

PROBLEM SHEET 4.

Due by the lecture on Friday 20th March 2009.

Exercise 1. * (Bonus exercise) Assume X, Y are Banach spaces and $T : X \rightarrow Y$ is a bounded linear operator. Assume further that $(x_n) \subset X$ and $x \in X$ are such that $x_n \xrightarrow{\text{weakly}} x$. Prove that

- (1) $Tx_n \xrightarrow{\text{weakly}} Tx$ in Y ;
- (2) If T is compact then $Tx_n \rightarrow Tx$ (converges in norm).

Exercise 2. Let X be a Banach space, $(x_n) \subset X$, $y_1, y_2 \in X$ and $(f_n) \subset X^*$, $g_1, g_2 \in X^*$.

- (1) Show that if $x_n \xrightarrow{\text{weakly}} y_i$, $i = 1, 2$, then $y_1 = y_2$. (The weak limit is unique.)
- (2) Show that if $f_n \xrightarrow{\text{weak}^*} g_i$, $i = 1, 2$, then $g_1 = g_2$. (The weak-star limit is unique.)
- (3) Show that if (x_n) converges weakly, then $(\|x_n\|)$ is bounded.
- (4) Show that if (f_n) converges weak-star, then $(\|f_n\|)$ is bounded.

Exercise 3. Let X be a Banach space; $(x_n) \subset X$, $x \in X$ and $(f_n) \subset X^*$, $f \in X^*$. Assume $f_n \xrightarrow{\text{weak}^*} f$ and $x_n \rightarrow x$. Prove that $f_n(x_n) \rightarrow f(x)$.

[Hint: You might find it useful to use one of the statements from the previous exercise.]

Exercise 4. This problem will use Alaoglu's Theorem to construct an element of $(\ell^\infty)^*$ that does not belong to ℓ^1 .

- (1) For each $n \in \mathbb{N}$, define $f_n : \ell^\infty \rightarrow \mathbb{C}$ by

$$f_n(x) = \frac{x_1 + \cdots + x_n}{n}, \quad x = (x_1, x_2, \dots) \in \ell^\infty.$$

Show that $f_n \in (\ell^\infty)^*$ and $\|f_n\| \leq 1$.

- (2) Use Alaoglu's Theorem to show that there exists an $f \in (\ell^\infty)^*$ equal to a weak-star limit of a subsequence of (f_n) .
- (3) Show that $f \neq \hat{a}$ for any $a \in \ell^1$, where \hat{a} is the image of a under the natural embedding of ℓ^1 into $(\ell^1)^{**} = (\ell^\infty)^*$.

Exercise 5. Let (e_n) be an orthonormal system in the Hilbert space H . Let $(a_n) \in \ell_2(\mathbb{R})$ be such that $a_n \geq 0$. Define $K \subset H$ to be the set of vectors that can be written $x = \sum_{n=1}^{\infty} b_n e_n$, where $b_n \in \mathbb{C}$, $|b_n| \leq a_n$. Show that the series converges indeed to an element $x \in H$. Then show that K is compact.