Lecture 5.

http://web.mat.bham.ac.uk/∼malevao/MSM3G06/

Hardy's inequality for series.

Theorem

If $p > 1$, $a_n \geq 0$ is a non-zero sequence and $A_n = a_1 + \cdots + a_n$, then

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p. \tag{1}$$

The constant is the best possible.

Hilbert's double series theorem.
If $\sum_{n=1}^{\infty} a_n^2 < \infty$, then the double series $\sum_{n,m=1}^{\infty} \frac{a_n a_m}{n+m}$ converges whenever $a_n \geq 0$ for all n.

Proof of Hardy's inequality for series.

Let $\Phi_n = \frac{1}{n^p} + \frac{1}{(n+1)^p} + \frac{1}{(n+2)^p} + \dots$.

$$\sum_{n=1}^{N}\Big(\frac{A_n}{n}\Big)^p \le \sum_{n=1}^{N}(A_n^p - A_{n-1}^p)\Phi_n.$$

$$A_n^p - A_{n-1}^p = p\xi^{p-1}a_n \le pA_n^{p-1}a_n$$

$$\Phi_n \le \frac{p}{p-1}\frac{1}{n^{p-1}}$$