show that $f(z') - q(-x_0 + z') \leq -f(y') + q(x_0 + y')$ for all $y', z' \in Y$ and then choose a_0 satisfying

$$f(z') - q(-x_0 + z') \leq a_0 \leq -f(y') + q(x_0 + y')$$

for all $y', z' \in Y$. This would imply (1.1) for all $t \in \mathbb{R}$ and therefore the conclusion of the Hahn-Banach Theorem holds.

Note that $f(z') - q(-x_0 + z') \leq -f(y') + q(x_0 + y')$ is equivalent to $f(y' + z') \leq q(x_0 + y') + q(-x_0 + z')$. But

$$q(x_0 + y') + q(-x_0 + z') \geq q(y' + z') \geq f(y' + z').$$

$\square$

In order to prove the general case of Theorem 1.1, we need Zorn's lemma. We will define the objects needed and will state the lemma.

Definition. Assume P is a set. A *partial order* on P is a binary relation $\preceq$ such that

(1) if $a \in P$, then $a \preceq a$;
(2) if $a, b, c \in P$, $a \preceq b$ and $b \preceq c$, then $a \preceq c$;
(3) if $a, b \in P$, $a \preceq b$ and $b \preceq a$, then $a = b$.

The pair $(P, \preceq)$ is called then a *partially ordered set* or a *poset.* A subset C of P is a *chain* or a *totally ordered set* if for all $a, b \in C$ we have $a \preceq b$ or $b \preceq a$. An element $m \in P$ is a *maximal element* of P if $m \preceq a$ implies $a = m$. An element $b \in P$ is called an *upper bound* for a set $S \subseteq P$ if $s \preceq b$ for all $s \in S$.

Theorem (Zorn's Lemma). *If every chain in a non-empty partially ordered set P has a supremum in P, then P has at least one maximal element.*

We now return to the proof of Hahn-Banach Theorem.

Proof of the general case of Theorem 1.1. Let $\mathcal{F}$ be the collection of all pairs (Z, g) where $Z \supseteq Y$ is a linear subspace of X, $g : Z \to \mathbb{R}$ is a linear functional with $g|Y = f$ and $g \leq q$ on Z. The set $\mathcal{F}$ is not empty since $(Y, f) \in \mathcal{F}$. We define a partial order on $\mathcal{F}$ by

$$(Z_1, g_1) \preceq (Z_2, g_2) \qquad \text{if} \qquad Z_1 \subseteq Z_2 \quad \text{and} \quad g_2|_{Z_1} = g_1.$$

This makes $(\mathcal{F}, \preceq)$ a partially ordered set. Suppose $\mathcal{C} = \{(Z_\alpha, g_\alpha), \alpha \in A\}$ is a chain in $\mathcal{F}$. If $Z = \bigcup_{\alpha \in A} Z_\alpha$ then Z is a linear subspace of X. Take any $x \in Z$, find $\alpha \in A$ such that $x \in Z_\alpha$, define $g(x) = g_\alpha(x)$. Then $g : Z \to \mathbb{R}$ is well-defined, it is a linear functional and and $g \leq q$ on Z. Thus $(Z, g) \in \mathcal{F}$ and is an upper bound for $\mathcal{C}$. By Zorn's Lemma, $\mathcal{F}$ has a maximal element (X', F). By Lemma 1.3 we have $X' = X$. $\square$

Chapter 2. BAIRE CATEGORY THEOREM AND ITS CONSEQUENCES

Section 1. **Baire Category Theorem.**

Reminder. A topological space is a space (set) X with *topology.* The family of *open* subsets of X defines the topology; a subset is *closed* if its complement is open. The *closure* of S is the smallest (w.r.t. inclusion) closed set containing S; equivalently it is the intersection of all closed sets containing S. *Interior* of set S is the biggest (w.r.t. inclusion) open set contained in S; equivalently it is the union of all open subsets of S.

A pair (X, ρ) is called a *metric space* if X is a set and $\rho : X \times X \to \mathbb{R}$ is a function such that

1. $\rho(x, y) \geq 0$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;
2. $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
3. $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ for all $x, y, z \in X$.

Open and closed balls:

$$\begin{aligned} \text{open: } & B(x, r) = \{y \in X \colon \rho(x, y) < r\}; \\ \text{closed: } & \overline{B}(x, r) = \{y \in X \colon \rho(x, y) \leq r\}. \end{aligned}$$

A set S is open if and only if for every $x \in S$ there exists $r_x > 0$ such that $B(x, r_x)$, the ball centered at x of radius r_x, is a subset of S.

A metric is called *complete* if every Cauchy sequence (x_n) converges in X. A subset $S \subseteq X$ is called *dense* in X if its closure $\overline{S}$ is equal to X, equivalently, if every point from X can be approximated by points from S, i.e. for every $x \in X$ there exists a sequence $(x_n) \subseteq S$ such that $\lim x_n = x$.

A vector space X over field $\mathbb{F}$ (equal to $\mathbb{R}$ or $\mathbb{C}$) is called a *normed space* if there is a map $\| \cdot \| : X \to \mathbb{R}$ such that

1. $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
2. $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{F}$ and $x \in X$;
3. $\|x\| + \|y\| \geq \|x + y\|$ for all $x, y \in X$.

Any normed space is a metric space; $\rho(x, y) = \|x - y\|$.

A normed space X is called a *Banach space* if it is complete, i.e. if every Cauchy sequence $(x_n) \subseteq X$ is convergent. Examples of Banach spaces:

1. finite-dimensional spaces;
2. ℓ^p, $1 \leq p \leq \infty$; c_0;
3. L^p, $1 \leq p \leq \infty$;

(4) $C(K)$, K is a compact;

(5) the space $\mathcal{B}(X, Y)$ of bounded linear operators from a normed space X into a Banach space Y;

(6) if X is a normed space, then the space of continuous linear functionals X^* is a Banach space;

(7) direct sum of two Banach spaces X and Y:

$$X \oplus Y = \{(x, y)\colon x \in X, y \in Y\}, \qquad \|(x, y)\| = \|x\| + \|y\|.$$

A subset S of a vector space is called *convex* if for any pair of points $x, y \in S$ and any $t \in [0, 1]$ the point $tx + (1 - t)y$ belongs to S.

Theorem 2.1 (Baire Category Theorem)**.** *Let X be a complete metric space, let $G_1, G_2, \ldots$ be open dense subsets of X. Then their intersection $\bigcap G_n$ is dense in X.*

Lemma 2.2 (Principle of nested balls)**.** *Let X be a complete metric space, $\{\overline{B}(x_i, r_i)\}$ be a sequence of closed balls such that*

$$\begin{cases} \overline{B}(x_i, r_i) \supseteq \overline{B}(x_{i+1}, r_{i+1}) \text{ for every } i \geq 1; \\ r_i \to 0 \end{cases} \tag{2.1}$$

Then the intersection $\bigcap \overline{B}(x_i, r_i)$ is not empty and consists of a unique point.

Proof. We prove first that the intersection $\bigcap \overline{B}(x_i, r_i)$ cannot contain two distinct points. Assume towards the contradiction that $e, e' \in \bigcap \overline{B}(x_i, r_i)$. Then $e, e' \in \overline{B}(x_i, r_i)$ for every $i \geq 1$, which implies $\operatorname{dist}(e, e') \leq 2r_i$ for every i. Since $r_i \to 0$, we conclude $\operatorname{dist}(e, e') = 0$, thus $e = e'$.

Now we show that the intersection $\bigcap \overline{B}(x_i, r_i)$ is not empty. We first prove that the sequence (x_i) is Cauchy. Indeed, let $\varepsilon > 0$. Choose N such that $r_i < \varepsilon/2$ for all $i > N$. If $n, m > N$, then

$$\begin{cases} \overline{B}(x_n, r_n) \subseteq \overline{B}(x_N, r_N) \\ \overline{B}(x_m, r_m) \subseteq \overline{B}(x_N, r_N) \end{cases} \quad \text{which implies} \quad \begin{cases} \operatorname{dist}(x_n, x_N) \leq r_N < \varepsilon/2 \\ \operatorname{dist}(x_m, x_N) \leq r_N < \varepsilon/2 \end{cases}$$

which implies $\operatorname{dist}(x_n, x_m) < \varepsilon$.

Since X is complete, every Cauchy sequence converges, denote the limit of (x_i) by c. If $n \geq i$, then $x_n \in \overline{B}(x_n, r_n) \subseteq \overline{B}(x_i, r_i)$. Now let i be fixed and let $n \to \infty$. Since every x_n belongs to $\overline{B}(x_i, r_i)$ which is a closed set, we conclude $c = \lim x_n \in \overline{B}(x_i, r_i)$. Since this is true for every i, we get $c \in \bigcap \overline{B}(x_i, r_i)$. $\square$

Exercise 1 (Problem sheet 1)**.** Prove the following:

(1) The Principle of nested balls Lemma 2.2 is in fact a characteristic property of completeness of metric spaces:
Prove that if X is a metric space such that any nested sequence of closed balls $\overline{B}(x_i, r_i)$ such that $r_i \to 0$ has a non-empty intersection, then X is complete.

(2) If $r_i \not\to 0$ then there exists a complete space X and a sequence of closed balls $\{\overline{B}(x_i, r_i)\}$ such that $\overline{B}(x_i, r_i) \supseteq \overline{B}(x_{i+1}, r_{i+1})$ for every $i \geq 1$ such that the intersection $\bigcap \overline{B}(x_i, r_i)$ is empty.
(Hint: consider $X = \mathbb{N}$ and introduce metric, in which infinite rays $\mathbb{N} \cap [n, +\infty)$ are closed balls of <u>finite</u> radius.)

(3) If X is a Banach space and a sequence of closed balls $\{\overline{B}(x_i, r_i)\}$ is such that for every $i \geq 1$, $\overline{B}(x_i, r_i) \supseteq \overline{B}(x_{i+1}, r_{i+1})$, then the intersection $\bigcap \overline{B}(x_i, r_i)$ is not empty even if $r_i \not\to 0$.

Proof of Theorem 2.1 (Baire Category Theorem). Choose any $x_0 \in X$, $r_0 > 0$. We now show that the intersection $B(x_0, r_0) \cap (\bigcap G_n)$ is not empty.

Since G_1 is dense, there exists $x_1 \in G_1 \cap B(x_0, r_0)$. Since $G_1 \cap B(x_0, r_0)$ is open, there exists $\rho_1 > 0$ such that $B(x_1, \rho_1) \subseteq G_1 \cap B(x_0, r_0)$. Let $r_1 = \rho_1/2$, then $\overline{B}(x_1, r_1) \subseteq B(x_1, \rho_1) \subseteq G_1 \cap B(x_0, r_0)$.

Assume $x_1, \ldots, x_n$ and $r_1, \ldots, r_n$ are constructed and

$$\begin{aligned} &\overline{B}(x_i, r_i) \subseteq G_i \text{ for every } 1 \leq i \leq n; \\ &\overline{B}(x_i, r_i) \supseteq \overline{B}(x_{i+1}, r_{i+1}) \text{ for every } 0 \leq i \leq n-1; \\ &r_{i+1} \in (0, r_i/2) \text{ for every } 0 \leq i \leq n-1. \end{aligned}$$

Since G_{n+1} is dense, there exists $x_{n+1} \in G_{n+1} \cap B(x_n, r_n)$. Since $G_{n+1} \cap B(x_n, r_n)$ is open, we can choose $\rho_{n+1} \in (0, r_n)$ such that $B(x_{n+1}, \rho_{n+1}) \subseteq G_{n+1} \cap B(x_n, r_n)$. Let $r_{n+1} = \rho_{n+1}/2$, then $r_{n+1} \in (0, r_n/2)$ and $\overline{B}(x_{n+1}, r_{n+1}) \subset B(x_{n+1}, \rho_{n+1}) \subseteq G_{n+1} \cap B(x_n, r_n)$.

Therefore the sequence of closed balls $\overline{B}(x_i, r_i)$ satisfies (2.1). Then by Lemma 2.2 the intersection $\bigcap \overline{B}(x_i, r_i)$ is a single point c. Since $c \in \overline{B}(x_i, r_i) \subseteq G_i$ for every i, we conclude that $c \in \bigcap G_i$. Recalling $c \in B(x_0, r_0)$, we get $B(x_0, r_0) \cap (\bigcap G_i)$ is not empty. Since $x_0 \in X$ and $r > 0$ are arbitrary, we conclude that $\bigcap G_i$ is dense in X. □

Theorem 2.3. *Let X be a complete metric space, $F_1, F_2, \ldots$ be closed subsets of X such that their union $\bigcup F_i$ is equal to X. Then at least one of F_i's has non-empty interior.*

Proof. Let $G_i = X \setminus F_i$ for each $i \geq 1$. Each G_i is an open set; if each G_i is dense in X, then by Theorem 2.1 the intersection $\bigcap G_i$ is not empty. This means that the union $\bigcup F_i$ is not whole X (explain why). Contradiction. Therefore there exists n such that G_n is not dense. In other words, there is an open ball $B(x, r)$ which does not intersect G_n. But $G_n \cap B(x, r) = \emptyset$ implies $F_n \supseteq B(x, r)$ (explain why). The latter means that the interior of F_n contains $B(x, r)$ (see definition of interior), and so interior of F_n is not empty. ☐

Definition. Let X be a topological space, $A \subseteq X$. We say that the set A

- is *nowhere dense* if its closure $\overline{A}$ has empty interior;
- is *of 1st category* if there exist nowhere dense sets $A_n \subseteq X$ such that $A \subseteq \bigcup A_n$;
- is *of 2nd category* if it is not of 1st category.

Exercise 2 (Problem sheet 1)**.** Let X be a topological space, $A, B, C, D \subseteq X$.

1. If A is nowhere dense, then its closure $\overline{A}$ is nowhere dense;
2. If A is nowhere dense, then its complement $X \setminus A$ is *everywhere dense*, which means it is dense in the whole space;
3. If $A \subseteq B$ and A is of 2nd category, then B is of 2nd category;
4. If $C \supseteq D$ and C is of 1st category, then D is of 1st category.

Theorem 2.4. *Any complete metric space is of 2nd category.*

Proof. Assume X is complete and is of 1st category. Then there exist nowhere dense sets A_n such that $\bigcup A_n = X$. Let $F_n = \overline{A}_n$. Then each F_n is closed and $\bigcup F_n = X$. Since X is complete, by Theorem 2.3 there exists n such that F_n has non-empty interior. This contradicts that each A_n is nowhere dense which means that the interior of $\overline{A}_n$ is empty. ☐

Exercise 3 (Problem sheet 1)**.** Determine whether the following subsets of $[0, 1]$ with the standard topology are of 1st or of 2nd category: $[0, 1]$, $\mathbb{Q} \cap [0, 1]$, $[0, 1] \setminus \mathbb{Q}$, Cantor set.

(In order to define a Cantor set, one starts by deleting the open middle third $(1/3, 2/3)$ from the interval $[0, 1]$, leaving two line segments: $[0, 1/3] \cup [2/3, 1]$. Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $[0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. This process is continued to infinity, the Cantor set contains all points in the interval $[0,1]$ that are not deleted at any step in this infinite process.)

Exercise 4 (Problem sheet 1)**.** $X=[0,1]$, λ is a Lebesgue measure. Prove that:

(1) for every $\varepsilon>0$ there exists a set of 1st category of measure bigger than $1-\varepsilon$;
(2) there exists $A\subseteq[0,1]$ of 1st category with $\lambda(A)=1$;
(3) $[0,1]$ is of 2nd category;
(4) there exists $B\subseteq[0,1]$ of 2nd category with $\lambda(B)=0$.

Exercise 5. Let $S=\bigcap_{m\geq1}\bigcup_{q=r_n\in\mathbb{Q}}(q-2^{-(m+n)},q+2^{-(m+n)})$. Prove the set S is of 2nd category and is of measure 0.

Example 2.5. *There exists a function $f:[0,1]\to\mathbb{R}$ which is continuous and is nowhere differentiable. Moreover, the set of such functions is dense in $C[0,1]$, the space of all continuous functions acting from $[0,1]$ to $\mathbb{R}$.*

Proof. Let $X=C[0,1]$. We consider the following subsets of X. For each $n\geq1$, let

$$F_n=\left\{f\in X\colon\begin{array}{l}\exists t\in[0,(n-1)/n]\text{ such that}\\ |f(t+h)-f(t)|\leq nh\text{ for every }h\in[0,1-t]\end{array}\right\}.$$

For each $n\geq1$, the set F_n is closed. Indeed, assume $f_i\in F_n$ and $f_i\to f$ in X (that is, the sequence (f_i) converges to f in sup-norm). For each i, there is a point $t_i\in[0,(n-1)/n)]$ such that $|f_i(t_i+h)-f(t_i)|\leq nh$ whenever $h\in[0,1-t_i]$. The sequence $(t_i)\subset[0,(n-1)/n]$ is bounded, therefore by Bolzano-Weierstrass theorem it has a convergent subsequence, say $t_{i_k}\to t\in[0,(n-1)/n)]$. Assume $0<h<1-t$, hence $1-h>t$. There exists k_0 such that $1-h>t_{i_k}$ for all $k>k_0$, therefore, $0<h<1-t_{i_k}$ for all $k>k_0$. Then

$$\begin{aligned}&|f(t+h)-f(t)|\\ &\leq|f(t+h)-f(t_{i_k}+h)|+|f(t_{i_k}+h)-f_{i_k}(t_{i_k}+h)|\\ &\qquad\qquad+|f_{i_k}(t_{i_k}+h)-f_{i_k}(t_{i_k})|+|f_{i_k}(t_{i_k})-f(t_{i_k})|+|f(t_{i_k})-f(t)|\\ &\leq|f(t+h)-f(t_{i_k}+h)|+\|f-f_{i_k}\|+nh+\|f_{i_k}-f\|+|f(t_{i_k})-f(t)|.\end{aligned}$$

If we let $k\to\infty$ then the continuity of f at t and at $t+h$ and the convergence of f_{i_k} in the norm to f gives the inequality $|f(t+h)-f(t)|\leq nh$ for every $0<h<1-t$ and thus $f\in F_n$. If $h=1-t$, the inequality

$$|f(1)-f(t)|\leq|f(1)-f_i(1)|+|f_i(1)-f_i(t_i)|+|f_i(t_i)-f(t_i)|+|f(t_i)-f(t)|$$

for every i proves $|f(1)-f(t)|\leq n(1-t)$. Hence F_n is closed.

At the same time, for each $n \geq 1$, the set F_n is nowhere dense. Since F_n is closed it is enough to show that the interior of F_n is empty, in other words, F_n does not contain any open ball $B(f, r)$.

Remark 2.6. We note first that the set of piecewise linear continuous functions is dense in $C[0,1]$.

Indeed, let $f \in C[0,1]$ and $\varepsilon > 0$ be arbitrary. Since f is continuous on $[0,1]$, it is uniformly continuous, therefore, there exists δ such that $|f(t) - f(t')| < \varepsilon/2$ whenever $|t - t'| < \delta$. Choose N such that $1/N < \delta$ and let $f_1(t)$ be a piecewise linear continuous function with $f_1(k/N) = f(k/N)$ for all $k = 0, 1, \ldots N$. Then for any $t \in [k/N, (k+1)/N]$ we have

$$\begin{aligned} &|f(t) - f_1(t)| \leq |f(t) - f(k/N)| + |f_1(t) - f_1(k(N))| \\ \leq &|f(t) - f(k/N)| + |f_1((k+1)/N) - f_1(k(N))| \\ = &|f(t) - f(k/N)| + |f((k+1)/N) - f(k(N))| < \varepsilon. \end{aligned}$$

We now return to the proof that each F_n has empty interior. It is enough to show that for any piecewise linear continuous function f and any $\varepsilon > 0$ there exists $g \notin F_n$ such that $\|g - f\| < \varepsilon$. Then F_n cannot contain a ball $B(f, \varepsilon)$.

Fix any piecewise linear continuous function f and $\varepsilon > 0$. Let M be the maximum among absolute values of “slopes” of linear pieces of f. Choose $m \in \mathbb{N}$ such that $m\varepsilon > n + M$. We now define $g(t) = f(t) + \varepsilon s(mt)$, where $s(t)$ is the “saw-tooth” function, $s(t) = \inf_{k \in \mathbb{Z}} |t - k|$. It is clear that $g \in C[0,1]$ and $\|g - f\| \leq \varepsilon/2 < \varepsilon$. We can also say that the right-hand side derivative of g exists at every point $t \in [0,1]$ and is bigger than n since $|\varepsilon m s'_+(mt)|$ is bigger than $M + n$.

We now show that the set of continuous functions f that have a point $t \in [0,1)$ for which there exists a finite right derivative $f'_+(t) = \lim_{h \to 0+} \frac{f(t+h) - f(t)}{h}$, is a subset of $\bigcup F_n$. Let f be a continuous function with the finite right derivative at a point $t \in [0,1)$. Let N be such that $(N-1)/N > t$ and $N > |f'_+(t)|$. Choose $\delta > 0$ such that $|f(y) - f(t)| < N|y - t|$ for $y \in (t, t+\delta]$. We can choose such a δ because the function $\varphi(y) = \frac{|f(y) - f(t)|}{|y - t|}$ is continuous on $[t, 1]$ and $\varphi(t) < N$. Let $n > N$ be also bigger than $\max_{y \geq t+\delta} \frac{|f(y) - f(t)|}{\delta}$. Then $f \in F_n$. (Check that all conditions in the definition of F_n are satisfied.)

Since X is complete, it is of 2nd category. Therefore by Theorem 2.4 the union $\bigcup F_n$ cannot cover the whole X. It follows there exists $f \in X \setminus \bigcup F_n$. This function is continuous but has no right derivatives at any point $t \in [0,1]$. $\square$

Exercise 6 (Problem sheet 1)**.** The set of continuous nowhere differentiable real valued functions on $[0,1]$ is dense in $C[0,1]$.

Remark 2.7. Banach spaces are of 2nd category.

Theorem 2.8 (Principle of uniform boundedness for functions)**.** *Let E be a set of 2nd category in a metric space X, let $\mathcal{F}$ be a family of continuous functions from X to $\mathbb{R}$ such that the set $\{f(x) : f \in \mathcal{F}\}$ is bounded for every $x \in E$. Then the elements of $\mathcal{F}$ are uniformly bounded in some nonempty ball $B(x_0, r)$, i.e. $|f(x)| \le n$ holds for some n, all $f \in \mathcal{F}$ and all $x \in B(x_0, r)$.*

Proof. For each $n \ge 1$ denote $F_n = \{x \in X \colon |f(x)| \le n \text{ for all } f \in \mathcal{F}\}$. The set F_n is equal to $\bigcap_{f\in\mathcal{F}} f^{-1}[-n,n]$, the intersection of closed sets. Therefore each F_n is closed. Since $E \subseteq \bigcup_{n\ge1} F_n$ is of 2nd category we conclude (by definition) $\exists n$ such that F_n has nonempty interior, i.e. $F_n \supset B(x_0, r)$. ☐

Corollary 2.9. *The statement of Theorem 2.8 holds if X is a complete metric space and $\mathcal{F}$ is pointwise bounded on X.*

Definition. Let X, Y be normed spaces. A family $\mathcal{F}$ of bounded linear operators is *uniformly bounded* if $\sup\{\|U\| \colon U \in \mathcal{F}\} < \infty$. A family $\mathcal{F}$ is *pointwise bounded on* $E \subseteq X$ if $\sup\{\|Ux\| \colon U \in \mathcal{F}\} < \infty$ for every $x \in E$.

Remark 2.10. If X, Y are normed spaces and $\mathcal{F} \subseteq \mathcal{B}(X,Y)$ is a uniformly bounded family of bounded linear operators then it is pointwise bounded on any subset of X.

Theorem 2.11 (Banach-Steinhaus Theorem, Principle of uniform boundedness for operators)**.**
Let X, Y be normed spaces, $E \subseteq X$ be of 2nd category, $\mathcal{F} \subseteq \mathcal{B}(X,Y)$. If $\mathcal{F}$ is pointwise bounded on E then $\mathcal{F}$ is uniformly bounded.
In other words, if

$$\sup\{\|Ux\| \colon U \in \mathcal{F}\} < \infty$$

for every $x \in E$, then

$$\sup\{\|U\| \colon U \in \mathcal{F}\} < \infty.$$

Proof. For each $U \in \mathcal{F}$ consider a continuous function $f_U : X \to \mathbb{R}$ defined by $f_U(x) = \|Ux\|$. Since the conditions of Theorem 2.8 are satisfied for $\mathcal{F}' = \{f_U \colon U \in \mathcal{F}\}$, we conclude there exists a nonempty ball $B(x_0, r)$ such that $|f_U(x)| = \|Ux\| \le n$

for any $U \in \mathcal{F}$ and any $x \in B(x_0, r)$. This implies $\|U\| \leq N = n/r$ (check it) for all $U \in \mathcal{F}$. $\square$

Corollary 2.12. *The statement of Theorem 2.11 is true in particular if X is a Banach space and $E = X$ (any Banach space is of 2nd category).*

Lemma 2.13 (Principle of condensation of singularities)**.** *Let X, Y be normed spaces, X be of 2nd category.*

1. *If $U_n \in \mathcal{B}(X, Y)$ are such that $\sup \|U_n\| = \infty$ then $\exists x_0 \in X$ such that $\sup \|U_n x_0\| = \infty$;*
2. *If $U_{m,n} \in \mathcal{B}(X, Y)$ are such that $\sup_n \|U_{m,n}\| = \infty$ for every m, then $\exists x_0 \in X$ such that $\sup_n \|U_{m,n} x_0\| = \infty$ for every m.*

Proof. 1. Easily follows from Theorem 2.11.

2. Let $E_m = \{x \in X : \sup_n \|U_{m,n}(x)\| < \infty\}$. For every $m \geq 1$, the set E_m cannot be of 2nd category (if it were, we would have $\sup_n \|U_{m,n}\| < \infty$, a contradiction). Therefore, each E_m is of 1st category. Hence $E = \bigcup_{m=1}^{\infty} E_m$ is of 1st category. Since X is of 2nd category, we conclude there exists $x_0 \in X \setminus \bigcup_{m=1}^{\infty} E_m$. Since $x \notin E_m$ for every m, we have $\sup_n \|U_{m,n} x_0\| = \infty$ for all $m \geq 1$. $\square$

Lemma 2.14. *Let X, Y be normed spaces, X be of 2nd category and Y be complete. If $\{U_n\} \subseteq \mathcal{B}(X, Y)$ is a sequence of linear bounded operators, then the following are equivalent:*

1. *(U_n) converges pointwise on X;*
2. *(U_n) converges pointwise on some $E \subseteq X$, such that E is dense in X and $(\|U_n\|)$ is a bounded sequence.*

If this is true, the pointwise limits of $U_n(x)$ define a bounded linear operator.

Proof. (1) $\Rightarrow$ (2): Follows from Theorem 2.11 with $E = X$.
(2) $\Rightarrow$ (1): Since Y is complete, it is enough to check that $(U_n(x))_{n \geq 1}$ is Cauchy for every $x \in X$. Fix any $\varepsilon > 0$, $x \in X$. Find $x' \in E$ such that $\|x - x'\| < \varepsilon$. Then we can write

$$\begin{aligned} \|U_n x - U_m x\| &= \|U_n(x - x') - U_m(x - x') + U_n x' - U_m x'\| \\ &\leq \|U_n\| \|x - x'\| + \|U_m\| \|x - x'\| + \|U_n x' - U_m x'\| \leq 3C\varepsilon \end{aligned}$$

provided $C = \sup_k \|U_k\|$ and $n, m > N$ where N is such that $\|U_n x' - U_m x'\| < C\varepsilon$ for $n, m > N$. Since ε is arbitrary, we get that $(U_n x)$ is Cauchy.

Let $U_0(x) = \lim_n U_n(x)$. Since all U_n are linear, we conclude U_0 is linear. For every $x \in X$ and $n \geq 1$ we have $\|U_n(x)\| \leq C\|x\|$, therefore $\|U_0(x)\| \leq C\|x\|$. $\square$

Remark 2.15. (1) $\Rightarrow$ (2) holds for any normed Y (did not use its completeness), (2) $\Rightarrow$ (1) holds for any normed X (did not use it is of 2nd category).

Remark 2.16. Condition 2 of Lemma 2.14 is equivalent to

(2') There exists $U_0 \in \mathcal{B}(X, Y)$ such that $U_0(x) = \lim U_n(x)$ for all $x \in E$.

Example 2.17. *There exists a 2π-periodic continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that its Fourier series diverges at zero.*

Proof. We prove this statement in several steps.

1. Consider the space $C_{2\pi}$ of continuous 2π-periodic functions $f : \mathbb{R} \to \mathbb{C}$. This space equipped with the supremum norm is a Banach space (proof: Exercise, Problem sheet 2).

2. The Fourier coefficient $\hat{f}(n)$ is defined by the equality:

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt,$$

the partial sum $S_n(f)(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikx}$. Then $S_n : C_{2\pi} \to C_{2\pi}$ is a linear bounded operator satisfying $S_n(f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t) dt$, where

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}.$$

Things to check: for every $f \in C_{2\pi}$ its image $S_n(f)$ also belongs to $C_{2\pi}$, $S_n(f)$ is equal to the integral above, S_n is a linear and bounded operator. This is Exercise (Problem sheet 2).

3. Let $\varphi_n(f) = S_n(f)(0)$. We show that $\|\varphi_n\| < \infty$ for every n and $\|\varphi_n\| \to \infty$.

Lemma 2.18. *If $D : \mathbb{R} \to \mathbb{R}$ is a continuous 2π-periodic function and $\Phi_D(f) = \int_{-\pi}^{\pi} f(t) D(t) dt$, then $\Phi_D \in (C_{2\pi})^*$ and $\|\Phi_D\| = \int_{-\pi}^{\pi} |D(t)| dt$.*

Proof. (a) $\Phi_D(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
(b) $|\Phi_D(f)| = |\int_{-\pi}^{\pi} f(t) D(t) dt| \leq \int_{-\pi}^{\pi} |f(t)||D(t)| dt \leq \|f\|_\infty \int_{-\pi}^{\pi} |D(t)| dt$, so

$$\|\Phi\| \leq \int_{-\pi}^{\pi} |D(t)| dt.$$

(c) We use the fact that for any ε there exists a step function $E(t) = \sum_{k=1}^{m} a_k\chi_{A_k}$, where A_k are disjoint closed intervals and subsets of $(-\pi, \pi)$ such that $\int_{-\pi}^{\pi} |D(t) - E(t)|dt < \varepsilon$.

Let E_n be a sequence of such step functions with the property that $\int_{-\pi}^{\pi} |D(t) - E_n(t)|dt \to 0$. Fix n and assume $E_n(t) = \sum_{k=1}^{m} a_k\chi_{A_k}$. Take $f_n \in C_{2\pi}$ such that $f_n(t) = \text{sign}(a_k)$ whenever $t \in A_k$ and $f_n(t) \in [-1, 1]$ for all $t \in [-\pi, \pi]$. Then $\|f_n\|_\infty = 1$ and $\Phi_{E_n}(f_n) = \int_{-\pi}^{\pi} |E_n(t)|dt$, hence $\|\Phi_{E_n}\| = \int_{-\pi}^{\pi} |E_n(t)|dt$. But $\|\Phi_{E_n} - \Phi_D\| \le \int_{-\pi}^{\pi} |D(t) - E_n(t)|dt \to 0$, therefore $\|\Phi_D\| = \int_{-\pi}^{\pi} |D(t)|dt$. □

By Lemma 2.18, $\|\varphi_n\| = \int_{-\pi}^{\pi} |D_n(-t)|dt$.

Our next aim is to show that

$$D_n(t) = \frac{\sin\left((n + \frac{1}{2})t\right)}{\sin(t/2)}.$$

Indeed,

$$D_n(t) = 1 + \sum_{k=1}^{n} (e^{ikt} + e^{-ikt}) = 1 + \sum_{k=1}^{n} 2\cos(kt).$$

Then

$$\sin(t/2)D_n(t) = \sin(t/2) + \sum_{k=1}^{n} 2\cos(kt)\sin(t/2)$$
$$= \sin(t/2) + \sum_{k=1}^{n} \big[\sin((k + \frac{1}{2})t) - \sin((k - \frac{1}{2})t)\big] = \sin((n + \frac{1}{2})t).$$

Therefore, $\|\varphi_n\| = \int_{-\pi}^{\pi} \frac{|\sin(n+1/2)t|}{\sin(t/2)}dt$. Why $\|\varphi_n\| \to \infty$?

$$\int_{-\pi}^{\pi} \frac{|\sin(n + 1/2)t|}{\sin(t/2)}dt = 2\int_{0}^{\pi} \frac{|\sin(n + 1/2)t|}{\sin(t/2)}dt \ge 4\int_{0}^{\pi} \frac{|\sin(n + 1/2)t|}{t}dt$$
$$= 4\int_{0}^{(n+1/2)\pi} \frac{|\sin t|}{t}dt \ge 4\sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{t}dt$$
$$\ge 4\sum_{k=1}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin t|dt = \frac{8}{\pi}\sum_{k=1}^{n} \frac{1}{k} \to \infty$$

as $n \to \infty$.

4. By Banach-Steinhaus Theorem (Theorem 2.11), we deduce that there exists a function $f \in C_{2\pi}$ such that $\sup\{\varphi_n(f)\}$ is infinite. This means that the Fourier series for the function f diverges at 0. □

Section 2. Approximation of integrals.

Let $[a, b]$ be an interval of finite length. If we pick distinct points $a \le t_{n,1} < t_{n,2} < \cdots < t_{n,n} \le b$ and some coefficients $A_{n,1}, \dots, A_{n,n} \in \mathbb{R}$, we can define an "integral sum"

$$\sigma_n(f) = \sum_{1 \le k \le n} A_{n,k} f(t_{n,k}).$$

Exercise 7 (Problem sheet 2)**.** For every $n \ge 1$, σ_n is a linear functional on $C[a, b]$. Its norm $\|\sigma_n\|$ is equal to $\sum_{1 \le k \le n} |A_{n,k}|$.

Theorem 2.19. *$\sigma_n(f) \to \int_a^b f(t)\, dt$ for every $f \in C[a, b]$ if and only if*

1. *there exists C such that $\sum_{1 \le k \le n} |A_{n,k}| \le C$ for every n*
2. *$\sigma_n(P) \to \int_a^b P(t)\, dt$ for every polynomial P.*

Proof. This Theorem follows from Lemma 2.14. (Note that the set of polynomials is dense in $C[a, b]$.) □

Remark 2.20. If $A_{n,k} \ge 0$ then second condition implies the first.

Proof. Let $P \equiv 1$. Then $\int_a^b P(t) dt = b - a$, $\sigma_n(P) = \sum_{1 \le k \le n} A_{n,k} = \sum_{1 \le k \le n} |A_{n,k}| \to b - a$, therefore, the sequence $(\sum_{1 \le k \le n} |A_{n,k}|)_{n \ge 1}$ is bounded. □

Also, in case $A_{n,k} \ge 0$ we do not need to require $(t_{n,k})_{k=1}^n$ to be distinct.

Proposition 2.21. *If $A_{n,k} \ge 0$ and $a \le t_{n,1} \le \cdots \le t_{n,n} \le b$ for every $k, n \ge 1$ then TFAE*

1. *$\sigma_n(f) \to \int_0^1 f(t)\, dt$ for every $f \in C[0, 1]$*
2. (a) $\sum_{1 \le k \le n} A_{n,k} \to 1$
   (b) $\sum_{1 \le k \le n} A_{n,k} e^{2\pi i m t_{n,k}} \to 0$ *for every* $m \in \mathbb{N}$
3. *for any interval $\Delta \subset [0, 1]$ such that $|\Delta| < 1$ one has $\sum_{t_{n,k} \in \Delta} A_{n,k} \to |\Delta| = \int_0^1 \chi_\Delta\, dt$.*

Proof. (1) ⇒ (2): Consider $f_a \equiv 1$ and $f_b(t) = e^{2\pi i m t}$.
(2) ⇒ (3): We need to show that $\sum_{k=1}^n A_{n,k} \chi_\Delta(t_{n,k}) \to \int_0^1 \chi_\Delta dt$, i.e. $\sigma_n(\chi_\Delta) \to \int_0^1 \chi_\Delta dt$.

Let $\widetilde{C}[0, 1]$ be a linear subspace of $C[0, 1]$ which consists of functions $f \in C[0, 1]$ such that $f(0) = f(1)$. In the same way as $C_{2\pi}$, the space $\widetilde{C}[0, 1]$ can be considered as a Banach space with the sup-norm induced by $C[0, 1]$. The set $\mathcal{T}$ of trigonometric polynomials $T(u) = \sum_{|k| \le N} c_k e^{2\pi i k u}$ is dense in $\widetilde{C}[0, 1]$. By $(2a)$ and $(2b)$ we have

$\sigma_n(T) \to \int_0^1 T(u)du$ for any $T \in \mathcal{T}$. Therefore, by Lemma 2.14 we have $\sigma_n(f) \to \int_0^1 f(u)du$ for any $f \in \widetilde{C}[0,1]$. Let $\varepsilon > 0$. Find nonnegative functions $g, h \in \widetilde{C}[0,1]$ such that $g \le \chi_\Delta \le h$ and $\int_0^1 (h-g) < \varepsilon$. Then we have $\sigma_n(g) \le \sigma_n(\chi_\Delta) \le \sigma_n(h)$ (since all coefficients nonnegative) and $\int_0^1 g \le \int_0^1 \chi_\Delta \le \int_0^1 h$. Choose N such that $|\sigma_n(g) - \int_0^1 g| < \varepsilon$ and $|\sigma_n(h) - \int_0^1 h| < \varepsilon$ for all $n > N$. Then for all $n > N$

$$\int_0^1 g - \varepsilon \le \sigma_n(g) \le \sigma_n(\chi_\Delta) \le \sigma_n(h) \le \int_0^1 h + \varepsilon.$$

Since $\int_0^1 (h-g) < \varepsilon$, we conclude that $\int_0^1 (h - \chi_\Delta) < \varepsilon$ and $\int_0^1 (\chi_\Delta - g) < \varepsilon$, therefore,

$$\int_0^1 \chi_\Delta - 2\varepsilon \le \sigma_n(\chi_\Delta) \le \int_0^1 \chi_\Delta + 2\varepsilon$$

for all $n > N$. This proves $\sigma_n(\chi_\Delta) \to \int_0^1 \chi_\Delta$.

(3) $\Rightarrow$ (1): Fix any $f \in C[0,1]$ and let $\varepsilon > 0$. Since f is continuous on $[0,1]$, it is uniformly continuous. Therefore, there exists $m \in \mathbb{N}$ such that $|x - x'| < 1/m$ implies $|f(x) - f(x')| < \varepsilon$. Divide the interval $[0,1]$ into m intervals Δ_i of length $1/m$ each. Choose N such that for all $n > N$

$$\Big|\sum_{t_{n,k} \in \Delta_i} A_{n,k} - |\Delta_i|\Big| < \varepsilon/m.$$

Then

$$\begin{aligned}
\Big|\int_0^1 f(t)dt - \sum_{k=1}^n A_{n,k} f(t_{n,k})\Big| &= \Big|\sum_{i=1}^m \int_{\Delta_i} f(t)dt - \sum_{i=1}^m \sum_{t_{n,k}\in\Delta_i} A_{n,k} f(t_{n,k})\Big| \\
&= \Big|\sum_{i=1}^m f(c_i)|\Delta_i| - \sum_{i=1}^m \sum_{t_{n,k}\in\Delta_i} A_{n,k} f(t_{n,k})\Big| \\
&= \Big|\sum_{i=1}^m f(c_i)\big[|\Delta_i| - \sum_{t_{n,k}\in\Delta_i} A_{n,k}\big] - \sum_{i=1}^m \sum_{t_{n,k}\in\Delta_i} A_{n,k}\big[f(t_{n,k}) - f(c_i)\big]\Big| \\
&\le \sum_{i=1}^m \|f\|\frac{\varepsilon}{m} + \sum_{i=1}^m (|\Delta_i| + \frac{\varepsilon}{m})\varepsilon,
\end{aligned}$$

since $|f(t_{n,k}) - f(c_i)| < \varepsilon$. Thus we get

$$\Big|\int_0^1 f(t)dt - \sigma_n(f)\Big| \le \|f\|\varepsilon + \varepsilon + \varepsilon^2 \le \varepsilon(\|f\| + 2)$$

if we assume $\varepsilon \in (0,1)$. Since ε is arbitrary we get $\int_0^1 f(t)dt = \lim_n \sigma_n(f)$. $\square$

Definition. We say that a sequence (t_n) is uniformly distributed in $[a,b]$ if

$$\frac{\#\{k \le n\colon t_k \in \Delta\}}{n} \to |\Delta|$$

for any interval $\Delta \subseteq [a,b]$.

Theorem 2.22 (Weyl)**.** *A sequence (t_k) is uniformly distributed in $[0,1]$ if and only if*

$$\frac{1}{n}\sum_{1\le k\le n} e^{2\pi i m t_k} \underset{n\to\infty}{\longrightarrow} 0$$

for every $m \ge 1$.

Proof. "**if**": Let $A_{n,k} = 1/n$ and $t_{n,k} = t_n$ for any $1 \le k \le n$. Then $\sum_{k=1}^n A_{n,k} = 1$ and $\sum_{k=1}^n A_{n,k} e^{2\pi i m t_{n,k}} \to 0$, so condition (2) of Proposition 2.21 is satisfied. Then condition (3) of Proposition 2.21 is satisfied, which means that $(1/n)\sum_{t_{n,k}\in\Delta} 1 = \frac{\#\{k\le n\colon t_k\in\Delta\}}{n} \to |\Delta|$.

"**only if**": If condition (3) of Proposition 2.21 is satisfied, then condition (2) of Proposition 2.21 is satisfied. ☐

Example 2.23. (1) *If α is irrational then $t_k = \alpha k - [\alpha k]$ is uniformly distributed in $[0,1]$.*

(2) *If α is irrational then $t_k = \alpha k^2 - [\alpha k^2]$ is uniformly distributed in $[0,1]$.*

(3) *Let a_k be the first digit of 2^k. Is it true that digits from 1 to 9 appear in the sequence (a_k) equally often?*

Proof. (1):

$$\frac{1}{n}\sum_{k=1}^n e^{2\pi i m(\alpha k - [\alpha k])} = \frac{1}{n}\sum_{k=1}^n e^{2\pi i m \alpha k} = \frac{1}{n}\left(e^{2\pi i m\alpha}\frac{e^{2\pi i m n\alpha}-1}{e^{2\pi i m\alpha}-1}\right) \to 0,$$

since the absolute value of the expression in brackets is bounded.

(2): Exercise.

(3): How to determine the first digit of 2^k? We will "see" this digit if we divide 2^k by the biggest power of 10 which does not exceed 2^k. But $\log_{10}(2^k) = k\log_{10} 2$, so if we denote $\alpha = \log_{10} 2$ (which is an irrational number) we get that the first digit is equal to $d \in \{1,2,3,4,5,6,7,8,9\}$ if $k\alpha - [k\alpha]$ belongs to the half-open segment $\Delta_d = [\log_{10} d, \log_{10}(d+1))$.

By (1) the sequence $t_k = k\alpha - [k\alpha]$ is uniformly distributed. Therefore the frequency with which the first digit of 2^k is equal to d with k from 1 to n, which is equal to

$\frac{1\le k\le n:\, t_k\in\Delta_k}{n}$, tends to the length of this half-open segment $|\Delta_d| = \log_{10}(1+1/d)$. Therefore, 1 is the most frequent first digit, 2 is the next and so on. $\square$

Section 3. Open mapping theorem.

Definition. Let X, Y be any topological spaces. The mapping $F : X \to Y$ is called *open* if for any open set $G \subseteq X$, the set $f(G)$ is open.

Remark 2.24. If X, Y are normed spaces, then a linear operator $U : X \to Y$ is open if and only if the image of the unit ball $UB(0,1)$ is open.

Lemma 2.25. *Let X, Y be normed spaces, $U : X \to Y$ a linear operator. The following are equivalent:*

1. *U is an open mapping,*
2. *$\exists C$ such that $\forall y \in Y \exists x \in X$ with $Ux = y$ and $\|x\| \le C\|y\|$,*
3. *0 is an interior point of $UB(0,1)$.*

Proof. (1) $\Rightarrow$ (2): Since U is open, the image of the unit ball in X contains an open ball of some radius r in Y:

$$UB(0,1) \supseteq B(0,r).$$

Take any nonzero $y \in Y$. Since $z = (r/2)(y/\|y\|) \in B(0,r)$, there exists $x' \in B(0,1)$ such that $Ux' = z$. Denoting $x = \frac{2\|y\|}{r}x'$ we get $Ux = y$ and $\|x\| < 2\|y\|/r$. Therefore $C = 2/r$ satisfies (2).

(2) $\Rightarrow$ (3): For any $y \in B(0,1/C)$ there exists x such that $Ux = y$ and $\|x\| \le C\|y\| < 1$. Therefore, $UB(0,1) \supseteq B(0,1/C)$ and so 0 is an interior point of $UB(0,1)$.

(3) $\Rightarrow$ (1): We remarked that it is enough to prove that $UB(0,1)$ is open. Let $\rho > 0$ be such that $UB(0,1) \supset B(0,\rho)$. Fix any point $y \in UB(0,1)$. Let $x \in B(0,1)$ be such that $Ux = y$. There exists $r > 0$ such that $B(x,r) \subseteq B(0,1)$. Then

$$\begin{aligned} UB(0,1) &\supset UB(x,r) = U(x + rB(0,1)) = Ux + rUB(0,1) \\ &\supset Ux + rB(0,\rho) = B(Ux, r\rho). \end{aligned}$$

Therefore, each point of $UB(0,1)$ is its interior point, which implies that $UB(0,1)$ is an open set. $\square$

Corollary 2.26. *Every open linear operator is surjective.*

Corollary 2.27. *X, Y are normed spaces, $U \in \mathcal{B}(X,Y)$ is an open linear operator. If X is complete then Y is complete.*

Proof. Let (y_n) be a Cauchy sequence in Y. We will prove this sequence converges.

Choose $(n_k) \uparrow$ such that $\|y_{n_{k+1}} - y_{n_k}\| \leq 2^{-k}$. Let C be a constant from condition (2) of Lemma 2.25. Find $x_k \in X$ such that $Ux_k = y_{n_{k+1}} - y_{n_k}$ and $\|x_k\| \leq C\|y_{n_{k+1}} - y_{n_k}\|$ for all $k \geq 1$. Let x_0 be such that $U(x_0) = y_{n_1}$. Then $\|x_k\| \leq C2^{-k}$ for all $k \geq 1$ and so $\sum_{k\geq 0} \|x_k\| < \infty$. Hence the sequence $z_n = \sum_{k=0}^{n} x_k$ is Cauchy. Since X is complete we conclude there exists $\lim z_n = x$. Note that

$$Uz_m = U(\sum_{k=0}^{m} x_k) = y_{n_1} + \sum_{k=1}^{m}(y_{n_{k+1}} - y_{n_k}) = y_{n_1} + y_{n_{m+1}} - y_{n_1} = y_{n_{m+1}}.$$

When $m \to \infty$ we know $z_m \to x$; since U is bounded we get $Uz_m \to Ux$. Therefore, $y_{n_{m+1}} \to Ux$ as $m \to \infty$. Since (y_n) is a Cauchy sequence, this implies $y_n \to Ux$. □

Theorem 2.28 (Open mapping theorem). *X, Y are Banach spaces, $U \in \mathcal{B}(X,Y)$. If $U(X)$ is of second category then U is an open mapping.*

Proof. We first show that for any $r > 0$, 0_Y is the interior point of $\overline{UB_X(0,r)}$.

Indeed, if $A_r = \overline{UB_X(0,r)}$, then $\bigcup(nA_r) \supset U(X)$ as for any $y(= Ux) \in U(X)$ we can find a number $n > \|x\|/r$ and then $y = nU(x/n) \in nA_r$. Note that it is impossible that each set nA_r is nowhere dense as this would imply that $U(X)$ is of 1st category. Hence exists n such that the interior of $\overline{nA_r} = nA_r$ is not empty. Therefore interior of A_r is not empty. Since the set A_r is convex and symmetric about 0 the fact that $\mathrm{Int}A_r \neq \emptyset$ implies $A_r \supset B(0,\rho)$ for some $\rho > 0$. Indeed, if $B(x_0,\rho) \subseteq A_r$ for some x_0 then $B(-x_0,\rho) \subseteq A_r$ and so by convexity of A_r this implies that $B(0,\rho) \subseteq A_r$.

Now we will prove that $\overline{UB_X(0,1/2)} \subseteq UB_X(0,1)$. This would imply that 0_Y is an interior point of $UB_X(0,1)$ which by Lemma 2.25 gives us that U is an open mapping.

Assume $y_1 \in \overline{UB_X(0,1/2)}$. Since 0_Y is an interior point of $UB_X(0,1/4)$ we conclude that

$$\left(y_1 - \overline{UB_X(0,1/4)}\right) \cap UB_X(0,1/2) \neq \emptyset.$$

Let $x_1 \in B_X(0,1/2)$ be such that $Ux_1 \in \left(y_1 - \overline{UB_X(0,1/4)}\right)$. Then $Ux_1 = y_1 - y_2$ for some $y_2 \in \overline{UB_X(0,1/4)}$. In the same way we construct inductively two sequences $(x_n) \subseteq X$ and $(y_n) \subseteq Y$ such that

- $x_n \in B_X(0,2^{-n})$, $y_n \in \overline{UB_X(0,2^{-n})}$,
- $y_{n+1} = y_n - Ux_n$.

Since $\sum_{n=1}^{\infty}\|x_n\| < \infty$ and X is complete, we conclude that the point $x = \sum_{n=1}^{\infty} x_n$ exists and belongs to $B_X(0,1)$. Also $\sum_{k=1}^{n} Ux_k = y_1 - y_{n+1} \to y_1$, since $\|y_{n+1}\| \leq \|U\|2^{-n-1}$. Therefore, $y_1 = Ux \in B_X(0,1)$. ☐

Corollary 2.29. *X, Y are Banach spaces, $U \in \mathcal{B}(X,Y)$. If U is surjective then it is an open mapping.*

Example. *Since $C[a,b] \subsetneq L^p[a,b]$ we conclude $C[a,b]$ is of 1st category in $L^p[a,b]$.*

Exercise 8 (Problem sheet 3)**.**
(1) X, Y are Banach spaces, $U \in \mathcal{B}(X,Y)$. Assume there are constants $\theta \in (0,1)$ and $C > 0$ such that

$$\forall y \in Y \quad \exists x \in X \quad \text{such that } \|x\| \leq C\|y\| \text{ and } \|y - Ux\| \leq \theta\|y\|. \qquad (*)$$

Then for every $y \in Y$ there is an $x \in X$ such that $Ux = y$ and $\|x\| \leq \frac{C}{1-\theta}\|y\|$.
(2) What can you say about operator U if it satisfies condition $(*)$?
(3) For any separable complete space X there exists a surjective (open) linear mapping $U : \ell^1 \to X$.

Section 4. Inverse mapping and closed graph theorems.

Theorem 2.30 (Inverse mapping theorem)**.** *X, Y are Banach spaces, $U \in \mathcal{B}(X,Y)$ is bijective. Then U^{-1} is a bounded linear operator.*

Proof. The mapping $V = U^{-1} : Y \to X$ is linear (check it). In order to prove that V is continuous, we show the set $V^{-1}(G)$ is open whenever G is. By Corollary 2.29 U is an open mapping, then $V^{-1}(G) = U(G)$ is open. ☐

Exercise 9 (Problem sheet 3)**.**
(1) The statement of Theorem 2.30 is not true for normed but not complete spaces.
Hint: Consider $X = C[a,b]$ with two norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$.

(2) If $\|\cdot\|, |\cdot|$ are two norms on X which both make it complete and $\|x\| \leq C|x|$ for all x, then these norms are equivalent.

Definition. If X, Y are Banach spaces we can consider their direct sum, the space $Z = X \times Y$ with the product topology. There are many equivalent norms on Z that induce the product topology. We will usually consider the norm $\|(x,y)\|_Z = \|x\|_X + \|y\|_Y$. Other equivalent norms include, for example, $(\|x\|_X^2 + \|y\|_Y^2)^{1/2}$ and

$\max\{\|x\|_X, \|y\|_Y\}$. There are two canonical projections from Z: $P_X(x,y) = x$ and $P_Y(x,y) = y$. For a subset L of X and a function $f : L \to Y$, its graph Γ_f is the set $\{(x, f(x)) \colon x \in L\} \subseteq Z$. We say f has closed graph if Γ_f is a closed subset of Z.

Remark. Some simple properties of direct sum of Banach spaces X and Y:

1. $z_n = (x_n, y_n) \to (x,y)$ in Z if and only if $x_n \to x$ in X and $y_n \to y$ in Y,
2. The sequence $z_n = (x_n, y_n)$ is Cauchy if and only if both sequences (x_n) and (y_n) are Cauchy,
3. Z is a Banach space,
4. The projections P_X and P_Y are continuous (bounded) linear operators from Z to X and Y resp.

Lemma 2.31. *X, Y are Banach spaces, $L \subseteq X$ is a linear subspace, $U : L \to Y$ is a linear operator. Then U has closed graph if and only if the conditions $x_n \in L$, $x_n \to x$, $Ux_n \to y$ imply that $x \in L$ and Ux=y.*

Proof. "**only if**": Assume $x_n \in L$, $x_n \to x$, $Ux_n \to y$. If we denote $z_n = (x_n, Ux_n)$ then $z_n \to (x,y)$. Since $z_n \in \Gamma_U$ for all n and Γ_U is closed, we conclude $(x,y) \in \Gamma_U$ and thus $y = Ux$.

"**if**": Assume $z_n \in \Gamma_U$, $z_n \to z$. For each n, there exists $x_n \in L$ such that $z_n = (x_n, Ux_n)$; z as an element of Z is equal to (x,y) for some $x \in X$ and $y \in Y$. Since $(x_n, Ux_n) \to (x,y)$ we conclude that $x_n \to x$, $Ux_n \to y$. By hypothesis this implies $x \in L$ and $Ux = y$ which means that $z = (x,y) = (x, Ux) \in \Gamma_U$. □

Exercise 10 (Problem sheet 3)**.** (1) Let $X = Y = L^2(\mathbb{R})$ and

$$L = \{x \in L^2(\mathbb{R}) \colon \int_{-\infty}^{\infty} t^2 |x(t)|^2 dt < \infty\}.$$

Then $Ux = t \cdot x$ has a closed graph but U is not continuous.

(2) With the same spaces X and Y and subspace L consisting of all functions from $L^2(\mathbb{R})$ with compact support, the operator U defined as above does not have a closed graph.

Recall, for any function $\varphi : T \to \mathbb{R}$ defined on a topological space T its support suppφ is the closed set $\overline{\{x \in T \colon \varphi(t) \neq 0\}}$.

Theorem 2.32 (Closed graph theorem)**.** *X, Y are Banach spaces, $U : X \to Y$ is a linear operator. Then the graph of U is closed if and only if U is a continuous (bounded) operator.*

Proof. Suppose first that U is continuous. Then trivially Lemma 2.31 implies that Γ_U is closed.

Suppose now that Γ_U is closed. Let $Z = X \times Y$ be the direct sum of X and Y. We know Z is a Banach space, and therefore (since Γ_U is a closed linear subspace of Z) Γ_U is itself a Banach space. Let $P_1 : \Gamma_U \to X$ be a linear operator defined by $P_1(x, Ux) = x$. Note that P_1 is bijective and is a projection (and so is continuous by Remark after the definition of the direct sum). Then by the Inverse mapping theorem (Theorem 2.30) the operator P_1^{-1} is bounded (continuous). Thus $U : X \to Y$ is the composition of the continuous linear operator $P_1^{-1} : X \to \Gamma_U$ and the continuous linear operator $P_2 : \Gamma_U \to Y$ defined by $P_2(x, Ux) = Ux$. The operator U is therefore continuous. $\square$

Chapter 3. LOCALLY CONVEX SPACES AND WEAK TOPOLOGIES

Section 1. Locally convex spaces.

Definition. A topological vector space (TVS) is a vector space that is also a topological space such that the linear structure and the topological structure are vitally connected.

We say that X is a *topological vector space (TVS)* if

(1) the map $X \times X \to X$ defined by $(x, y) \mapsto x + y$ is continuous;
(2) the map $\mathbb{F} \times X \to X$ defined by $(\alpha, x) \mapsto \alpha x$ is continuous.

Exercise 1. Any normed space is a topological vector space.

Definition. If X is a vector space over $\mathbb{F}$ (which is equal to $\mathbb{R}$ or $\mathbb{C}$), a *seminorm* is a function $p : X \to [0, +\infty)$ having the properties:

(1) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$;
(2) $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{F}$ and $x \in X$.

Remark 3.1.

(1) Any seminorm is a sublinear functional.
(2) If p is a seminorm then $p(0) = 0$.
(3) If a seminorm p is such that $p(x) = 0$ implies $x = 0$, then p is a norm.

Definition. Suppose X is a vector space and $\mathcal{P}$ is a family of seminorms on X. We now define $\tau_{\mathcal{P}}$, the *topology determined by a family of seminorms* $\mathcal{P}$. By definition, it is a topology generated by sets $B(x_0, p, \varepsilon) = \{x \in X \colon p(x - x_0) < \varepsilon\}$, where $p \in \mathcal{P}$, $x_0 \in X$ and $\varepsilon > 0$. In other words, a set $S \subseteq X$ is open ($S \in \tau_{\mathcal{P}}$) if for every $x_0 \in S$ there exist $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $\bigcap_{i=1}^n B(x_0, p_i, \varepsilon_i) \subset S$.

Note that the requirement that for every $x \in S$ there exist $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $\bigcap_{i=1}^n B(x, p_i, \varepsilon_i) \subset S$, is equivalent to the following: for every $x_0 \in S$ there exist $p_1, \ldots, p_n \in \mathcal{P}$ and an $\varepsilon > 0$ such that

$$V_{p_1,\ldots,p_n}(x_0, \varepsilon) = \{x \in X \colon p_j(x - x_0) < \varepsilon \text{ for all } 1 \leq j \leq n\} \subseteq S.$$

(We take $\varepsilon = \min_{1 \leq j \leq n} \varepsilon_j$.)

Remark. Let $\mathcal{P}$ be a collection of seminorms. Then $(X, \tau_{\mathcal{P}})$ is a TVS.

Exercise 2. Let X be a TVS. Prove the following:

(1) If f is a linear functional on X, then $p(x) = |f(x)|$ is a seminorm.

(2) If $\tau_{\mathcal{P}}$ is the topology defined by a family $\mathcal{P}$ of seminorms on X, then for each $p \in \mathcal{P}$, p is continuous in $\tau_{\mathcal{P}}$.

(3) If $p_1, \dots, p_n$ are continuous seminorms, then $p_1(x)+\cdots+p_n(x)$ and $\max_i(p_i(x))$ are continuous seminorms.

(4) If p, q are two seminorms, such that

$$\{x \colon p(x) < 1\} \subseteq \{x \colon q(x) < 1\},$$

then $p(x) \geq q(x)$ for all $x \in X$.

Lemma 3.2. *Let $(X, \tau_{\mathcal{P}})$ be a TVS and $f : X \to \mathbb{F}$ be a linear functional. The following statements are equivalent:*

(1) *f is continuous,*

(2) *there are $p_1, \dots, p_n \in \mathcal{P}$ and positive α such that $|f(x)| \leq \alpha \sum_{k=1}^{n} p_k(x)$ for all x.*

Proof. (1) $\Rightarrow$ (2): The set $U = \{x : |f(x)| < 1\}$ is an open neighbourhood of 0, therefore, there are $p_1, \dots, p_n \in \mathcal{P}$ and $\varepsilon > 0$ such that $V_{p_1,\dots,p_n}(0, \varepsilon) \subseteq U$. Let $p = \varepsilon^{-1}(p_1 + \cdots + p_n)$. If $p(x) < 1$ then $\sum_{k=1}^{n} p_k(x) < \varepsilon$ then $p_k(x) < \varepsilon$ for each k, then $x \in V_{p_1,\dots,p_n}(0, \varepsilon) \subseteq U$, so $|f(x)| < 1$. This means that the seminorms p and $q(x) = |f(x)|$ satisfy the conditions of Exercise 2, then $|f(x)| \leq \varepsilon^{-1}(p_1(x)+\cdots+p_n(x))$ for all $x \in X$.

(2) $\Rightarrow$ (1): Choose any points $x_0, x \in X$. Then $|f(x_0) - f(x)| = |f(x_0 - x)| \leq \alpha \sum_{k=1}^{n} p_k(x_0 - x)$ implies that whenever $x \in V_{p_1,\dots,p_n}(x_0, \varepsilon/(\alpha n))$ we get $|f(x_0) - f(x)| < \varepsilon$. Thus f is continuous. ☐

Definition. A *locally convex space* (LCS) is a TVS whose topology is defined by a family of seminorms $\mathcal{P}$ such that $\bigcap_{p\in\mathcal{P}}\{x \in X \colon p(x) = 0\} = \{0\}$.

Remark 3.3. The topology in a LCS is Hausdorff, i.e. every pair of points can be separated by open neighbourhoods.

Indeed, let $x \neq y$ be two points in X. Then $x - y \neq 0$, so there is a seminorm p such that $p(x - y) \neq 0$. Then $V_p(x, p(x - y)/3) \cap V_p(y, p(x - y)/3) = \emptyset$.

Example 3.4. (1) *Let X be a compact, consider the space $C(X)$ of all functions continuous on X. For any compact $K \subseteq X$ and any $f \in C(X)$, let $p_K(f) = \sup\{|f(x)| \colon x \in K\}$. Then $\{p_K\}_{K\subseteq X \text{ is compact}}$ is a family of seminorms that makes $C(X)$ into a LCS.*

(2) *Let X be a normed space. For any $x^* \in X^*$ define $p_{x^*}(x) = |x^*(x)|$. Then p_{x^*} is a seminorm and collection of all p_{x^*}, $x^* \in X^*$ makes X into a LCS.*
The topology defined on X by $(p_{x^*})_{x^* \in X^*}$ is called the weak topology and is denoted by $\sigma(X, X^*)$.

(3) *Let X be normed space and $x \in X$. Define $p_x(x^*) = |x^*(x)|$. Then p_x is a seminorm on X^* and collection of all p_x, $x \in X$ makes X^* into a LCS.*
The topology defined on X^* by $(p_x)_{x \in X}$ is called the weak-star (or weak) topology and is denoted by $\sigma(X^*, X)$.*

Remark. 1. On the weak topology.
It is weaker than the strong=norm topology, i.e. every set open in weak topology is open in norm topology. Indeed, an open neighbourhood in weak topology is a set of the form $V_{x_1^*,\dots,x_n^*}(x_0, \varepsilon) = \{x \in X : |x_j^*(x - x_0)| < \varepsilon \text{ for all } j = 1, \dots, k\}$. Since each x_j^* is continuous in the norm topology on X, the set $V_{x_1^*,\dots,x_n^*}(x_0, \varepsilon)$ is the intersection of open (in strong topology) sets. Thus it is open in strong topology.

Definition. For a sequence (x_n) of points in X and $x \in X$ we say $x_n \xrightarrow{\text{weakly}} x$ if $x^*(x_n) \to x^*(x)$ for every $x^* \in X^*$.

2. On the weak-star topology.

Definition. For a sequence (x_n^*) of linear continuous functionals in X^* and $x^* \in X^*$ we say $x_n^* \xrightarrow{\text{weak}-\text{star}} x^*$ if $x_n^*(x) \to x^*(x)$ for every $x \in X$.

Remark. Note that on space X^* we have 3 topologies: strong (=norm) topology; weak-star topology $\sigma(X^*, X)$ and weak topology $\sigma(X^*, X^{**})$. These topologies are all different.

Section 2. Metrizable and non-metrizable LCS.

Definition. For a topological space (X, τ) we say that τ is metrizable if there is a metric d on X such that the topology is defined by d coincides with τ.

Theorem 3.5. *A LCS is metrizable if and only if its topology is determined by a countable family of seminorms.*

Proof. 1. Assume first that there is a countable family of seminorms $\mathcal{P} = \{p_1, p_2, \dots\}$ on X, such that $\bigcap_{n \geq 1} \{x \colon p_n(x) = 0\} = \{0\}$. Then we show that the topology $\tau_{\mathcal{P}}$ is metrizable and the metric can be defined by

$$\rho(x,y)=\sum_{n\geq 1}2^{-n}\frac{p_n(x-y)}{1+p_n(x-y)}.$$

We check that ρ is a metric. It is clear that $\rho(x,y)\geq 0$, $\rho(x,y)=\rho(y,x)$ for any $x,y\in X$.

If $\rho(x,y)=0$ then $p_n(x-y)=0$ for every n, then $(x-y)\in\bigcap_{n\geq 1}\{x\colon p_n(x)=0\}$, therefore, $x-y=0$.

We now check the triangle inequality. Since the function $\frac{s}{s+1}$ increases on $[0,+\infty)$, we conclude $p_n(u+v)\leq p_n(u)+p_n(v)$ implies

$$\frac{p_n(u+v)}{1+p_n(u+v)}\leq\frac{p_n(u)+p_n(v)}{1+p_n(u)+p_n(v)}\leq\frac{p_n(u)}{1+p_n(u)}+\frac{p_n(v)}{1+p_n(v)}.$$

Now if we let $u=x-z$ and $v=z-y$ we get $\rho(x,y)\leq\rho(x,z)+\rho(z,y)$.

It remains to check that the topology τ defined by ρ coincides with $\tau_{\mathcal{P}}$. Consider the bijective mapping

$$I:(X,\tau_{\mathcal{P}})\to(X,\tau).$$
$$x\mapsto x$$

We will show that I and I^{-1} are continuous.

I is continuous. For $x\in X$, let $V_n(x)=V_{p_1,\ldots,p_n}(x,1/n)$. If $y\in V_n(x)$ then $p_k(x-y)<1/n$ for all $k=1,\ldots,n$, therefore, $\rho(x,y)<1/n+1/2^n$. Let $\varepsilon>0$ and $y\in B(x,\varepsilon)$. Let N be such that $\rho(x,y)+1/N+1/2^N<\varepsilon$. Then $V_N(y)\subseteq B(x,\varepsilon)$. Hence $I^{-1}(B(x,\varepsilon))$ is open in $\tau_{\mathcal{P}}$.

I^{-1} is continuous. Let G be any set open in $\tau_{\mathcal{P}}$. We want to show that $I(G)$ is open in metric. Fix any $x\in G$. There exists n such that $V_n(x)\subseteq G$. (There are $p_{k_1},\ldots,p_{k_m}\in\mathcal{P}$ and $\varepsilon>0$ such that $V_{p_{k_1},\ldots,p_{k_m}}(x,\varepsilon)\subseteq G$, then taking $n>\max\Big\{1/\varepsilon,\max\{k_i,1\leq i\leq m\}\Big\}$ we get $V_n(x)\subseteq V_{p_{k_1},\ldots,p_{k_m}}(x,\varepsilon)\subseteq G$.) Assume $\varepsilon_1\in(0,1)$ is such that $\varepsilon_1/(1-\varepsilon_1)<1/n$. Let $\varepsilon=\varepsilon_1/2^n$. Note that the function $\frac{s}{1-s}$ is increasing on $s\in(0,1)$, so $\frac{2^k\varepsilon}{1-2^k\varepsilon}<1/n$ for all $k\leq n$. Note that if $\rho(x,y)<\varepsilon$ then $\frac{1}{2^k}\frac{p_k(x-y)}{1+p_k(x-y)}<\varepsilon$ for every $k=1,\ldots,n$. Then simple calculations show $p_k(x-y)<\frac{2^k\varepsilon}{1-2^k\varepsilon}$ for every $k=1,\ldots,n$. Hence $B(x,\varepsilon)\subseteq V_n(x)\subseteq G$.

2. We now check that if a LCS is metrizable then its topology is determined by a countable family of seminorms. Assume X is a LCS with the topology determined by the family of seminorms $\mathcal{P}$.

Assume ρ is the metric on X which defines the topology $\tau_{\mathcal{P}}$. Since X is LCS there exist continuous seminorms $p_1, \dots, p_{k_n} \in \mathcal{P}$ and $\varepsilon_n > 0$ such that $V_{p_1,\dots,p_{k_n}}(0, \varepsilon_n) \subseteq B(0, 1/n)$. Define $q_n(x) = (p_1(x) + \dots + p_{k_n}(x))/\varepsilon_n$.

- q_n is a seminorm (see Exercise).
- q_n is continuous in $\tau_{\mathcal{P}}$ since each p_i is continuous in $\tau_{\mathcal{P}}$ (see Exercise).
- $q_n(x) < 1$ implies $p_k(x) < \varepsilon_n$ for every $k = 1, \dots, k_n$, and so

$$x \in V_{p_1,\dots,p_{k_n}}(0, \varepsilon_n) \subseteq B(0, 1/n).$$

We now prove that the topology on X w.r.t. the metric ρ coincides with the topology $\tau_{\mathcal{Q}}$ determined by the family of seminorms $\{q_n\}_{n \geq 1}$.

First, consider any ball $B(x_0, r)$ open in metric ρ. Choose $n > 1/r$. Then

$$V_{q_n}(x_0, 1) = \{x : q_n(x - x_0) < 1\} \subseteq B(x_0, 1/n) \subseteq B(x_0, r).$$

Conversely, let $G \ni x_0$ be any set open in topology $\tau_{\mathcal{Q}}$. Then there exist $q_{k_1}, \dots, q_{k_m}$ and $\varepsilon > 0$ such that $V_{q_{k_1},\dots,q_{k_m}}(x_0, \varepsilon) \subseteq G$. Since each q_{k_i} is continuous in topology $\tau_{\mathcal{P}}$ (which coincides with the topology defined by metric ρ), we conclude that $V_{q_{k_1},\dots,q_{k_m}}(x_0, \varepsilon)$ is open in (X, ρ). $\square$

Example 3.6 (Non-metrizable topology)**.** *Let* $X = \mathbb{R}^{[0,1]}$ *be a linear space of all real-valued functions defined on* $[0, 1]$. *Consider the family of seminorms* $\mathcal{P} = \{p_t \colon p_t(x) = |x(t)|, t \in [0, 1]\}$. *Let* $\tau_{\mathcal{P}}$ *be a topology determined by* $\mathcal{P}$. *Then* $\tau_{\mathcal{P}}$ *is non-metrizable.*

Proof. The collection of sets $\cap_{j=1}^{n}\{x \in X \colon p_{t_j}(x - x_0) < \varepsilon_j\}$ forms a neighbourhood basis for a point x_0, thus any set $G \subseteq X$ is open in $\tau_{\mathcal{P}}$ if and only if for any $x_0 \in G$ there are $t_1, \dots, t_n$ and $\varepsilon > 0$ such that

$$V_{t_1,\dots,t_n}(x_0, \varepsilon) = \{x \in X \colon p_{t_j}(x - x_0) < \varepsilon, 1 \leq j \leq n\}$$

is a subset of G.

If $\tau_{\mathcal{P}}$ is metrizable, every point $x_0 \in X$ has a countable basis of open neighbourhoods $B(x_0, 1/n)$. But this we mean that if G is any open set containing x_0, then there exists $n \geq 1$ such that $B(x_0, 1/n) \subseteq G$.

Since $B(0, 1/n)$ is open in $\tau_{\mathcal{P}}$, there exists a finite set of points $T_n \subset [0, 1]$ and $\varepsilon_n > 0$ such that $V_{T_n}(0, \varepsilon_n) \subseteq B(0, 1/n)$. Take any $t^* \in [0, 1] \setminus \bigcup_{n \geq 1} T_n$ and consider $G = V_{t^*}(0, \varepsilon)$. Since this set is open, there exists m such that $B(0, 1/m) \subseteq G$, then

$V_{T_m}(0,\varepsilon_m) \subseteq V_{t^*}(0,\varepsilon)$, a contradiction since the function

$$f(t) = \begin{cases} 0, & \text{if } t \neq t^*; \\ 2\varepsilon, & \text{if } t = t^* \end{cases}$$

□

belongs to $V_{T_m}(0,\varepsilon_m)$ but does not belong to $V_{t^*}(0,\varepsilon)$.

Section 3. Weak and weak* topologies.

Proposition 3.7. *Let X be a normed space, $(x_n) \subseteq X$, $x \in X$, $(f_n) \subseteq X^*$, $f \in X^*$. Then*

1. *If $x_n \to x$ in norm, then $x_n \xrightarrow{\text{weakly}} x$.*
2. *If $f_n \to f$ in norm, then $f_n \xrightarrow{\text{weak-star}} f$.*
3. *If $x_n \xrightarrow{\text{weakly}} x$ in X then $\|x\| \leq \liminf \|x_n\|$.*
4. *If $f_n \xrightarrow{\text{weak-star}} f$ in X^* then $\|f\| \leq \liminf \|f_n\|$.*

Proof. (1) Note first that "$x_n \to x$ in norm" means $\|x_n - x\| \to 0$. Let $g \in X^*$ be arbitrary. Then $|g(x_n - x)| \leq \|g\|\|x_n - x\| \to 0$, therefore $g(x_n) \to g(x)$. By definition, this means $x_n \xrightarrow{\text{weakly}} x$.

(2) Again, "$f_n \to f$ in norm" means $\|f_n - f\| \to 0$. Let $y \in X$ be arbitrary. Then $|(f_n - f)(y)| \leq \|y\|\|f_n - f\| \to 0$, therefore $f_n(y) \to f(y)$. By definition, this means $f_n \xrightarrow{\text{weak-star}} f$.

(3) A corollary to Hahn-Banach Theorem (Corollary 6.12 from the part 1 of Topics in Analysis) says that there exists $\varphi \in X^*$ such that $\|\varphi\| = 1$, $\varphi(x) = \|x\|$. Since $x_n \xrightarrow{\text{weakly}} x$, we conclude $\varphi(x_n) \to \varphi(x) = \|x\|$. Therefore,

$$\|x\| = \lim |\varphi(x_n)| = \liminf |\varphi(x_n)| \leq \liminf \|\varphi\|\|x_n\| = \liminf \|x_n\|.$$

(4) By definition $\|f\| = \sup_{\|y\|\leq 1} |f(y)|$. Let $\varepsilon > 0$, find $y \in X$ of norm not exceeding 1 such that $|f(y)| > \|f\| - \varepsilon$. Then

$$\|f\| - \varepsilon < |f(y)| = \lim |f_n(y)| = \liminf |f_n(y)| \leq \liminf \|f_n\|\|y\| \leq \liminf \|f_n\|.$$

Since ε is arbitrary, we conclude $\|f\| \leq \liminf \|f_n\|$. □

Exercise 3. In Hilbert space,

1. $e_n \to 0$ in $\sigma(H, H^*)$
2. $\|x_n - x_0\| \to 0$ if and only if $\begin{cases} x_n \xrightarrow{\text{weakly}} x_0 \text{ and} \\ \|x_n\| \to \|x_0\|. \end{cases}$

Definition. Let X be a vector space over $\mathbb{F}$. A linear subspace L of X is called a *hyperplane* if $\dim(X/L) = 1$.

Proposition 3.8. *A linear subspace in X is a hyperplane if and only if it is the kernel of a non-zero functional. Two linear functionals have the same kernels if and only if one is a non-zero multiple of the other.*

Proof. If $f : X \to \mathbb{F}$ is a linear functional and $f \not\equiv 0$ then $\ker f$ is a hyperplane. Conversely, if L is a hyperplane, let $Q : X \to X/L$ be the natural map and let $T : X/L \to \mathbb{F}$ be an isomorphism. Then $f := T \circ Q$ is a linear functional on X and $\ker f = L$.

Assume now f, g are linear functionals on X with $\ker f = \ker g$. Let $x_0 \in X$ be such that $f(x_0) = 1$, then $\beta = g(x_0) \neq 0$. If $x \in X$ and $\alpha = f(x)$, then $x - \alpha x_0 \in \ker f = \ker g$, so $g(x) = g(x_0)\alpha = g(x_0)f(x)$. Thus $g = \beta f$. ☐

Lemma 3.9. *X is a real TVS, G is open convex nonempty, $0 \notin G$. Then there exists a closed hyperplane L in X such that $G \cap L = \emptyset$.*

Proof. Pick any $x_0 \in G$, then $H = x_0 - G$ is an open convex set containing 0. Then $q(x) = \inf\{t \colon t \geq 0 \text{ and } x \in tH\}$ satisfies $H = \{x \colon q(x) < 1\}$.

Indeed, this will follow from the claim that for any nonzero $x_0 \in X$, the intersection $H \cap [0, +\infty)x_0$ is a half open interval $[0, a_{x_0})x_0$ for some $a > 0$ because if $x \in H$ then $1 \in [0, a_x)$, and so $a_x > 1$; we have therefore $q(x) \leq q < 1$ where $1/q \in (1, a_x)$.

We now prove the claim. The intersection $H \cap [0, +\infty)x_0$ is convex. Further, assume V is any open set containing 0. Since the multiplication by scalar $M : \mathbb{R} \times X \to X$ defined by $M(\lambda, x) = \lambda x$ is continuous and $M(V) \ni 0 = 0 \cdot x_0$, we conclude there exist $\varepsilon > 0$ and $V_1 \ni x_0$ open such that $\lambda x \in V$ for all $\lambda \in (-\varepsilon, \varepsilon)$ and all $x \in V_1$. In particular, $x_0/n \in V$ for sufficiently large n. Now assume $tx_0 \in H$ for some $t \geq 0$. Since H is open, there exists $V \ni 0$ open such that $tx_0 + V \subset H$. Then $(t + 1/n)x_0 \in H$ for sufficiently large n. This proves that $H \cap [0, +\infty)x_0$ is a half open interval.

Also, q is a nonnegative sublinear functional. Property $q(x + y) \leq q(x) + q(y)$ follows from $t_1H + t_2H = (t_1 + t_2)H$ for all $t_1, t_2 \geq 0$, which is true because H is convex. Since $x_0 \notin H$, we conclude $q(x_0) \geq 1$.

Let $Y = \mathbb{R}x_0$, $\varphi : Y \to \mathbb{R}$ is defined by $\varphi(sx_0) = sq(x_0)$. If $s \geq 0$, then $\varphi(sx_0) = sq(x_0) = q(sx_0)$; if $s < 0$ then $\varphi(sx_0) = sq(x_0) < 0 \leq q(sx_0)$. So $\varphi \leq q$ on Y. Let $f : X \to \mathbb{R}$ be a linear functional such that $f|_Y = f$ and $f \leq q$ on X (we use the

Hahn-Banach Theorem, Theorem 1.1). Put $L = \ker f$. Note that since $f \leq q$, we have that f is continuous at 0: for any $\varepsilon > 0$ preimage $f^{-1}(-\varepsilon, \varepsilon) \supseteq q^{-1}[0, \varepsilon) = \varepsilon H$ contains an open neighbourhood of 0. Since f is linear it is continuous at all points.

Now assume $x \in G$. Then $x_0 - x \in H$, so

$$f(x_0) - f(x) = f(x_0 - x) \leq q(x_0 - x) < 1.$$

Therefore $f(x) > f(x_0) - 1 = q(x_0) - 1 \geq 0$, hence $f(x) > 0$ for all $x \in G$ which implies $L \cap G = \emptyset$. □

Corollary 3.10. *If X is a real TVS, $A, B \subseteq X$ are disjoint convex sets and A is open, then there exists a continuous linear functional $f : X \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that*

$$f(a) < \alpha \quad \forall a \in A; \qquad\qquad f(b) \geq \alpha \quad \forall b \in B.$$

If B is also open, then f and α can be chosen so that

$$f(a) < \alpha < f(b) \quad \forall a \in A, b \in B.$$

Proof. The open set $G = A - B = \{a - b \colon a \in A, b \in B\}$ satisfies the hypothesis of Lemma 3.9 (check it). Let $f : X \to \mathbb{R}$ be a continuous linear functional such that $L = \ker f$ is the closed hyperplane from Lemma 3.9 such that $G \cap L = \emptyset$. Note that $f(G)$ is a convex subset of $\mathbb{R}$ and $0 \notin f(G)$. Therefore, either $f(x) > 0$ for all $x \in G$ or $f(x) < 0$ for all $x \in G$. Suppose $f(x) < 0$ for all $x \in G$ (otherwise replace f with $-f$), then $f(a) < f(b)$ for all $a \in A, b \in B$. Therefore, for $\alpha = \sup_{a \in A} f(a)$ we have $f(a) \leq \alpha \leq f(b)$ for all $a \in A, b \in B$. Further, since A is open, we conclude $f(A)$ is open (check it) and therefore, $f(a) < \alpha$ for all $a \in A$. If B is open we have $f(b) > \alpha$ for all $b \in B$. □

Corollary 3.11. *Let X be a real LCS, $A \subseteq X$ be a closed convex subset and $x_0 \notin A$. Then there exists a continuous linear functional f such that* $\sup_{x \in A} f(x) < f(x_0)$.

Proof. Since $X \setminus A$ is an open set containing x_0, there exists an open neighbourhood U of x_0 such that $U \subseteq X \setminus A$. Because X is LCS, there is a continuous seminorm p on X such that $\{x \colon p(x) < 1\} \subseteq (U - x_0)$. Let $U' = \{x \colon p(x) < 1/2\}$. Note that $(U' + x_0) \cap (U' + A) = \emptyset$ and $U' + x_0$ and $U' + A$ are open convex subsets of X that contain x_0 and A. The result then follows from Corollary 3.10. □

Theorem 3.12. *X is a normed space, $A \subseteq X$ is convex. The following are equivalent:*

(1) *A is closed in norm topology,*

(2) *A is weakly closed.*

Proof. Any weakly closed set is closed in norm topology, see Remark after the definition of weak topology (Example 3.4).

Assume A is closed in norm topology. Assume $x_0 \notin A$ then by Corollary 3.11 there exists a continuous linear functional f such that $\sup_{x \in A} f(x) = \alpha < f(x_0)$. Now we are going to find a weakly open neighbourhood of x_0 disjoint from A.

If X is a real space, let $V = \{x \colon f(x) > \alpha\}$.

If X is a complex space, consider $g(x) = f(x) - if(ix)$, this is a continuous linear (complex) functional (check it). Therefore, g is continuous in $\sigma(X, X^*)$ and thus its real part, $\mathrm{Re} g$ is continuous in weak topology. Therefore, $V = \{x \colon f(x) > \alpha\}$ is open in weak topology. □

Example 3.13. *The conclusion of Theorem 3.12 does not hold if A is not convex and X is any infinite-dimensional space.*

Indeed, we show now that the unit sphere $S = \{x \colon \|x\| = 1\}$ is not weakly closed in any infinite-dimensional normed space. Let $V \ni 0$ be any weakly open set. There exist $f_1, \dots, f_n \in X^*$ and $\varepsilon > 0$ such that

$$V_{f_1, \dots, f_n}(0, \varepsilon) = \{x \colon |f_j(x)| < \varepsilon \ \forall 1 \le j \le n\} \subseteq V.$$

We now show that $L = \bigcap_{j=1}^n \ker f_j$ is an infinite-dimensional linear subspace of X. Indeed, L is the kernel of the linear operator $F = (f_1, \dots, f_n) : X \to \mathbb{R}^n$. Its kernel cannot be finite-dimensional as the image is finite-dimensional and X is infinite-dimensional.

Therefore, there is $x \neq 0$ such that $x \in L$. Then $\frac{x}{\|x\|} \in L \cap S$. Since $L \subset V$, we conclude $V \cap S \neq \emptyset$.

Remark. This implies that the weak topology coincides with the norm topology if and only if the normed space is finite-dimensional:

The closed unit sphere is not closed in weak topology when X is any infinite-dimensional normed space.

Proposition 3.14. *X is a vector space, $f_1, \dots, f_n, f$ are linear functionals on X, such that $\ker f \supset \cap_{j=1}^n \ker f_j$. Then there exist scalars $\alpha_1, \dots, \alpha_n$ such that $f = \sum_{j=1}^n \alpha_j f_j$.*

Proof. Without loss of generality we may assume that $\bigcup j \neq k \ker f_j \neq \bigcup j = 1^n \ker f_j$ for every k (otherwise consider $\{f_1, \dots, f_n\} \setminus \{f_k\}$. Choose $y_k \in \bigcup j \neq k \ker f_j \setminus \bigcup j = 1^n \ker f_j$, then $f_k(y_k) \neq 0$ and $f_j(y_k) = 0$ for every $j \neq k$. Then for every x and for every k, the vector $x - \sum_{j=1}^n \frac{f_j(x)}{f_j(x_j)} x_j$ belongs to $\ker(f_k)$. Thus

$$x - \sum_{j=1}^n \frac{f_j(x)}{f_j(x_j)} x_j \in \cap_{j=1}^n \ker f_j \subseteq \ker f,$$

so $f(x) = \sum_{j=1}^n \frac{f_j(x)}{f_j(x_j)} f(x_j) = \sum_{j=1}^n \frac{f(x_j)}{f_j(x_j)} f_j(x)$. Therefore, $f = \sum_{j=1}^n \frac{f(x_j)}{f_j(x_j)} f_j$. □

Lemma 3.15. *Let X be a LCS. Then*

(1) $(X, wk)^* = X^*$
(2) $(X^*, wk^*)^* = X$

Proof. 1. Since every weakly open set if open in the norm topology, each $f \in (X, \mathrm{wk})^*$ (linear functional continuous in (X, wk)), belongs to X^*. Conversely: if $f \in X^*$ then it is continuous in (X, wk) and so belongs to $(X, \mathrm{wk})^*$.

2. If $x \in X$ then it is weak* continuous, and so $x \in (X^*, \mathrm{wk}^*)^*$. Conversely: if $\Phi \in (X^*, \mathrm{wk}^*)^*$, then by Lemma 3.2 there are $x_1, \dots, x_n \in X$ such that $|\Phi(x^*)| \leq \sum_{k=1}^n |x_k(x^*)|$ for all $x^* \in X^*$. This implies $\bigcap_{k=1}^n \ker x_k \subseteq \ker\Phi$ hence by Proposition 3.14, $\Phi = \sum \alpha_k x_k$ for some $\alpha_k \in \mathbb{R}$. This means $\Phi \in X$. □

Theorem 3.16 (Alaoglu)**.** *X is a normed space. Then the closed unit ball of X^*, $B_{X^*} = \{f \in X^* \colon \|f\| \leq 1\}$ is w^*-compact.*

Remark. We prove the theorem only in the case when X is a *separable* space, that is, there exists a sequence $(x_n) \subseteq X$ such that $\overline{\{x_n \colon n \geq 1\}} = X$.

We will also show that in case X is separable, the weak-star topology is metrizable when restricted on the unit ball B_{X^*}.

NB: (without proof) The weak-star topology on X^* is never metrizable if X is infinite-dimensional.

Lemma 3.17. *Let $y_n = (y_{n,1}, y_{n,2}, \dots, y_{n,k}, \dots)$ be a sequence of elements of $\mathbb{R}^{\mathbb{N}}$. If for every k there exists C_k such that $|y_{n,k}| \leq C_k$ for all $n \geq 1$ then there exists a subsequence of y_n which converges in each coordinate.*

Proof. Since $(y_{n,1})$ is a bounded sequence of real numbers, we can find a subsequence of y_n whose first coordinates converge. Denote elements of this subsequence by $y_n^{(1)}$. Since $y_{n,2}^{(1)}$ is a bounded sequence of real numbers, we can find a subsequence of $y_n^{(1)}$

whose second coordinates also converge. Denote elements of this subsequence by $y_n^{(2)}$. Let us continue in this way: for each $k \geq 1$, the sequence $y_n^{(k)}$ is a subsequence of $y_n^{(k-1)}$ and has the property that for every $1 \leq i \leq k$, the ith coordinates of $y_n^{(k)}$ converge.

Let $z_k = y_k^{(k)}$. This is a subsequence of (y_n) and for all $i \geq 1$, the ith coordinates of z_k converge. $\square$

Proof of Theorem 3.16. Let (x_n) be a sequence of points in X which forms a dense subset of X. Let $p_n(f) = |f(x_n)|$, $\mathcal{P}_0 = \{p_n\}$ and σ_0 be the topology on X^* defined by $\mathcal{P}_0$. Note that σ_0 is *weaker* than the weak-star topology, that is the collection of open sets in σ_0 is a subset of the collection of open sets in the weak-star topology.

Note that if $f \in X^*$ and $p_n(f) = 0$ for every n then $f(x_n) = 0$ for every n and so $f(x) = 0$ for every $x \in X$ as f is continuous on X and (x_n) is dense in X. Therefore, (X^*, σ_0) is a metrizable LCS.

We now check that σ_0 coincides with σ^* on B_{X^*}. For this we check that the bijective mapping

$$\begin{aligned} I : (B_{X^*}, \sigma_0) &\to (B_{X^*}, \sigma^*) \\ f &\mapsto f \end{aligned}$$

is a homeomorphism, i.e. I and I^{-1} are continuous.

Since σ_0 is weaker than the weak-star topology we get that I^{-1} is continuous.

We now check that I is continuous. Since σ_0 is metrizable it is enough to prove that $(f_n) \to f_0$ in σ_0, $f_n, f_0 \in B_{X^*}$ implies $(f_n) \to f_0$ in the weak-star topology.

Note that $(f_n) \to f_0$ in σ_0 means $p_m(f_n - f_0) \to 0$ $\forall m \geq 1$, which is equivalent to $f_n(x_m) \to f_0(x_m)$ $\forall m \geq 1$. Thefore we have a sequence of linear functionals from X to $\mathbb{R}$ with bounded norms that converge to f_0 on a set dense in X. By Lemma 2.14 this implies $f_n(x) \to f_0(x)$ for all $x \in X$, therefore, $f_n \to f_0$ in the weak-star topology.

We are left to prove that (B_{X^*}, weak^*) is compact. Since this space is metrizable, it is enough to show that every sequence of points from (B_{X^*}, weak^*) has a convergent subsequence. Let (f_n) be a sequence of points in (B_{X^*}, weak^*), put

$$y_n = (f_n(x_1), f_n(x_2), \ldots, f_n(x_k), \ldots),$$

where (x_k) is a dense subset of X as before. For any k we have $|y_{n,k}| = |f_n(x_k)| \leq \|f_n\|\|x_k\| \leq \|x_k\| = C_k$ for all $n \geq 1$, therefore we can apply Lemma 3.17 and find a subsequence (y_{n_i}) such that for every $k \geq 1$, the sequence $y_{n_i,k} = f_{n_i}(x_k)$ converges.

Thus we get a sequence of linear functionals (f_{n_i}) which converges on a dense subset of X. Applying Lemma 2.14 again, we get that there exists $f_0 \in X^*$ such that $f_{n_i} \to f_0$ in the weak-star topology. Since $\|f_0\| \le \liminf \|f_{n_i}\|$ (see Proposition 3.7), we conclude $\|f_0\| \le 1$, thus $f_0 \in B_{X^*}$. $\square$

Definition. If X is a normed space, we can consider its dual X^*. It is a Banach space (see Proposition 6.3, part 1 of *Topics in Analysis*) and we can consider its dual, $(X^*)^* \equiv X^{**}$ and X^{**} is a Banach space. This space is called the *second dual* of X.

Remark 3.18. If $x \in X$ then x defines a continuous linear functional on X^*, i.e. an element $\hat{x}$ of X^{**}: we define $\hat{x}$ by $\hat{x}(f) = f(x)$ for every $f \in X^*$. Note that by Corollary 6.12 in part 1 of *Topics in Analysis* we have

$$\|\hat{x}\| = \sup_{\|f\| \le 1} f(x) = \|x\|$$

for all $x \in X$. The map

$$I : X \to X^{**}$$
$$x \mapsto \hat{x}$$

is called the *natural map* or *natural embedding* of X into its second dual.

Definition. A normed space X is called *reflexive* if $X^{**} = \{\hat{x} \colon x \in X\}$.

Remark 3.19. One has to be very careful in interpreting the definition of a reflexive space. It is clear from the definition that the reflexive space X has to be isomorphic to its second dual X^{**}. It in **not true** however, that a Banach space X that is isometric to X^{**} is reflexive. The definition of reflexivity stipulates that the isometry be the natural embedding of X into X^{**}.

Lemma 3.20. *If X is a normed space, then the closed unit ball of X is $\sigma(X^{**}, X^*)$ dense is the closed unit ball of X^{**}.*

Proof. Let B be the $\sigma(X^{**}, X^*)$-closure of the closed unit ball of X in X^{**}. Clearly B is contained in the closed unit ball of X^{**}. Assume there is $x_0^{**} \notin B$ which belongs to the closed unit ball of X^{**}. Then there are $f_1, \dots, f_n \in X^*$ and an $\varepsilon > 0$ such that the set $V = V_{f_1,\dots,f_n}(x_0^{**}, \varepsilon) = \{x^{**} \in X^{**} \colon (x^{**} - x_0^{**})(f_i) < \varepsilon \text{ for all } i = 1, \dots, n\}$ does not intersect B.

Define $\varphi : X^{**} \to \mathbb{R}^n$ by $\varphi(x^{**}) = (x^{**}(f_1), \dots, x^{**}(f_n))$. Then $a = (a_1, \dots, a_n) = \varphi(x_0^{**})$ does not belong to $\overline{\varphi(B_X)}$, where B_X is the closed unit ball of X. Since $\overline{\varphi(B_X)}$

is convex closed set, Corollary 3.11 (which reduces in this case to a simple fact in the usual finite-dimensional inner product space), tells us there exist $b = (b_1, \dots, b_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that

$$\langle b, a\rangle > \sup_{x\in B_X} \langle b, \varphi(x)\rangle = \sup_{x\in B_X} (\sum_{i=1}^{n} b_i f_i(x)).$$

Consider now a continuous linear functional $f = \sum_{i=1}^{n} b_i f_i \in X^*$. Note that

$$x_0^{**}(f) > \sup_{x\in B_X} f(x)$$

Then

$$\|f\| = \sup_{x\in B_X} |f(x)| = \sup_{x\in B_X} f(x) < x_0^{**}(f) \le \|x_0^{**}\|\ \|f\| \le \|f\|,$$

a contradiction. □

Theorem 3.21. *X is a Banach space. The following are equivalent:*

1. *X is reflexive,*
2. *$\sigma(X^*, X) = \sigma(X^*, X^{**})$,*
3. *X^* is reflexive,*
4. *the closed unit ball B of X is weakly compact.*

Proof. (1) ⇒ (4): By Alaoglu's Theorem, the closed unit ball of X^{**} is $\sigma(X^{**}, X^*)$ compact. Since $X^{**} = X$, we get that B is $\sigma(X, X^*)$ compact.

(4) ⇒ (1): We note first that $\sigma(X^{**}, X^*)|_X = \sigma(X, X^*)$. By (4), the set B is $\sigma(X^{**}, X^*)$ closed in the closed unit ball of X^{**}, but by Lemma 3.20 the set B is $\sigma(X^{**}, X^*)$ dense in the closed unit ball of X^{**}, therefore, B coincides with the closed unit ball of X^{**}. We thus get $X = X^{**}$.

(1) ⇒ (2): Clear since $X = X^{**}$.

(2) ⇒ (3): By Alaoglu's Theorem, the closed unit ball of X^* is compact. Since we have already proved (4) ⇒ (1), we get that X^* is reflexive.

(3) ⇒ (1): Since B is norm closed in X^{**} it is also $\sigma(X^{**}, X^{***})$ closed in X^{**} by Theorem 3.12. Since $X^* = X^{***}$ by (3), this implies B is $\sigma(X^{**}, X^*)$ closed in X^{**}. But since it is $\sigma(X^{**}, X^*)$ dense in the closed unit ball of X^{**}, the closed unit balls of X and X^{**} coincide and thus we get $X = X^{**}$. □

Chapter 4. COMPACT OPERATORS

Section 1. **General properties of compact operators.**

Definition. Let (X, ρ) be a metric space, $A \subseteq X$. We say A is

(1) *compact* if every open covering of A contains a finite subcovering;
(2) *precompact*, if the closure of A is compact;
(3) *bounded* if there exists $x \in X$ and $R > 0$ such that $A \subseteq B(R)$;
(4) *totally bounded* if for any $\varepsilon > 0$ there exist n and $x_1, \dots, x_n \in X$ such that $A \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon)$. The set $(x_i)_{1 \le i \le N}$ is called an ε-*net* for A.

Proposition 4.1 (Exercise). *Let (X, ρ) be a metric space, $A \subseteq X$. Then*

(1) *A is compact if and only if it is closed and any sequence $(x_n) \subseteq A$ has a convergent subsequence;*
(2) *A is precompact if and only if any sequence $\{x_n\} \subseteq A$ has a convergent subsequence;*
(3) *if A is compact then A is closed and totally bounded.*

Definition. Let X, Y be Banach spaces and B_X be the closed unit ball of X. A linear operator $U : X \to Y$ is called *compact* if $U(B_X)$ is precompact.

Proposition 4.2 (Elementary properties of compact operators). *Let X, Y be Banach spaces and $U : X \to Y$ be a linear operator. Then*

(1) *if U is compact then U is bounded (continuous);*
(2) *if U is compact and $A \subseteq X$ is bounded then $U(A)$ is precompact;*
(3) *U is compact if and only if for any bounded sequence $(x_n) \subseteq X$ there exists a subsequence (x_{n_k}) such that Ux_{n_k} converges;*
(4) *the identity operator $I : X \to X$ is compact if and only if X is a finite dimensional space.*

Proof. 1. Let B_X be the closed unit ball of X. Since $U(B_X)$ is precompact, it is totally bounded, therefore bounded. So we conclude that there exists r such that $U(B_X) \subseteq B(0, r)$. This means $\|Ux\| \le r$ for any $\|x\| \le 1$. Thus $\|U\| \le r$.

2. Let $r > 0$ be such that $A \subseteq rB_X$. Then the closure of $U(A)$ is a closed subset of compact $r\overline{U(B_X)}$. Thus $\overline{U(A)}$ is compact.

3. The condition that any bounded sequence (x_n) has a subsequence (x_{n_k}) such that Ux_{n_k} converges is equivalent to $U(B_X)$ being a precompact. This is a definition of U being a compact operator.

4. By definition, the identity operator is compact if and only if the closed unit ball of X is compact. This is equivalent to X being a finite-dimensional space. $\square$

Proposition 4.3. *If X is a complete metric space then its subset A is totally bounded if and only if its closure is compact.*

Proof. We only need to prove that in complete metric spaces any totally bounded subset is precompact.

Let $F = \overline{A}$. First show that the complete metric space F is totally bounded. Let $\varepsilon > 0$. Choose $x_1, \dots, x_N \in X$ a finite $\varepsilon/3$-net for A. Let $y_i \in A$ be such that $\rho(x_i, y_i) < \varepsilon/3$. [If such y_i does not exist we do not need the ball $B(x_i, \varepsilon/3)$ in the covering of A.] Then for every $y \in F$ there exists a point $x \in A$ such that $\rho(x, y) < \varepsilon/3$, then choosing i such that $\rho(x_i, x) < \varepsilon/3$ we get $\rho(y, y_i) < \varepsilon$. Thus F has a finite ε-net $(y_i)_{1 \le i \le N}$.

Now show that F is a sequential compact. Take any sequence of points (x_n) in F. Since F is totally bounded we can find a finite covering of F by balls of radius 1. Consider a ball B_1 which contains infinitely many members of (x_n), let $x^{(1)}_{n_k} \in B_1$ be a subsequence of (x_n). Consider now covering of F by balls of radius $1/2$ and find a ball B_2 which contains infinitely many members of $x^{(1)}_k$. Denote this subsequence by $x^{(2)}_k$. On next step we consider balls of radius $1/3$ and so on. Let $y_k = x^{(k)}_k$. Then for $n, m > N$ the distance between y_n and y_m is not bigger than $1/N$, which means (y_k) is Cauchy. Since F is complete, (y_k) converges. $\square$

Definition. A family $\mathcal{F}$ of continuous functions $x : K \to \mathbb{C}$ on a compact space K is called *equicontinuous* if for every $\varepsilon > 0$ and every $t \in K$ there is an open neighbourhood U_t of t such that $|x(t) - x(t')| < \varepsilon$ for all $t' \in U_t$ and $x \in \mathcal{F}$.

If K has a metric ρ then this condition takes the following form: for every $\varepsilon > 0$ and every $t \in K$ there exists $\delta_t > 0$ such that if $x \in \mathcal{F}$ and $\rho(t, t') < \delta$ then $|x(t) - x(t')| < \varepsilon$. Since K is compact, we may find finitely many points $t_1, \dots, t_n \in K$ such that $\bigcup_{1 \le i \le n} B(t_i, \delta_{t_i})$ covers K. If we set $\delta = \min\{\delta_{t_i}, 1 \le i \le n\}$ then we get the following definition of an equicontinuous family of functions:
For every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in \mathcal{F}$ and for any pair of points $t, t' \in K$ with $\rho(t, t') < \delta$ we have $|x(t) - x(t')| < \varepsilon$.

Example 4.4. *A family of functions satisfying Lipschitz condition with the same Lipschitz constant $|x(t) - x(t')| \le L\mathrm{dist}(t, t')$ is equicontinuous.*

Theorem 4.5 (Arzela-Ascoli)**.** *(K, ρ) is a compact space with metric ρ, $A \subseteq C(K)$. The following are equivalent:*

(1) A is precompact,

(2) A is bounded and equicontinuous.

Example 4.6. *The family $\mathcal{F}$ consisting of all constant functions is equicontinuous but not bounded.*

The family $\mathcal{F} = \{\sin(nt), t \in [-\pi, \pi]\}$ is bounded but not equicontinuous.

Proof. (1) $\Rightarrow$ (2) $\overline{A}$ is compact; since $\|x\|$ is a continuous function, we conclude $\|x\|$ is bounded on $\overline{A}$. Thus A is bounded.

Fix any $\varepsilon > 0$. Let $x_1, \dots x_n \in X$ be such that $A \subseteq \bigcup_{k=1}^{n} B(x_k, \varepsilon/3)$. For each k find $\delta_k > 0$ such that $\rho(t, t') < \delta_k$ implies $|x_k(t) - x_k(t')| < \varepsilon/3$. Let $\delta = \min_{1 \le k \le n} \delta_k$ and assume $\rho(t, t') < \delta$. Consider any $x \in A$ and find $1 \le k \le n$ such that $\|x - x_k\| < \varepsilon/3$. Then

$$|x(t) - x(t')| \le |x(t) - x_k(t)| + |x_k(t) - x_k(t')| + |x_k(t') - x(t')| < \varepsilon$$

as $|x(t) - x_k(t)| \le \|x - x_k\| < \varepsilon/3$, $|x_k(t) - x_k(t')| < \varepsilon/3$ and $|x_k(t') - x(t')| \le \|x - x_k\| < \varepsilon/3$. This proves A is equicontinuous.

(2) $\Rightarrow$ (1): Assume $M > 0$ is such that $|x(t)| \le M$ for all $x \in A$, $t \in K$. Fix any $\varepsilon > 0$. Let $\delta > 0$ be a number given by the definition of equicontinuous family of functions on a metric space K. Since K is compact there exists finitely many sets of diameter less than or equal to δ that cover K. Moreover, we may assume these sets $E_1, \dots, E_m$ are disjoint.

Let L be a linear span of $(\chi_{E_i})_{1 \le i \le m}$ and $B = \{\sum_{i=1}^{m} a_i \chi_{E_i} \colon |a_i| \le M, 1 \le i \le m\} = B(0, M) \cap L$. Note that for any $x \in A$ there is $y \in B$ such that $\|x - y\| \le \varepsilon$. Indeed, let $y = \sum_{i=1}^{m} x(t_i) \chi_{E_i}$. Then $y \in B$ and for a fixed $t \in K$ there exists a unique i such that $t \in E_i$. Then $y(t) = x(t_i)$ and so $|x(t) - y(t)| = |x(t) - x(t_i)| < \varepsilon$ as $\rho(t, t_i) \le \operatorname{diam}(E_i) < \delta$. Since this inequality holds for every $t \in K$, we conclude $\|x - y\| = \sup |x(t) - y(t)| \le \varepsilon$.

We remark that B is bounded and is a subset of a finite-dimensional space $L \subseteq L^\infty(K)$, therefore, B is a totally bounded subset of $L^\infty(K)$. Then A is is a totally bounded subset of $L^\infty(K)$ (we replace the balls of radius ε for B by balls of radius 2ε with the same centres; the new balls cover A). Since $L^\infty(K)$ is complete, by Proposition 4.3 we conclude that $\overline{A} \subset C(K)$ is compact in $L^\infty(K)$, therefore, it is compact in $C(K)$. $\square$

Definition. If X, Y are normed spaces and $U \in \mathcal{B}(X, Y)$ is a bounded linear operator, then the (bounded) linear operator $U^* : Y^* \to X^*$ defined by $(U^* y^*)(x) = y^*(Ux)$ is called the *adjoint* of U.

Theorem 4.7 (Schauder)**.** *X, Y are Banach spaces, $U : X \to Y$ is compact if and only if its adjoint U^* is compact.*

Proof. 1. Assume U is compact. Then $K = \overline{U(B_X)}$ is compact, where B_X is the closed unit ball of X. Therefore, for any $g \in Y^*$

$$\|U^* g\| = \sup_{\|x\| \le 1} |U^* g(x)| = \sup_{\|x\| \le 1} |g(Ux)| = \sup_{y \in U(B_X)} |g(y)| = \sup_{y \in K} |g(y)| = \|g|_K\|_{C(K)}.$$

Assume $(g_n) \subset Y^*$ is bounded. Let C be such that $\|g_n\| \le C$ for all $n \ge 1$. Consider $h_n = g_n|_K$, then $h_n \in C(K)$ and $\|U^* g_n - U^* g_m\| = \|U^*(g_n - g_m)\| = \|h_n - h_m\|_{C(K)}$.

We now check that $\mathcal{H} = (h_n)$ is bounded and is equicontinuous family of functions from $C(K)$:
(a): $|h_n(y)| \le \|g_n\| \|Ux\| \le C\|U\|$;
(b): $|h_n(y) - h_n(y')| = |h_n(y - y')| \le \|g_n\| \|y - y'\| \le C\|y - y'\|$, therefore we have that (h_n) satisfy Lipschitz condition with the same Lipschitz constant C, and so (h_n) is equicontinuous by Example 4.4.

By Arzela-Ascoli's Theorem (Theorem 4.5) this implies that $\mathcal{H}$ is precompact, therefore there is a convergent subsequence (h_{n_k}). This implies $g_{n_k}|_K$ is Cauchy, therefore, $U^* g_{n_k}$ is Cauchy, therefore, $U^* g_{n_k}$ converges as X^* is complete.

2. Now assume U^* is compact. Then $U^{**} : X^{**} \to Y^{**}$ is compact. Note that $U^{**}\pi_X = \pi_Y U$, where $\pi_M : M \to M^{**}$ is the natural embedding. Assume (x_n) is a bounded sequence in X. Then $\hat{x}_n = \pi_X(x_n) \in X^{**}$ is bounded as $\|\hat{x}_n\| = \|x_n\|$. There exists a subsequence $(\hat{x}_{n_k})$ such that $U^{**}\hat{x}_{n_k}$ converges. But $\pi_Y U x_{n_k} = U^{**}\hat{x}_{n_k}$ and $\|\pi_Y U x_{n_k} - \pi_Y U x_{n_m}\| = \|\pi_Y(U x_{n_k} - U x_{n_m})\| = \|U x_{n_k} - U x_{n_m}\|$, therefore $(U x_{n_k})$ is Cauchy. It converges as Y is complete. ☐

Proposition 4.8 (Further elementary properties of compact operators)**.** *Let X, Y, Z be Banach spaces, $U, V \in \mathcal{B}(X, Y)$, $W \in B(Y, Z)$.*

1. *If U, V are compact operators, then $aU + bV$ is a compact operator for any $a, b \in \mathbb{C}$;*
2. *If at least one of U, W is compact, then WU is compact;*
3. *If the sequence of linear operators U_n converges to U_0 and U_n is compact for every $n \ge 1$ then U_0 is compact.*

Proof. 1. Let $T = aU + bV$ and (x_n) is a bounded sequence of points. There exists a subsequence (x_{n_k}) such that Ux_{n_k} converges. Further, there exists its subsequence $y_i = x_{n_{k_i}}$ such that $V(y_i)$ converges. Then Ty_i converges.

2. Let (x_n) be a bounded sequence. If U is compact, there exists a subsequence (x_{n_k}) such that Ux_{n_k} converges. Since W is continuous, WUx_{n_k} converges. If W is compact we note first that since U is bounded, (Ux_n) is bounded. Therefore, there is a subsequence such that WUx_{n_k} converges.

3. Let B_X be the closed unit ball of X and let $A = U_0(B_X)$. Since Y is complete, it is enough to show that A is totally bounded. Fix any $\varepsilon > 0$ and choose m such that $\|U_m - U_0\| < \varepsilon/2$. Let $C = U_m(B_X)$. Since U_m is compact we have that C is precompact, therefore it is totally bounded. Let $y_1, \dots, y_n \in Y$ be such that $C \subseteq \bigcup_{i=1}^{n} B(y_i, \varepsilon/2)$. Fix any $y \in A$. There exists $x \in B_X$ such that $y = U_0x$. Let $z = U_mx \in C$. There exists $1 \le i \le n$ such that $\|z - y_i\| < \varepsilon/2$. Then $\|y - y_i\| < \varepsilon/2 + \|y - z\| = \varepsilon/2 + \|U_0x - U_mx\| \le \varepsilon/2 + \|U_0 - U_m\|\|x\| < \varepsilon$. Thus A is totally bounded. By Proposition 4.3 this implies A is compact, since Y is a Banach space. □

Section 2. Approximation property.

Definition. Let X, Y be Banach spaces. We say that an operator $U \in \mathcal{B}(X, Y)$ is of *finite rank* if $\dim(U(X)) < \infty$.

Example 4.9. *Any operator of finite rank between Banach spaces is compact.*

Proof. Since the range of U is finite-dimensional, the closure of a bounded set $U(B_X)$ is compact. □

Corollary 4.10. *Let X, Y be Banach spaces. If U_n are finite rank operators and $U_n \to U_0$ then U_0 is compact.*

Proof. Any operator of finite rank is a compact operator, then use Proposition 4.8. □

Remark. When X is any Banach space and H is a Hilbert space, it is true that any compact operator from X to H is a limit of finite rank operators. Whether this is true in general for Banach spaces was an unsolved question for many years until in 1973 Enflo gave a counter-example.

Theorem 4.11. *If X is a Banach space and H is a Hilbert space and $U_0 : X \to H$ is a compact operator, then there is a sequence of finite rank operators $U_n : X \to H$ such that $\|U_n - U_0\| \to 0$.*

Definition. A Banach space X is said to have the *approximation property*, if, for every compact set $K \subset X$ and every $\varepsilon > 0$, there is a finite rank linear operator $A : X \to X$ such that $\|Ax - x\| < \varepsilon$ for every $x \in K$.

Proposition 4.12. *Any Hilbert space H has the approximation property.*

Proof. Let $K \subset H$ be compact and $\varepsilon > 0$. There exist $x_1, \dots, x_n \in K$ such that $K \subseteq \bigcup_{i=1}^n B(x_i, \varepsilon/2)$. Let $P : H \to H$ be an orthogonal projection on the linear span $\langle x_1, \dots, x_n \rangle$, this is a finite rank operator. Take any $x \in K$. There exists i such that $\|x - x_i\| < \varepsilon/2$. Then

$$\|Px - x\| \leq \|Px - Px_i\| + \|Px_i - x_i\| + \|x_i - x\| \leq \|x - x_i\| + 0 + \|x_i - x\| < \varepsilon.$$

$\square$

Proof of Theorem 4.11. We prove a stronger statement:
If X, Y are Banach spaces, Y has the approximation property and $U_0 : X \to Y$ is a compact operator, then there is a sequence of finite rank operators $U_n : X \to Y$ such that $\|U_n - U_0\| \to 0$.

Let B_X be the closed unit ball of X. Let $\varepsilon > 0$. Since $K = \overline{U_0(B_X)}$ is compact, there exists a finite rank (bounded linear) operator $A : Y \to Y$ such that $\|Ay - y\| < \varepsilon$ for every $y \in K$. Let $V_\varepsilon = AU_0$. Then V is a finite rank operator and $\|V_\varepsilon x - U_0 x\| = \|Ay - y\| < \varepsilon$ for every $x \in B_X$ (here $y = U_0 x \in K$). Let $U_n = V_{1/n}$; hence $\|U_n - U_0\| \leq 1/n$. $\square$

Example 4.13. *Let X, Y be Banach spaces; $f_1, \dots, f_n \in X^*$ and $y_1, \dots, y_n \in Y$. Then*

$$(*) \qquad Ux = \sum_{k=1}^{n} f_k(x) y_k$$

defines a finite rank operator. Moreover, any operator $U : X \to Y$ of finite rank has this form.

Proof. 1. It is clear that U is a bounded linear operator of finite rank.

2. Let U be a bounded linear operator of finite rank. Note that $U(X)$ is a linear subspace of Y; since it is finite-dimensional, we can find a (linearly independent) basis

$y_1, \dots, y_n$ of $U(X)$, WLOG $\|y_k\| = 1$ for each k. For each $x \in X$, define $f_k(x)$ to be scalars so that the formula (*) is satisfied. It is clear each f_k is linear; let us show each f_k is continuous. Assume $x \to 0$. We need to show that $f_k(x) \to 0$ for every k. Since U is a linear bounded operator we have $Ux \to 0$. Note that the operator $V : \mathbb{R}^n \to U(X)$ defined by $V(a_1, \dots, a_n) = \sum_{k=1}^n a_k y_k$ is continuous and the $(n-1)$-dimensional sphere is compact, therefore, $\inf_{\|a\|=1} \|V(a)\| = \|V(a_0)\| = m$ for some $\|a_0\| = 1$ and $m > 0$ since y_k are linearly independent. Then $m(\sum_{k=1}^n |f_k(x)|^2)^{1/2} \leq \|\sum_{k=1}^n f_k(x) y_k\| = \|Ux\| \to 0$, so $|f_k(x)| \to 0$ for each k. ☐

Example 4.14 (Hilbert Schmidt operators)**.**

Let $X = L^2(T, \mu)$, $Y = L^2(S, \nu)$, μ is a σ-finite measure on T. Then any functional f_k on X will have the form

$$f_k(x) = \int_T x(t) a_k(t) \, d\mu(t)$$

for some $a_k \in L^2(T, \mu)$. Then any finite rank linear bounded operator from X to Y has the form

$$(Ux)(s) = \sum_{k=1}^n \left(\int_T x(t) a_k(t) \, d\mu(t) \right) y_k(s) = \int_T \left(\sum_{k=1}^n y_k(s) a_k(t) \right) x(t) \, d\mu(t).$$

We call the function $K_n(s,t) = \sum_{k=1}^n y_k(s) a_k(t)$ the *degenerate kernel.*

Let now $K \in L^2(S \times T, \nu \times \mu)$. The operator

$$\begin{aligned} U : L^2(T, \mu) &\to L^2(S, \nu) \\ x(t) &\mapsto \int_T K(s,t) x(t) \, d\mu(t) \end{aligned}$$

is called Hilbert Schmidt operator. It is compact and $\|U\| \leq \|K\|_2$.

Proof. Note first that the step functions (those of the form $\sum_{k=1}^n a_k \chi_{A_k \times B_k}(s,t) = \sum_{k=1}^n a_k \chi_{A_k}(s) \chi_{B_k}(t)$) are dense in $L^2(S \times T)$. Assume K_n are step functions such that $K_n \to K$ in $L^2(S \times T)$. Let $U_n x(s) = \int_T K_n(s,t) x(t) \, d\mu(t)$. Since each K_n is the degenerate kernel, each U_n is a finite rank operator.

It remains to note that $\|U_n - U\| \leq \|K_n - K\|_2 \to 0$, therefore U is compact by Corollary 4.10. ☐

Section 3. **Fredholm alternative.** In this section, we will look at *spectrum* of a compact operator, i.e. those points $\lambda \in \mathbb{C}$ for which $T = (U - \lambda I)$ is not invertible. For this, we will study the properties of the operator T and see in what sense we can speak about its "invertibility".

In what follows, we denote by I the identity operator on a Banach space X.

Lemma 4.15. *Let X be a Banach space, $U : X \to X$ be a compact operator. If $T = I - U$, $(y_n) \subseteq T(X)$ is a bounded sequence, then there is a bounded sequence $(x_n) \subseteq X$ such that $Tx_n = y_n$.*

Proof. There is a sequence $(\widetilde{x}_n) \subseteq X$ such that $T\widetilde{x}_n = y_k$. Let $L = \ker T$ and $d_n = \operatorname{dist}(\widetilde{x}_n, L) = \inf_{z\in L} \|\widetilde{x}_n - z\|$. Since T is continuous, L is closed. Let $(z_n) \subseteq L$ be such that $\|\widetilde{x}_n - z_n\| \leq 2d_n$. Let $x_n = \widetilde{x}_n - z_n$. We know $Tx_n = y_n$ and $d_n \leq \|x_n\| \leq 2d_n$. Now we prove $(\|x_n\|)$ is bounded.

If $(\|x_n\|)$ is not bounded, we pass to its unbounded subsequence, so without loss of generality assume $\|x_n\| \to \infty$. Since $Tx_n = y_n$ we get

$$\frac{x_n}{\|x_n\|} - U\frac{x_n}{\|x_n\|} = \frac{y_n}{\|x_n\|} \to 0,$$

since (y_n) is bounded. Let $w_n = \frac{x_n}{\|x_n\|}$. Since (w_n) is bounded, there exists its subsequence such that Uw_{n_k} converges to some v. Then $\lim_k w_{n_k} = v$, and so $Uv = \lim_k Uw_{n_k} = v$. Then $Tv = v - Uv = 0$, so $v \in L$. This implies $\operatorname{dist}(w_{n_k}, L) \to 0$. This contradicts to the fact that for any $z \in L$

$$\|w_n - z\| = \left\| \frac{x_n}{\|x_n\|} - z \right\| = \frac{\left\| x_n - \|x_n\| z \right\|}{\|x_n\|} = \frac{\left\| \widetilde{x}_n - (z_n + \|x_n\| z) \right\|}{\|x_n\|} \geq \frac{d_n}{\|x_n\|} \geq \frac{1}{2}.$$

☐

Proposition 4.16. *Let X be a Banach space and $U : X \to X$ be a compact operator; let I denote the identity operator on X. Then*

1. *$(I - U)(X)$ is closed,*
2. *$\dim \ker(I - U) < \infty$.*

Proof. 1. Assume $y_n \in (I - U)(X)$ are such that $y_n \to y_0$. Since every convergent sequence is bounded we can apply Lemma 4.15 and find a bounded sequence (x_n) such that $(I - U)x_n = y_n$. Since U is a compact operator, there exists a subsequence (x_{n_k}) such that Ux_{n_k} converges. Then $x_{n_k} = Ux_{n_k} + y_{n_k}$ converges too. Let $x_0 = \lim_k x_{n_k}$. Then $x_0 = Ux_0 + y_0$ which implies $y_0 = (I - U)x_0 \in (I - U)(X)$.

2. Let $L = \ker(I - U)$, then $U|_L = I|_L$. Since restriction of a compact operator is a compact operator, we conclude $I|_L$ is compact. By Proposition 4.2 this implies L is finite-dimensional. $\square$

Proposition 4.17. *X is a Banach space, $U : X \to X$ is a compact operator, $T = I - U$. The following are equivalent:*

$$\begin{array}{lll} 1)\exists T^{-1} & 2)\ker T = \{0\} & 3) T \textit{ is surjective} \\ 1')\exists (T^*)^{-1} & 2')\ker T^* = \{0\} & 3') T^* \textit{ is surjective} \end{array}$$

Proof. It is clear that (1) $\Leftrightarrow$ (1′), (1) $\Leftrightarrow \begin{cases}(2)\\(3)\end{cases}$ and (1′) $\Leftrightarrow \begin{cases}(2')\\(3')\end{cases}$. We will show that (2) $\Rightarrow$ (3) $\Rightarrow$ (2′) $\Rightarrow$ (3′) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3): Assume T is not surjective so that $T(X) = X_1 \subsetneq X = X_0$. Then since T is injective we get $T(X_1) = X_2 \subsetneq T(X_0) = X_1$. If we continue in this way, we get a sequence $X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \dots$. By Proposition 4.16 these spaces are all closed and therefore $\exists x_n \in X_n$ such that $\|x_n\| = 1$ and $\operatorname{dist}(x_n, X_{n+1}) > 1/2$ (choose any $z_n \in X_n \setminus X_{n+1}$, then consider $y_n \in X_{n+1}$ such that $\|z_n - y_n\| < 2\operatorname{dist}(z_n, X_{n+1})$, let $x_n = (z_n - y_n)/\|z_n - y_n\|$).

We now show that (Ux_n) does not have a convergent subsequence. This will contradict to the fact that U is a compact operator and $(\|x_n\|)$ is bounded (see Proposition 4.2). Let $k > m$, then

$$Ux_k - Ux_m = (x_k - (x_k - Ux_k)) - (x_m - (x_m - Ux_m)) = -x_m + w,$$

where $w \in X_{m+1}$, since $x_k \in X_k \subseteq X_{m+1}$, $Tx_k \in X_{k+1} \subseteq X_{m+1}$ and $Tx_m \in X_{m+1}$. Then $\|Ux_k - Ux_m\| = \|-x_m + w\| = \|x_m - w\| \geq \operatorname{dist}(x_m, X_{m+1}) > 1/2$. This implies (Ux_n) cannot have a convergent subsequence.

(3) $\Rightarrow$ (2′): Assume $T^*g = 0$ and take any $y \in Y$. There exists $x \in X$ such that $Tx = y$. Then $g(y) = g(Tx) = (T^*g)(x) = 0$. This implies $g = 0$.

(2′) $\Rightarrow$ (3′): Note that $T^* = I_{X^*} - U^*$. Schauder Theorem (Theorem 4.7) implies that U^* is compact, therefore by (2) $\Rightarrow$ (3) the operator T^* is surjective.

(3′) $\Rightarrow$ (2): Assume $Tx = 0$ and take any $f \in X^*$. There exists $g \in Y^*$ such that $f = T^*g$. Then $f(x) = (T^*g)(x) = g(Tx) = g(0) = 0$. This implies $x = 0$. $\square$

Theorem 4.18 (Fredholm Alternative)**.** *Let X be a Banach space and $U : X \to X$ be a compact operator; $T = I - U$. Consider the following equations:*

$$(+) \qquad Tx = y$$

$$(++) \qquad T^*g = 0$$

1. *If* (++) *has only trivial solution, then* (+) *has a unique solution for every $y \in X$.*
2. *If* (++) *has a nontrivial solution, then* (+) *has a solution if and only if $g(y) = 0$ for every g, a solution of* (++)*.*

Proof. 1. If (++) has only trivial solution, then $\ker T^* = \{0\}$, therefore by Proposition 4.17 T is invertible and thus (+) has a unique solution $x = T^{-1}y$.

2. Assume (++) has a nontrivial solution. If x is a solution of (+) and $g \in Y^*$ is a solution of (++), then $g(y) = g(Tx) = (T^*g)x = 0$. If $g(y) = 0$ for all $g \in \ker T^*$ then y must be in $T(X)$. Indeed, assume $y \notin A = T(X)$. By Corollary 3.11 we can separate y from $T(X)$ by a real-valued real linear bounded functional f, i.e. there exists $g \in X^*$ such that $\mathrm{Re}g(y) > \mathrm{Re}g(Tx) = \mathrm{Re}(T^*g)(x)$ for every $x \in X$. This would be possible only if $\mathrm{Re}(T^*g) = 0$ which implies $T^*g = 0$. But then $g(y) = 0$, a contradiction. ☐

Definition. Let X, Y be Banach spaces, $A \in \mathcal{B}(X, Y)$. We say a complex number λ belongs to the *spectrum* of A if $A - \lambda I$ is not invertible. We say λ is an *eigenvalue* of A if there is a nonzero $x \in X$ such that $Ax = \lambda x$. The spectrum of A is denoted by $\sigma(A)$, the set of eigenvalues is denoted by $\sigma_p(A)$ (the *point spectrum*).

Theorem 4.19 (Spectral theorem of compact operators)**.** *Let X be a Banach space (over $\mathbb{C}$) and $U \in \mathcal{B}(X)$ be a compact operator. Then the following are valid.*

1. *If $\lambda \neq 0$ is an eigenvalue for U, then the multiplicity of λ, which is the dimension of* $\ker(U - \lambda I)$*, is finite.*
2. *If $\lambda \neq 0$ belongs to $\sigma(U)$, then λ is an eigenvalue.*
3. *If dim$(X) = \infty$, then $0 \in \sigma(U)$, i.e. U is not invertible.*
4. *The set $\sigma(U)$ is either finite or countable; in the latter case $\sigma(U) = \{\lambda_n\}$ with $\lambda_n \to 0$.*

Proof. 1. Let $N_\lambda = \ker(U - \lambda I)$. Since $\lambda \neq 0$, we conclude $N_\lambda = \ker(I - \lambda^{-1}U)$, by Proposition 4.16 this space is finite-dimensional as $\lambda^{-1}U$ is a compact operator.

2. If λ is not an eigenvalue then $\ker(U-\lambda I)=\{0\}$. Again note that $\ker(U-\lambda I)=\ker(I-\lambda^{-1}U)$ then Proposition 4.17 implies that the operator $I-\lambda^{-1}U$ is invertible. This is equivalent to the operator $U-\lambda I$ being invertible which contradicts $\lambda \in \sigma(U)$.

3. Assume U is invertible. Then since $I=U^{-1}U$ and U is compact, Proposition 4.8 implies that I is compact. Then by Proposition 4.2 we get that X is finite-dimensional.

4. It is enough to prove that for any $\delta>0$ the set of eigenvalues λ such that $|\lambda|>\delta$, is finite.

Assume this is not true and there is a $\delta>0$ and an infinite collection of eigenvalues $(\lambda_k)_{k\geq 1}$ with absolute value of each λ_k bigger than δ and all λ_k distinct. Assume (e_k) are corresponding eigenvectors, by L_n denote the linear span of $\{e_1,\dots,e_n\}$. From the Linear Algebra we know that $L_k \subsetneq L_{k+1}$ for every $k\geq 0$ ($L_0=\{0\}$). Therefore, for every $k\geq 0$, there exists $x_{k+1}\in L_{k+1}$ such that $\|x_{k+1}\|=1$ and $\operatorname{dist}(x_{k+1},L_k)>1/2$.

We show now that the sequence $(Ux_k)_{k\geq 1}$ has no convergent subsequence. This contradicts to the property of compact operator that every bounded sequence of points in X has a subsequence such that its images under U converge (see Proposition 4.2).

Note first that $Ux_k-\lambda_k x_k\in L_{k-1}$ for any $k\geq 1$, thus $Ux_k=\lambda_k x_k+w_k$, $w_k\in L_{k-1}$. Then for any $n>k$, we have

$$Ux_n-Ux_k=\lambda_n x_n+w_n-\lambda_k x_k-w_k=\lambda_n x_n-z_{n-1},$$

where $z_{n-1}\in L_{n-1}$, since $w_n\in L_{n-1}$, $\lambda_k x_k\in L_k\subseteq L_{n-1}$ and $w_k\in L_{k-1}\subseteq L_{n-1}$. Then

$$\|Ux_n-Ux_k\|=\|\lambda_n x_n-z_{n-1}\|=|\lambda_n|\left\|x_n-\frac{1}{\lambda_n}z_{n-1}\right\|\geq \delta/2.$$

□

Corollary 4.20. *If* $\dim(X)=\infty$ *and* U *is a compact operator on* X *then one and only one of the following possibilities occurs:*

(1) $\sigma(U)=\{0\}$

(2) $\sigma(U)=\{0,\lambda_1,\dots,\lambda_n\}=\{0\}\cup\sigma_p(U)$,

(3) $\sigma(U)=\{0,\lambda_1,\lambda_2,\dots\}=\{0\}\cup\sigma_p(U)$ *and* $\lim\lambda_k=0$,

where $\lambda_k\neq 0$ *for* $1\leq k$ *and* $\dim\ker(U-\lambda_k I)<\infty$.