Lecture 4.

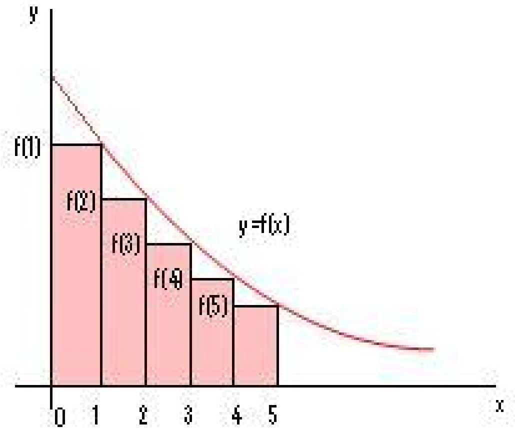
Carlson Inequality (1934).
Let (a_k) be a nonzero sequence of nonnegative numbers.

$$\left(\sum_{k=1}^{\infty} a_k\right)^4 < \pi^2 \left(\sum_{k=1}^{\infty} a_k^2\right)\left(\sum_{k=1}^{\infty} k^2 a_k^2\right) \qquad (1)$$

Proof.
Let λ be any positive number. For each $k \geq 1$, write

$$b_k = \frac{1}{\sqrt{\lambda + \frac{1}{\lambda}k^2}}, \qquad c_k = a_k\sqrt{\lambda + \frac{1}{\lambda}k^2}.$$

Then $a_k = b_k c_k$ for each $k \geq 1$.

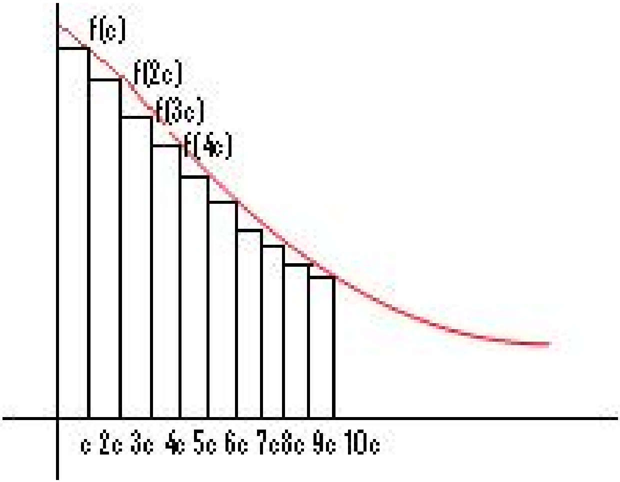


$$\int_0^{\infty} f(x)dx > \sum_{k=1}^{\infty} f(k).$$

Carlson inequality for functions

If $f : [0, +\infty) \to [0, +\infty)$ is integrable then

$$\left(\int_0^{\infty} f(x) dx \right)^4 \leq \pi^2 \int_0^{\infty} f^2(x) dx \int_0^{\infty} x^2 f^2(x) dx.$$



f(c)
f(2c)
f(3c)
f(4c)
c 2c 3c 4c 5c 6c 7c 8c 9c 10c