Finite-dimensional normed vector spaces

Proposition 3.5. *Let* $(X, \|\cdot\|)$ *be an* n*-dimensional normed vector space for some* $n \in \mathbb{N}$ *and let* $\{e_1, \ldots, e_n\}$ *be a basis for* X*. Then there exist constants* $a, b > 0$ *such that*

$$a \sum_{i=1}^{n} |a_i| \leqslant \left\| \sum_{i=1}^{n} a_i e_i \right\| \leqslant b \sum_{i=1}^{n} |a_i| \tag{3.4}$$

for all $(a_1, \ldots, a_n) \in \mathbb{R}^n$*.*

Proof. Note that $\|(a_1, \ldots, a_n)\|_1 = \sum_{i=1}^{n} |a_i|$ and if $\|(a_1, \ldots, a_n)\|_1 = 0$ then $a_i = 0$ for all $1 \leqslant i \leqslant n$ and (3.4) is satisfied with any a and b. Assume therefore $\|(a_1, \ldots, a_n)\|_1 \neq 0$. By homogeneity it suffices to show that

$$a \leqslant \left\| \sum_{i=1}^{n} a_i e_i \right\| \leqslant b \tag{3.5}$$

for all $(a_1, \ldots, a_n) \in \mathbb{R}^n$ with $\|(a_1, \ldots, a_n)\|_1 = 1$ since for $(\alpha_1, \ldots, \alpha_n) \neq \mathbf{0}$ inequality (3.4) follows from (3.5) applied to the renormalised vector $(\alpha_1', \ldots, \alpha_n')$ given by $\alpha_i' = \alpha_i / \|(\alpha_1, \ldots, \alpha_n)\|_1$ for each $i = 1, \ldots, n$.

To prove (3.5), define a mapping

$$F : K \to \mathbb{R}; \quad F(a_1, \ldots, a_n) = \left\| \sum_{i=1}^{n} a_i e_i \right\|$$

where

$$K := \{(a_1, \ldots, a_n) \in \mathbb{R}^n : \|(a_1, \ldots, a_n)\|_1 = 1\}.$$

It is easy to see that $K \subseteq \mathbb{R}^n$ is a closed and bounded set. It is therefore compact. We now show that $F(a_1, \ldots, a_n) > 0$ for every $(a_1, \ldots, a_n) \in K$. Indeed, since $\{e_1, \ldots, e_n\}$ is a linearly independent system it follows that

$$(a_1, \ldots, a_n) \in K \quad \Rightarrow \quad \sum_{i=1}^{i} a_i e_i \neq 0$$

and consequently $F(a_1, \ldots, a_n) > 0$ for all $(a_1, \ldots, a_n) \in K$.

By Proposition 2.3, F is continuous and therefore is bounded above and below and attains these bounds; i.e. there exist points $A, B \in K$ such that

$$F(A) \leqslant F(a_1, \ldots, a_n) \leqslant F(B)$$

for all $(a_1, \ldots, a_n) \in K$. Denote $a = F(A)$ and $b = F(B)$, then $a, b > 0$ and (3.5) is satisfied. □

Corollary 3.6. *Let* X *be a finite dimensional vector space, let* $\|\cdot\|$ *and* $|||\cdot|||$ *be two norms on* X. *Then* $\|\cdot\| \approx |||\cdot|||$.

Proof. Let $\{e_1, \dots, e_n\}$ be a basis for X. By Proposition 3.5, there are real positive numbers a, b and a', b' such that

$$a \sum_{i=1}^{n} |a_i| \leqslant \| \sum a_i e_i \| \leqslant b \sum_{i=1}^{n} |a_i| \qquad \text{and}$$
$$a' \sum_{i=1}^{n} |a_i| \leqslant ||| \sum_{i=1}^{n} a_i e_i ||| \leqslant b' \sum_{i=1}^{n} |a_i|.$$

Let $x \in X$, find $(a_1, \dots, a_n)$ such that $x = \sum_{i=1}^{n} a_i e_i$. Therefore,

$$\|x\| = \| \sum a_i e_i \| \leqslant b \sum_{i=1}^{n} |a_i| \leqslant \frac{b}{a'} ||| \sum a_i e_i ||| = \frac{b}{a'} |||x||| \text{ and}$$
$$\|x\| = \| \sum a_i e_i \| \geqslant a \sum_{i=1}^{n} |a_i| \geqslant \frac{a}{b'} ||| \sum a_i e_i ||| = \frac{a}{b'} |||x|||,$$

hence $\frac{a}{b'} |||x||| \leqslant \|x\| \leqslant \frac{b}{a'} |||x|||$ for all $x \in X$. □

Remark. Exercise 4b in PS2 claims that there are two non-equivalent norms on ℓ^1: namely, $\|\cdot\|_1$ and $\|\cdot\|_\infty$. Then Corollary 3.6 implies that ℓ^1 is infinite-dimensional. Finally, as every $\ell^p \supseteq \ell^1$ for $1 \leqslant p \leqslant \infty$ (exercise 5 PS1), we get that all ℓ^p are infinite-dimensional.

Corollary 3.7. *Let* $(X, \|\cdot\|)$ *be a finite dimensional normed vector space. Then* $(X, \|\cdot\|)$ *is a Banach space.*

Proof. Let $e_1, \dots, e_n$ be a basis for X and $\|\cdot\|_1$ be the ℓ^1-norm on X: $\| \sum a_i e_i \|_1 := \sum_{i=1}^{n} |a_i|$. Since $(X, \|\cdot\|_1)$ is a Banach space and any two norms on X are equivalent, we conclude (using Remark 4 after the definition of equivalent norms) that $(X, \|\cdot\|)$ is a Banach space too. □

Corollary 3.8. *Let* $(X, \|\cdot\|)$ *be a finite dimensional normed vector space and let* Y *be a linear subspace of* X. *Then* Y *is closed.*

Proof. By Corollary 3.7, $(Y, \|\cdot\|)$ is a Banach space. Thus Y is closed by Proposition 3.4. □

We conclude this section with a brief discussion on compactness.

Definition (Compact set)

Let $(X, \|\cdot\|)$ be a normed vector space and $C \subseteq X$. Then a set C is said to be *compact* if for every family of open sets $(U_\alpha)_{\alpha \in A}$ such that $\bigcup_{\alpha \in A} U_\alpha \supseteq C$ there exists a finite subfamily $(U_{\alpha_i})_{i=1,\dots,n}$ such that $\bigcup_{1 \leqslant i \leqslant n} U_{\alpha_i} \supseteq C$.

[Every open cover has a finite subcover.]

Theorem (Heine–Borel)**.** *Let $(X, \|\cdot\|)$ be a finite dimensional normed vector space and let $C \subseteq X$. Then C is compact if and only if C is closed and bounded.*

Remark. In general, compact sets in normed vector spaces are necessarily closed and bounded. However, away from the finite-dimensional case, the converse is not true in general. For example, let $X = \ell^2$, $C = \{e^{(n)}, n \geqslant 1\}$, where $e^{(n)} = (e_1^{(n)}, e_2^{(n)}, \dots)$ with

$$e_j^{(n)} = \delta_{n,j} = \begin{cases} 1 & \text{if } j = n; \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that C is bounded. Moreover, for all $n \neq m$ we have $\|e_n - e_m\|_2 = \sqrt{2}$ and so C is closed (the set of accumulation points of C is empty). However if we let $U_n = B(e^{(n)}, \sqrt{2}/3)$ be disjoint open balls around vectors $e^{(n)}$, then any finite union $U = \bigcup_{k=1}^N U_{n_k}$ will not cover the whole C (letting $m = \max\{n_1, \dots, n_N\}$ we get that $e^{(m+1)} \notin U$).

END OF LECTURE 9

Theorem 3.9. *Let $(X, \|\cdot\|)$ be a normed vector space. If $C = \overline{B}(0,1)$ is compact then X is finite dimensional.*

Proof. **The following proof has not been covered in the lectures**
Suppose that $C := \overline{B}(0,1)$ is compact. Clearly, $C \subseteq \bigcup_{x \in C} B(x, 1/2)$. By the definition of compactness, there exists a finite subset $\{x_1, \dots, x_n\}$ of C such that

$$C \subseteq \bigcup_{i=1}^n B(x_i, 1/2).$$

Let M be the linear span $\langle x_1, \dots, x_n \rangle$. Then we have

$$C \subseteq M + B\big(0, \frac{1}{2}\big) = M + \frac{1}{2}B(0,1) \subseteq M + \frac{1}{2}C,$$

where for $A, B \subseteq X$ we define $A + B = \Big\{ a + b \mid a \in A, b \in B \Big\}$. Therefore, using the fact that M is a linear subspace of X, we get

$$C \subseteq M + \frac{1}{2}\Big(M + \frac{1}{2}C\Big) = M + \frac{1}{4}C.$$