MSM2P01 SEQUENCES AND SERIES, EXERCISE SHEET 3.

Due by 12:00 Monday 23th February 2009.

(1) A sequence (a_n) is given by $a_1 = 2$,

$$a_{n+1} = \frac{a_n + \frac{2}{a_n}}{2}.$$

(a) Prove that

$$a_{n+1}^2 - 2 = \frac{1}{4}\frac{(a_n^2 - 2)^2}{a_n^2}$$

for all n.

(b) Deduce that $a_n > \sqrt{2}$ for all n.

(c) Using induction, show that

$$a_n^2 - 2 \leq \frac{8}{4^n}$$

for each $n \geq 1$.

(d) Using $a_n^2 - 2 = (a_n - \sqrt{2})(a_n + \sqrt{2})$ deduce that $a_n \to \sqrt{2}$ as $n \to \infty$.

(2) Suppose in each case that (a_n) is a sequence whose nth term is given by the expressions below. Prove that (a_n) converges and find the limit. If you use the continuity of any standard functions in your answer, state which function(s) you use that you know to be continuous and at what value $l \in \mathbb{R}$ or $(l, m) \in \mathbb{R}^2$ the continuity of that(those) function(s) is required.

(a) $\frac{7n-1}{\sqrt{n^2+n+18}}$; (b) $\frac{n^2-3}{\sqrt{n^5+7}}$; (c) $\frac{(n+1)^3+(1-n)^3}{(n+1)(1-n)}$, $n \geq 2$; (d) $\frac{2^n+1}{5^n-2}$.

(3) Prove that the following sequences (a_n) do not converge, using the following method.

Suppose first that the sequence in question converges to some $l \in \mathbb{R}$. Then find convergent sequences $(b_n), (c_n), \ldots$, and continuous functions $f, g, \ldots$ such that some expression involving $f, g, \ldots, (a_n), (b_n), (c_n), \ldots$ evaluates to a sequence that you in fact know to be non-convergent.

The convergence or nonconvergence of any sequences you use must be justified either by stating that they are "standard" convergent/nonconvergent sequences discussed in lectures (such as $\frac{1}{n}$ or $(-1)^n$) or by giving a proof.

(a) $(-1)^n(1-2^{-n})+1$; (b) $(-1)^n(2^{-n})+\frac{n^2}{1+n}$; (c) $\frac{1}{2}(\cos(n)+(-1)^n\frac{1}{n})$; (d) $\frac{\sin(n)+n\cos(n)}{1+n}$.

(4) The sequence (a_n) is defined inductively by $a_1 = a_2 = 1$ and $a_{n+2} = \frac{a_n+a_{n+1}+1}{4}$. Prove the following by induction on n.

(a) $a_n \leq 1$ for all n.

(b) $a_n \geq \frac{1}{2}$ for all n.

(c) Prove the following holds for all $n \geq 1$: $a_n \geq a_{n+1} \geq \frac{a_n+1}{3}$.

(d) Say whether (a_n) has a limit in $\mathbb{R}$ and if it does, find this limit.

(5) The sequence (c_n) is defined by $c_1 = 1$, $c_2 = \frac{1}{2}$, and

$$c_{n+2} = \frac{3c_{n+1} + c_n}{4}.$$

Prove the following.

(a) $\frac{1}{2} \leq c_n \leq 1$ for all n.

(b) $c_{n+1} - c_n = (-1)^n \frac{2}{4^n}$ for all n.

(c) The subsequences (c_{2n-1}) and (c_{2n}) are both monotonic. (Hint: use (b) above to find a formula for $c_{n+2} - c_n$.)

(d) Explain why these results imply that (c_{2n-1}) and (c_{2n}) both converge, and in fact both converge to the same limit $l \in \mathbb{R}$. Which theorem from the course allows you to deduce that l is the limit of the original sequence (c_n)?

(e) Find the limit l. Suggestion: Use (b) repeatedly to get an expression for c_n in terms of n. Sum this series then find the limit.

You can find all exercise sheets on
http://web.mat.bham.ac.uk/∼malevao/MSM2P01/