Lecture 3.

Definition 1. A function $f : [-\pi, \pi] \to \mathbb{C}$ is a *trigonometric polynomial* if

$$f(x) = \sum_{|n| \leq N} a_n e^{inx},$$

for some $a_n \in \mathbb{C}$ and $N \in \mathbb{N}$.

Definition 2. Given two trigonometric polynomials

$$f(x) = \sum_{|n| \leq N} a_n e^{inx} \qquad \text{and} \qquad F(x) = \sum_{|n| \leq N} A_n e^{inx}$$

for which

$$|a_n| \leq A_n \qquad \text{for all} \quad |n| \leq N$$

we say F is a *majorant* of f.

Proposition.
If $q = 2k$ for some $k \in \mathbb{N}$, then

$$\int_{-\pi}^{\pi} |f(x)|^q dx \leq \int_{-\pi}^{\pi} |F(x)|^q dx, \tag{1}$$

where f is a trigonometric polynomial and F is a majorant of f.

$$f(x) = 1 + re^{ix} - r^3 e^{3ix} \qquad \text{and} \qquad F(x) = 1 + re^{ix} + r^3 e^{3ix}$$

Then the reverse inequality holds for $q = 3$ when r is small:

$$\int_{-\pi}^{\pi} |f(x)|^q dx > \int_{-\pi}^{\pi} |F(x)|^q dx.$$

Hardy Littlewood Majorant Problem

Conjecture (Hardy-Littlewood)
For every $q \neq 2k$ there exists B_q such that if f is a trigonometric polynomial and F is its majorant then

$$\int_{-\pi}^{\pi} |f(x)|^q dx \leq B_q \int_{-\pi}^{\pi} |F(x)|^q dx.$$

Answer. Bachelis disproved Hardy and Littlewood's conjecture in 1973 by showing $B_q = \infty$ whenever $q/2 \notin \mathbb{N}$.