THE HISTORY AND CONTEXT OF MATHEMATICS, SPRING 2009: THE LIFE AND MATHEMATICS OF G.H. HARDY

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1. Mathematical preliminaries

1.1. **Complex exponentials.**

$$\int_{-\pi}^{\pi} e^{inx}dx = \begin{cases} 2\pi, & \text{if} n = 0, \\ 0, & \text{if} n = \pm 1, \pm 2, \dots . \end{cases} \tag{1.1}$$

Indeed, notice that if $n = 0$, then

$$\int_{-\pi}^{\pi} e^{inx}dx = \int_{-\pi}^{\pi} dx = 2\pi.$$

If $n = \pm 1, \pm 2, \dots$, then

$$\int_{-\pi}^{\pi} e^{inx}dx = \frac{e^{inx}}{in}|_{-\pi}^{\pi} = \frac{e^{in\pi} - e^{-in\pi}}{in} = 0,$$

since by the Euler's formula

$$e^{in\pi} = \cos n\pi + i \sin n\pi = \cos n\pi \quad \text{and} \quad e^{-in\pi} = \cos n\pi - i \sin n\pi = \cos n\pi.$$

1.2. **Mean Value Theorem.** Let $a < b$ be real numbers. Let f be a continuous function in the interval $[a, b]$ such that f is differentiable in (a, b). Then there exists $\xi \in (a, b)$ such that

$$f(b) - f(a) = f'(\xi)(b - a).$$

1.3. **Young's inequality.**

Proposition 1.1. *Assume $\alpha \in (0, 1)$ and $a, b > 0$. Then*

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1 - \alpha)b \tag{1.2}$$

and the equality is achieved if and only if $a = b$.

Proof. We may assume without loss of generality that $b > a$. Then by the mean value theorem applied to the function $f(x) = x^{1-\alpha}$, we have

$$b^{1-\alpha} - a^{1-\alpha} = (1 - \alpha)\xi^{-\alpha}(b - a) \qquad \text{for some} \qquad a < \xi < b.$$

Notice that since $\alpha > 0$ the function $\xi^{-\alpha}$ is decreasing in $\xi > 0$, so

$$b^{1-\alpha} - a^{1-\alpha} < (1 - \alpha)a^{-\alpha}(b - a).$$

Multiplying both sides by a^{α}, it easily follows that

$$b^{1-\alpha}a^{\alpha} < \alpha a + (1 - \alpha)b.$$

Finally notice that if $a = b$, then (1.2) is an identity, and the previous argument shows that if $a \neq b$, then the inequality (1.2) is strict. Hence, we conclude that equality in (1.2) is achieved if and only if $a = b$. □

Remark 1.2. Writing $p = 1/\alpha$ and replacing a^{α} by a' and $b^{1-\alpha}$ by b', we observe that (1.2) may be written as follows:

Proposition 1.3 (Young's inequality)**.** *If $p > 1$, p' is such that*

$$\frac{1}{p} + \frac{1}{p'} = 1$$

(that is, $p' = p/(p-1)$), and a and b are positive real numbers, then

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \tag{1.3}$$

where equality is achieved if and only if $a^p = b^{p'}$.

1.4. **Hölder's Inequality.**

Theorem 1.4. *If $p > 1$ and p' is such that $1/p + 1/p' = 1$ (that is, $p' = p/(p-1)$), and $a_1, \ldots, a_N$ and $b_1, \ldots, b_N$ are positive real numbers, then*

$$\sum_{n=1}^{N} a_n b_n \leq \left(\sum_{n=1}^{N} a_n^p\right)^{(1/p)} \left(\sum_{n=1}^{N} b_n^{p'}\right)^{(1/p')}. \tag{1.4}$$

There is strict inequality unless (a_n^p) and $(b_n^{p'})$ are proportional, that is unless $a_n^p/b_n^{p'}$ is independent of n.

Remark 1.5. (1) If $p = p' = 2$, Hölder's inequality becomes Cauchy-Schwartz inequality:

$$(\sum_{n=1}^{N} a_n b_n)^2 \leq \left(\sum_{n=1}^{N} a_n^2\right) \left(\sum_{n=1}^{N} b_n^2\right).$$

(2) By letting $N \to \infty$, it is easy to see that (1.4) extends to infinite sums, i.e.

$$\sum_{n=1}^{\infty} a_n b_n \leq \left(\sum_{n=1}^{\infty} a_n^p\right)^{(1/p)} \left(\sum_{n=1}^{\infty} b_n^{p'}\right)^{(1/p')},$$

for $p > 1$ and p' such that $1/p + 1/p' = 1$.

Proof of Hölder's inequality. Writing $\alpha = 1/p$ (and thus $1 - \alpha = 1/p'$) and defining a_n' and b_n' in terms of a_n and b_n as follows

$$a_n = (a_n')^\alpha \qquad b_n = (b_n')^{1-\alpha},$$

we notice that proving (1.4) is equivalent to proving that

$$\sum_{n=1}^{N} (a_n')^\alpha (b_n')^{1-\alpha} \leq \left(\sum_{n=1}^{N} a_n'\right)^\alpha \left(\sum_{n=1}^{N} b_n'\right)^{1-\alpha}. \tag{1.5}$$

There is strict inequality unless the sequence (a_n') is proportional to (b_n').

We will continue by proving (1.5).

`Case A.` We first prove the result in the case when $\sum_{n=1}^N a_n' = \sum_{n=1}^N b_n' = 1$. By using the inequality (1.2) in the Preliminaries, we have

$$\sum_{n=1}^{N} (a_n')^\alpha (b_n')^{1-\alpha} \leq \sum_{n=1}^{N} (\alpha a_n' + (1-\alpha) b_n') = \alpha \sum_{n=1}^{N} a_n' + (1-\alpha) \sum_{n=1}^{N} b_n'$$

$$= \alpha \cdot 1 + (1-\alpha) \cdot 1 = 1 = \left(\sum_{n=1}^{N} a_n'\right)^\alpha \left(\sum_{n=1}^{N} b_n'\right)^{1-\alpha}.$$

Here there is strict inequality unless $a_n' = b_n'$ for all n.

`Case B.` We will prove the general case using the result in `Case A.` To this end, let $\sum_{n=1}^N a_n' = A$ and $\sum_{n=1}^N b_n' = B$. Define

$$\tilde{a}_n = \frac{a_n'}{A} \qquad \text{and} \qquad \tilde{b}_n = \frac{b_n'}{B}.$$

Then, using $\sum_{n=1}^N \tilde{a}_n = \sum_{n=1}^N \tilde{b}_n = 1$ and `Case A`, we have

$$\begin{aligned}\sum_{n=1}^N (a_n')^\alpha (b_n')^{1-\alpha} &= \sum_{n=1}^N \left(\frac{a_n'}{A}A\right)^\alpha \left(\frac{b_n'}{B}B\right)^{1-\alpha} \\ &= A^\alpha B^{1-\alpha} \sum_{n=1}^N \tilde{a}_n^\alpha \ \tilde{b}_n^{1-\alpha} \\ &\leq A^\alpha B^{1-\alpha} \left(\sum_{n=1}^N \tilde{a}_n\right)^\alpha \left(\sum_{n=1}^N \tilde{b}_n\right)^{1-\alpha} \\ &= A^\alpha B^{1-\alpha} = \left(\sum_{n=1}^N a_n'\right)^\alpha \left(\sum_{n=1}^N b_n'\right)^{1-\alpha}.\end{aligned}$$

Notice that there is strict inequality unless $\tilde{a}_n = \tilde{b}_n$ for all n, that is unless (a_n') is proportional to (b_n'). This concludes the proof.

□

2. The Hardy-Littlewood Majorant Problem [HL, 1935]

2.1. **Motivation.** We will study here a conjecture stated by Hardy and Littlewood in 1935 (see [HL]). Although their original work concerns so-called "majorants" of Fourier series, here we will consider the (equivalent) analogue for trigonometric polynomials.

This problem can be seen as an example of the influence of Hardy and Littlewood's work on future generations of mathematicians.

2.2. **Statement of the problem.**

Definition 2.1. A function $f : [-\pi, \pi] \to \mathbb{C}$ is a *trigonometric polynomial* if

$$f(x) = \sum_{|n| \leq N} a_n e^{inx},$$

for some $a_n \in \mathbb{C}$ and $N \in \mathbb{N}$.

Definition 2.2. Given two trigonometric polynomials

$$f(x) = \sum_{|n| \leq N} a_n e^{inx} \qquad \text{and} \qquad F(x) = \sum_{|n| \leq N} A_n e^{inx}$$

for which

$$|a_n| \leq A_n \qquad \forall |n| \leq N,$$

we say F is a *majorant* of f.

Following Hardy and Littlewood, we consider relations of inequality between

$$\int_{-\pi}^{\pi} |f(x)|^q dx \qquad \text{and} \qquad \int_{-\pi}^{\pi} |F(x)|^q dx,$$

where $q \geq 2$, f is a trigonometric polynomial and F is a majorant of f.

There are special cases in which it is easy to establish the inequality. Hardy and Littlewood [HL] showed that

Proposition 2.3. *If $q = 2k$ for some $k \in \mathbb{N}$, then*

$$\int_{-\pi}^{\pi} |f(x)|^q dx \leq \int_{-\pi}^{\pi} |F(x)|^q dx, \tag{2.1}$$

where f is a trigonometric polynomial and F is a majorant of f.

Proof. First observe that if $f(x) = \sum_{|n|\leq N} a_n e^{inx}$, then

$$\int_{-\pi}^{\pi} |f(x)|^{2k} dx = \int_{-\pi}^{\pi} \left| \sum_{|n|\leq N} a_n e^{inx} \right|^{2k} dx = \int_{-\pi}^{\pi} \left(\sum_{|n|\leq N} a_n e^{inx} \overline{\sum_{|n|\leq N} a_n e^{inx}} \right)^k dx$$

$$= \int_{-\pi}^{\pi} \left(\sum_{|n|\leq N} a_n e^{inx} \sum_{|n|\leq N} \overline{a}_n e^{-inx} \right)^k dx$$

$$= \int_{-\pi}^{\pi} \left(\sum_{|n|\leq N} a_n e^{inx} \right)^k \left(\sum_{|n|\leq N} \overline{a}_n e^{-inx} \right)^k dx$$

$$= \int_{-\pi}^{\pi} \sum_{\substack{n_1,\dots,n_k \\ |n_j|\leq N}} a_{n_1} \cdots a_{n_k} e^{i(n_1+\cdots+n_k)x} \sum_{\substack{m_1,\dots,m_k \\ |m_j|\leq N}} \overline{a}_{m_1} \cdots \overline{a}_{m_k} e^{-i(m_1+\cdots+m_k)x} dx$$

$$= \sum_{n_1,\dots,n_k} \sum_{m_1,\dots,m_k} a_{n_1} \cdots a_{n_k} \overline{a}_{m_1} \cdots \overline{a}_{m_k} \int_{-\pi}^{\pi} e^{i(n_1+\cdots+n_k-m_1-\cdots-m_k)x} dx$$

$$= 2\pi \sum_{n_j,m_j: n_1+\cdots+n_k = m_1+\cdots+m_k} a_{n_1} \cdots a_{n_k} \overline{a}_{m_1} \cdots \overline{a}_{m_k},$$

where we have used the identity (1.1) in the Preliminaries in obtaining the last identity.

Thus,

$$\int_{-\pi}^{\pi} |f(x)|^{2k} dx = 2\pi \sum_{n_j,m_j: n_1+\cdots+n_k = m_1+\cdots+m_k} a_{n_1} \cdots a_{n_k} \overline{a}_{m_1} \cdots \overline{a}_{m_k}. \tag{2.2}$$

Next suppose that $f(x) = \sum_{|n|\leq N} a_n e^{inx}$ and $F(x) = \sum_{|n|\leq N} A_n e^{inx}$ for which

$$|a_n| \leq A_n \qquad \forall |n| \leq N \tag{2.3}$$

(that is to say, F is a majorant of f). Then, from (2.2) and (2.3), we get

$$\int_{-\pi}^{\pi} |f(x)|^{2k} dx = \left| 2\pi \sum_{n_j,m_j: n_1+\cdots+n_k = m_1+\cdots+m_k} a_{n_1} \cdots a_{n_k} \overline{a}_{m_1} \cdots \overline{a}_{m_k} \right|$$

$$\leq 2\pi \sum_{n_j,m_j: n_1+\cdots+n_k = m_1+\cdots+m_k} |a_{n_1}| \cdots |a_{n_k}| |\overline{a}_{m_1}| \cdots |\overline{a}_{m_k}|$$

$$\leq 2\pi \sum_{n_j,m_j: n_1+\cdots+n_k = m_1+\cdots+m_k} A_{n_1} \cdots A_{n_k} A_{m_1} \cdots A_{m_k}$$

$$= 2\pi \sum_{n_j,m_j: n_1+\cdots+n_k = m_1+\cdots+m_k} A_{n_1} \cdots A_{n_k} \overline{A}_{m_1} \cdots \overline{A}_{m_k}$$

$$= \int_{-\pi}^{\pi} |F(x)|^{2k} dx.$$

□

3. Carlson inequality, [HLP, 1952]

In [R], R. Rado, one of Hardy's former students, wrote:
"Hardy's interest in inequalities extended over the greater part of his mathematical career. Some of the investigations in this field were undertaken with a view to definite applications in other branches of mathematics, for instance, in the theory of functions or in the theory of Fourier series, but in most cases the problems were studied for their own sake. As a result of his work the subject was changed radically, and what had previously been a collection of isolated formulae became a systematic discipline."

The inequality we discuss in this section is named after Swedish professor Fritz Carlson, who proved in 1934 that

$$(\sum_{k=1}^{\infty} a_k)^4 < \pi^2 \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} k^2 a_k^2 \tag{3.1}$$

holds for any non-zero sequence (a_k) of non-negative numbers. By saying the sequence (a_n) is non-zero we mean that not all a_n are equal to 0.

However in 1936 Hardy presented two elementary proofs of the Carlson inequality. This had the greatest impact on other authors of the matter. We present one of the proofs here.

Remark 3.1 (Carlson inequality for functions)**.** There is an analogous inequality for integrals: For any nonnegative integrable function $f : [0, +\infty) \to [0, +\infty)$ the following inequality holds:

$$\left(\int_0^\infty f(x)dx\right)^4 \leq \pi^2 \int_0^\infty f^2(x)dx \int_0^\infty x^2 f^2(x)dx.$$

Hardy's proof of Carlson inequality. Let λ be any positive number. For each $k \geq 1$, write

$$b_k = \frac{1}{\sqrt{\lambda + \frac{1}{\lambda}k^2}}, \qquad c_k = a_k \sqrt{\lambda + \frac{1}{\lambda}k^2}.$$

Then $a_k = b_k c_k$ for each $k \geq 1$.

Now let us apply the Cauchy-Schwartz inequality (see Preliminaries, Remark 1.5) to the sum $\sum_{k=1}^{\infty} a_k$. We get

$$\left(\sum_{k=1}^{\infty} a_k\right)^2 = \left(\sum_{k=1}^{\infty} b_k c_k\right)^2 \leq \left(\sum_{k=1}^{\infty} b_k^2\right)\left(\sum_{k=1}^{\infty} c_k^2\right).$$

Note that

$$\sum_{k=1}^{\infty} c_k^2 = \left(\sum_{k=1}^{\infty} a_k^2(\lambda + \lambda^{-1}k^2)\right) = \lambda\left(\sum_{k=1}^{\infty} a_k^2\right) + \frac{1}{\lambda}\left(\sum_{k=1}^{\infty} k^2 a_k^2\right) = \lambda S + \frac{1}{\lambda}T,$$

where

$$S = \sum_{k=1}^{\infty} a_k^2 \qquad \text{and} \qquad T = \sum_{k=1}^{\infty} k^2 a_k^2.$$

The sum of the series $\sum_{k=1}^{\infty} b_k^2$ can be estimated as follows:

$$\sum_{k=1}^{\infty} b_k^2 = \sum_{k=1}^{\infty} \frac{1}{\lambda + \lambda^{-1}k^2} = \sum_{k=1}^{\infty} f(k),$$

where $f(x) = \dfrac{1}{\lambda + \lambda^{-1}x^2}$ for every $x \geq 0$.

The function $f(x) = \frac{1}{\lambda+\lambda^{-1}x^2} : [0, +\infty) \to \mathbb{R}$ is strictly decreasing, therefore (see picture above)

$$\sum_{k=1}^{\infty} f(k) < \int_0^{\infty} f(x)dx.$$

Calculating the integral:

$$\int_0^{\infty} f(x)dx = \int_0^{\infty} \frac{dx}{\lambda+\lambda^{-1}x^2} = \frac{1}{\lambda}\int_0^{\infty} \frac{dx}{1+(x/\lambda)^2} \underset{u=x/\lambda}{=} \int_0^{\infty} \frac{du}{1+u^2} = \arctan u\Big|_0^{\infty} = \pi/2.$$

Since $\lambda > 0$ and $S, T > 0$ (because (a_n) is a non-zero sequence), this implies

$$\left(\sum_{k=1}^{\infty} a_k\right)^2 < \frac{\pi}{2}\left(\lambda S + \frac{1}{\lambda}T\right).$$

Assume now that a non-zero sequence of non-negative numbers (a_n) is fixed. Then $S, T > 0$ and so we can let λ be equal $\sqrt{T/S}$. Then

$$\left(\sum_{k=1}^{\infty} a_k\right)^2 < \pi\sqrt{ST} = \pi\sqrt{\left(\sum_{k=1}^{\infty} a_k^2\right)\left(\sum_{k=1}^{\infty} k^2 a_k^2\right)},$$

therefore,

$$\left(\sum_{k=1}^{\infty} a_k\right)^4 < \pi^2\left(\sum_{k=1}^{\infty} a_k^2\right)\left(\sum_{k=1}^{\infty} k^2 a_k^2\right).$$

□

Remark 3.2 (Not examinable)**.** The constant π^2 in Carlson's inequality (3.1) is the best.

Proof. Let $c > 0$ be arbitrary, consider

$$a_n = \frac{1}{1+c^2n^2}$$

and denote

$$A = \sum_{k=1}^{\infty} a_k, \qquad S = \sum_{k=1}^{\infty} a_k^2, \qquad T = \sum_{k=1}^{\infty} k^2 a_k^2.$$

Then

$$A - S = \sum_{k=1}^{\infty}(a_k - a_k^2) = \sum_{k=1}^{\infty} a_k(1-a_k) = \sum_{k=1}^{\infty}\left(\frac{1}{1+c^2k^2}\cdot\frac{c^2k^2}{1+c^2k^2}\right) = c^2T.$$

This implies $A = c^2T + S$, therefore,

$$\frac{A}{\sqrt{ST}} = \frac{c^2T + S}{\sqrt{ST}} = c\left(c\sqrt{\frac{T}{S}} + c^{-1}\sqrt{\frac{S}{T}}\right) = c(x + x^{-1}) \geq 2c$$

and so

$$\frac{(A)^2}{\sqrt{ST}} \geq 2cA = 2c\sum_{k=1}^{\infty} a_k = 2\sum_{k=1}^{\infty} \frac{c}{1 + c^2k^2}$$

Note that as $c \to 0$, the expression in the RHS tends to $2\int_0^\infty \frac{dx}{1+x^2} = 2\frac{\pi}{2} = \pi$. Therefore the value of

$$\frac{(A)^2}{\sqrt{ST}}$$

may become arbitrarily close to π; so the value of $\frac{(A)^4}{ST}$ may become arbitrarily close to π^2. □

4. Hardy's inequality for series [H, 1920]

This section is devoted to stating and proving Hardy's inequality for series, also referred to as *the discrete Hardy inequality.*

4.1. **Statement of Hardy's inequality for series.**
Hardy's inequality for series reads (see [HLP]):

Theorem 4.1. *If $p > 1$, $a_n \geq 0$ is a non-zero sequence and $A_n = a_1 + \cdots + a_n$, then*

$$\sum_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p. \tag{4.1}$$

The constant is the best possible.

Roughly speaking, Hardy's inequality for series says that we can 'control' the infinite sum of the pth powers of the arithmetic means of the a_n's by the infinite sum of the pth powers of the a_n's themselves.

4.2. **Hardy's original motivation: Hilbert's double series theorem.**
Hardy's inequality for series was discovered in the course if trying to simplify the known proofs of a result on double series of positive numbers due to David Hilbert (see [Hi]) from the early 1900's. In its most simple form *Hilbert double series theorem* states that

If $\sum_{n=1}^{\infty} a_n^2 < \infty$, then the double series $\sum_{n,m=1}^{\infty} \frac{a_n a_m}{n+m}$ converges whenever $a_n \geq 0$ for all n.

At the time Hardy started his research several different proofs had been published but, quoting Hardy [H], "none of these proofs is as simple and elementary as might be desired". In 1920 Hardy observed that *Hilbert's theorem* is an immediate consequence of the discrete Hardy's inequality (i.e., Hardy's inequality for series). The argument is as follows:

We divide the double series into two parts, S_1 and S_2, about the "diagonal", and consider the part S_1 in which $m \leq n$. Then

$$\begin{aligned} S_1 &= \sum\sum_{m\leq n} \frac{a_n a_m}{n+m} \leq \sum\sum_{m\leq n} \frac{a_n a_m}{n} \\ &= \sum_{n=1}^{\infty} \frac{a_n}{n}\sum_{m=1}^{n} a_m = \sum_{n=1}^{\infty} a_n \frac{A_n}{n} \leq \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^2\right)^{1/2}, \end{aligned}$$

where we used the Hölder inequality (see Preliminaries, Theorem 1.4) in obtaining the last inequality.

Now, observe that, by hypothesis, $\sum_{n=1}^{\infty} a_n^2$ is convergent. On the other hand, the convergence of $\displaystyle\sum_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^2$ follows from that of $\displaystyle\sum_{n=1}^{\infty} a_n^2$ by using Hardy's inequality for series (see Theorem 4.1). Hence S_1 is convergent.

The convergence of $S_2 = \displaystyle\sum\sum_{m>n}\frac{a_n a_m}{n+m}$ can be proved in the same way. We leave details to the reader.

4.3. **Hardy's original proof** [H, 1920]**.** In this subsection we will give Hardy's original proof of his inequality for series. Some remarks will be given at the end of the proof.

Proof. Let $A_0 = 0$,

$$A_n = a_1 + \cdots + a_n \qquad \text{and} \qquad \Phi_n = \frac{1}{n^p} + \frac{1}{(n+1)^p} + \frac{1}{(n+2)^p} + \cdots.$$

Then

$$\begin{aligned}\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p &= \sum_{n=1}^{N} A_n^p(\Phi_n - \Phi_{n+1}) \\ &= \sum_{n=1}^{N}(A_n^p - A_{n-1}^p)\Phi_n + \sum_{n=1}^{N}(A_{n-1}^p\Phi_n - A_n^p\Phi_{n+1}).\end{aligned}$$

Notice that

$$\begin{aligned}&\sum_{n=1}^{N}(A_{n-1}^p\Phi_n - A_n^p\Phi_{n+1}) \\ &\quad = (A_0^p\Phi_1 - A_1^p\Phi_2) + (A_1^p\Phi_2 - A_2^p\Phi_3) + \cdots + (A_{N-1}^p\Phi_N - A_N^p\Phi_{N+1}) \\ &\quad = -A_N^p\Phi_{N+1} \le 0.\end{aligned}$$

Therefore,

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p = \sum_{n=1}^{N}(A_n^p - A_{n-1}^p)\Phi_n - A_N^p\Phi_{N+1} \le \sum_{n=1}^{N}(A_n^p - A_{n-1}^p)\Phi_n, \tag{4.2}$$

since $A_N^p\Phi_{N+1} \ge 0$.

Next, by the mean-value theorem (see Preliminaries), if $f(x) = x^p$, $p > 1$, then

$$f(A_n) - f(A_{n-1}) = f'(\xi)(A_n - A_{n-1}) = p\xi^{p-1}a_n$$

for some $\xi \in (A_{n-1}, A_n)$. Since the function ξ^{p-1} is an increasing function of ξ for $p > 1$, from the above identity we get

$$f(A_n) - f(A_{n-1}) \le pA_n^{p-1}a_n. \tag{4.3}$$

Now observe also that

$$\begin{aligned}&\frac{1}{n^p} + \frac{1}{(n+1)^p} + \frac{1}{(n+2)^p} + \cdots < \frac{1}{n^p} + \int_n^{\infty}\frac{dx}{x^p} \\ &= \frac{1}{n^p} + \frac{1}{p-1}\frac{1}{n^{p-1}} \le \frac{1}{n^{p-1}} + \frac{1}{p-1}\frac{1}{n^{p-1}} = \frac{p}{p-1}\frac{1}{n^{p-1}}.\end{aligned}$$

The first inequality follows from the fact that the function $f(x) = \frac{1}{x^p} : (0, +\infty) \to \mathbb{R}$ is strictly decreasing (see picture on page 6). Therefore,

$$\Phi_n = \frac{1}{n^p} + \frac{1}{(n+1)^p} + \frac{1}{(n+2)^p} + \cdots < \frac{p}{p-1}\frac{1}{n^{p-1}}. \tag{4.4}$$

Therefore, from (4.2), (4.3) and (4.4) we get

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p \le \sum_{n=1}^{N}(A_n^p - A_{n-1}^p)\Phi_n \le \sum_{n=1}^{N}(pa_nA_n^{p-1})\frac{p}{p-1}\frac{1}{n^{p-1}}$$

$$= \frac{p^2}{p-1}\sum_{n=1}^{N}a_n\left(\frac{A_n}{n}\right)^{p-1}. \tag{4.5}$$

Now, by using the Hölder's inequality, followed by Proposition 1.3 (see Preliminaries), with

$$a = \frac{p^2}{p-1}\left(\sum_{n=1}^{N}a_n^p\right)^{1/p} \qquad \text{and} \qquad b = \left(\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p\right)^{(p-1)/p},$$

from the above inequality it is easy to see that

$$\begin{aligned}\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p &\le \frac{p^2}{p-1}\sum_{n=1}^{N}a_n\left(\frac{A_n}{n}\right)^{p-1}\\ &\le \frac{p^2}{p-1}\left(\sum_{n=1}^{N}a_n^p\right)^{1/p}\left(\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^{(p-1)p'}\right)^{1/p'}\\ &= \frac{p^2}{p-1}\left(\sum_{n=1}^{N}a_n^p\right)^{1/p}\left(\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^{p}\right)^{(p-1)/p}\\ &\le \left(\frac{p^2}{p-1}\right)^p\frac{1}{p}\left(\sum_{n=1}^{N}a_n^p\right) + \frac{p-1}{p}\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p.\end{aligned}$$

(We used that $p' = p/(p-1)$.) From this we obtain

$$\left(1-\frac{p-1}{p}\right)\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p \le \left(\frac{p^2}{p-1}\right)^p\frac{1}{p}\left(\sum_{n=1}^{N}a_n^p\right),$$

or, equivalently,

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p \le \left(\frac{p^2}{p-1}\right)^p\sum_{n=1}^{N}a_n^p. \tag{4.6}$$

□

Remark 4.2 (see [HLP]). (1) Hardy was unable to fix the constant in (4.6). Notice how the constant obtained by Hardy is in general larger than the one stated in Theorem 4.1. This "imperfection" was fixed by Landau in 1926.

(2) There is an analogous inequality for integrals, usually referred to in the literature as the *continuous Hardy inequality* or *Hardy's inequality for integrals*:

$$\text{If } p \ge 1,\ f(x) \ge 0 \text{ and } F(x) = \int_0^x f(t)dt, \text{ then for nonzero } f,$$

$$\int_0^\infty\left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p\int_0^\infty f^p(x)dx.$$

The constant is the best possible.

(3) Many alternative proofs of Hardy's inequality, for both the discrete and continuous versions have been given by various authors, for example, Broadbent (1928), Knopp (1928), see [KMP].

(4) Hardy's inequalities, both for series and for integrals, have been generalised considerably and applied many times in analysis and in the theory of differentiable equations.

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