PROBLEM SHEET 3.

Due by the lecture on Friday 6th March 2009.

Exercise 1. (1) Show that the Inverse Mapping Theorem is not true for normed not complete spaces.

[Hint: Consider $X = Y = C[a,b]$ and equip one of the spaces with the $\|\cdot\|_\infty$-norm and the other with $\|\cdot\|_2$.]

(2) Assume X is a linear space. If $\|\cdot\|$ and $|\cdot|$ are two norms that make X complete and there exists $C > 0$ such that $\|x\| \le C|x|$ for every $x \in X$, then these norms are equivalent, i.e. there exists $C' > 0$ such that $|x| \le C'\|x\|$ for every $x \in X$.

Exercise 2. (1) Let $X = Y = L^2(\mathbb{R})$ and

$$L = \{x \in L^2(\mathbb{R})\colon \int_{-\infty}^{\infty} t^2|x(t)|^2 dt < \infty\}.$$

Prove that the linear operator $U : L \to L^2(\mathbb{R})$ defined by $Ux(t) = t \cdot x(t)$ has a closed graph but is not continuous.

[Hint: The following facts about convergence of functions in L^2 may be used without proof:

- If $f_n, f \in L^2(\mathbb{R})$ and $\|f_n - f\|_2 \to 0$ then for each $\varepsilon > 0$, the measure $\mu\{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon\}$ tends to 0 as $n \to \infty$. In other words, (f_n) converges to f *in measure.*
- If $f_n, f \in L^2(\mathbb{R})$ and (f_n) converges to f in measure, then there exists a subsequence f_{n_k} such that $f_{n_k}(x) \to f(x)$ for almost all $x \in X$.
- Fatou's Lemma.]

(2) With the same spaces X and Y and subspace M consisting of all functions from $L^2(\mathbb{R})$ with compact support, the operator $U : M \to L^2(\mathbb{R})$ defined by the same formula does not have a closed graph.

[For a function $\varphi : T \to \mathbb{R}$ its support suppφ is the closed set $\overline{\{x \in T\colon \varphi(t) \neq 0\}}$. In case $T = \mathbb{R}$, compact support is equivalent to "$\exists C > 0$ such that $\varphi(x) = 0$ for all $|x| > C$".]

Exercise 3. Let X be a TVS. Prove the following:

(1) If f is a linear functional on X, then $|f(x)|$ is a seminorm on X.

(2) If $\tau_{\mathcal{P}}$ is the topology defined by a family $\mathcal{P}$ of seminorms on X, then for each $p \in \mathcal{P}$, p is continuous in $\tau_{\mathcal{P}}$.

(3) If $p_1, \ldots, p_n$ are continuous seminorms, then $p_1(x) + \cdots + p_n(x)$ and $\max_i(p_i(x))$ are continuous seminorms.

(4) If p, q are two seminorms, such that

$$\{x\colon p(x) < 1\} \subseteq \{x\colon q(x) < 1\},$$

then $p(x) \ge q(x)$ for all $x \in X$.

Exercise 4. (1) If X is a normed space and $x_n \to x$ in norm, then $x_n \overset{weakly}{\longrightarrow} x$.

(2) If $X = \ell_2(\mathbb{N})$ and (e_n) is the standard basis of X, show that the sequence (e_n) weakly converges to 0.

Exercise 5. *(Bonus exercise.) Prove that in Hilbert space, $\|x_n - x_0\| \to 0$ if and only if

$$x_n \overset{weakly}{\longrightarrow} x_0 \quad \text{and} \quad \|x_n\| \to \|x_0\|.$$