**PROBLEM SHEET 1.**

*Due by the lecture on Friday 6th February 2009.*

**Exercise 1.** *(Bonus exercise.)

(1) The Principle of nested balls (Lemma 2.2) is in fact a characteristic property of completeness of metric spaces:
Prove that if $X$ is a metric space such that any nested sequence of closed balls $\overline{B}(x_i, r_i)$ such that $r_i \to 0$ has a non-empty intersection, then $X$ is complete.
(2) Show that if $r_i \not\to 0$ then there exists a complete space $X$ and a sequence of closed balls $\{\overline{B}(x_i, r_i)\}$ such that $\overline{B}(x_i, r_i) \supseteq \overline{B}(x_{i+1}, r_{i+1})$ for every $i \geq 1$ but the intersection $\bigcap \overline{B}(x_i, r_i)$ is empty.
(Hint: consider $X = \mathbb{N}$ and introduce metric, in which infinite rays $\mathbb{N} \cap [n, +\infty)$ are closed balls of <u>finite</u> radius.)
(3) Prove that if $X$ is a Banach space and a sequence of closed balls $\{\overline{B}(x_i, r_i)\}$ is such that for every $i \geq 1$, $\overline{B}(x_i, r_i) \supseteq \overline{B}(x_{i+1}, r_{i+1})$, then the intersection $\bigcap \overline{B}(x_i, r_i)$ is not empty even if $r_i \not\to 0$.

**Exercise 2.** Let $X$ be a topological space, $A, B, C, D \subseteq X$. Prove the following:

(1) If $A$ is nowhere dense, then its closure $\overline{A}$ is nowhere dense;
(2) If $A$ is nowhere dense, then its complement $X \setminus A$ is *everywhere dense*, which means it is dense in the whole space;
(3) If $A \subseteq B$ and $A$ is of 2nd category, then $B$ is of 2nd category;
(4) If $C \supseteq D$ and $C$ is of 1st category, then $D$ is of 1st category.

**Exercise 3.** Determine whether the following subsets of $[0,1]$ with the standard topology are of 1st or of 2nd category: $[0,1]$, $\mathbb{Q} \cap [0,1]$, $[0,1] \setminus \mathbb{Q}$, Cantor set. Prove your answers.

(In order to define a Cantor set, one starts by deleting the open middle third $(1/3, 2/3)$ from the interval $[0,1]$, leaving two line segments: $[0,1/3] \cup [2/3,1]$. Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $[0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$. This process is continued to infinity, the Cantor set contains all points in the interval $[0,1]$ that are not deleted at any step in this infinite process.)

**Exercise 4.** $X = [0,1]$, $\lambda$ is a Lebesgue measure. Prove that:

(1) for every $\varepsilon > 0$ there exists a set of 1st category of measure bigger than $1 - \varepsilon$;
(2) there exists $A \subseteq [0,1]$ of 1st category with $\lambda(A) = 1$;
(3) $[0,1]$ is of 2nd category;
(4) there exists $B \subseteq [0,1]$ of 2nd category with $\lambda(B) = 0$.

**Exercise 5.** Prove that the set of continuous nowhere differentiable real valued functions is dense in the space $C[0,1]$.

You can find all problem sheets, hints and other useful information on
http://web.mat.bham.ac.uk/~malevao/MSM4P13/