Example (Normed spaces)

1. For $p \in [1, \infty]$, the pair $(\mathbb{R}^n, \|\cdot\|_p)$ is a normed vector space where

$$\begin{cases} \|x\|_p = (\sum_{k=1}^n |a_k|^p)^{1/p} & \text{for } p \in [1, \infty), \\ \|x\|_\infty = \max\{|x_k| : k = 1, \dots, n\} & \text{for } p = \infty \end{cases}$$

for $x = (a_1, ..., a_n) \in \mathbb{R}^n$.

1a: if $p = 2$ then this is our usual distance in $\mathbb{R}^n$: $\|(a_1, \dots, a_n)\|_2 = \sqrt{|a_1|^2 + \cdots + |a_n|^2}$.

2. For $p \in [1, \infty)$, let ℓ^p denote the set of all infinite sequences $x = (a_k)_{k \in \mathbb{N}}$, where $a_k \in \mathbb{R}$ for each $k \in \mathbb{N}$, such that $\sum_{k \geqslant 1} |a_k|^p < \infty$. If $\|x\|_p := (\sum_{k=1}^\infty |a_k|^p)^{1/p}$ then $(\ell^p, \|\cdot\|_p)$ is a normed vector space.

For $p = \infty$ we let ℓ^∞ denote the set of all bounded sequences $x = (a_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}$ and for $x \in \ell^\infty$ define $\|x\|_\infty := \sup\{|a_k| : k \in \mathbb{N}\}$. Then $(l_\infty, \|\cdot\|_\infty)$ is a normed vector space. We shall prove this later.

Example Let $x = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots)$. Then $x \in \ell^\infty$ but $x \notin \ell^1$.

END OF LECTURE 2

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3. For $a, b \in \mathbb{R}$, we define two different norms on the same vector space $C[a, b]$:

$$\begin{cases} \|f\|_\infty & := \sup\{|f(t)| : t \in [a, b]\}; \\ \|f\|_1 & := \int_a^b |f(t)| dt. \end{cases}$$

Proof. We check that $\|\cdot\|_1$ is a norm.

(i) It is clear that $\int_a^b |f(t)| dt \geqslant 0$ for all $f \in C[a, b]$.

Assume $f \in C[a, b]$ and $\|f\|_1 = 0$. We want to show that $f(t) = 0$ for all t. Consider $h(t) = |f(t)|$. Assume there exists $t \in (a, b)$ such that $c = h(t) \neq 0$. Since h takes only nonnegative values, we get $c > 0$. By continuity of h

$$\exists \varepsilon > 0 \quad \forall x \in (t - \varepsilon, t + \varepsilon) \subseteq (a, b) \quad h(x) > c/2.$$

As $h \geqslant 0$ we get

$$0 = \int_a^b h(x) dx \geqslant \int_{t-\varepsilon}^{t+\varepsilon} h(x) dx \geqslant c\varepsilon > 0,$$

a contradiction. This proves $h(t) = |f(t)| = 0$ for all $t \in (a, b)$ thus $f \equiv 0$ on $[a, b]$ (by continuity at a and b).

(ii) Let $\lambda \in \mathbb{R}$ be arbitrary, then $\int_a^b |\lambda f(t)| dt = |\lambda| \int_a^b |f(t)| dt$.

(iii) Let $f, g \in C[a, b]$, then $\|f + g\|_1 = \int_a^b |f(t) + g(t)| dt \leqslant \int_a^b \Big(|f(t)| + |g(t)| \Big) dt = \|f\|_1 + \|g\|_1$.

We prove that $\|\cdot\|_\infty$ is a norm on $C[a, b]$ in Problem Sheet 1. □

Case study: The ℓ^p spaces.

In this section we give a proof that $(\ell^p, \|\cdot\|_p)$ is a normed vector space for $p \in [1,\infty]$.

We first remark that if $x \in \ell^p$ and $\|x\|_p = 0$ then $x = 0$.

Proposition 2.1 (Minkowski's inequality). *Let $x = (a_k)$ and $y = (b_k)$ be two elements of ℓ^p, where $p \in [1,\infty]$. Then the sequence $(a_k + b_k)$ defines an element of ℓ^p and*

$$\|x+y\|_p \leqslant \|x\|_p + \|y\|_p. \tag{2.1}$$

Proof of Proposition 2.1. Let $x = (a_k)$ and $y = (b_k)$.

When $p = \infty$, we have

$$|a_k + b_k| \leqslant |a_k| + |b_k| \leqslant \|x\|_\infty + \|y\|_\infty$$

for each $k \in \mathbb{N}$ and therefore $x + y$ is a bounded sequence (belongs to ℓ^∞) and $\|x+y\|_\infty = \sup_{k\in\mathbb{N}} |a_k + b_k| \leqslant \|x\|_\infty + \|y\|_\infty$.

Suppose now that $p \in [1,\infty)$ and recall $x = (a_k)$ and $y = (b_k)$. Note that if $\|x\|_p = 0$ or $\|y\|_p = 0$ then $x = 0$ or $y = 0$ respectively and (2.1) is trivially satisfied. So we assume that $\|x\|_p$ and $\|y\|_p$ are both nonzero and define

$$A_k := \frac{|a_k|}{\|x\|_p}, \quad B_k := \frac{|b_k|}{\|y\|_p}$$

for each $k \in \mathbb{N}$, then

$$\sum_{k\in\mathbb{N}} A_k^p = \Big(\sum_{k\in\mathbb{N}} |a_k|^p\Big)/\|x\|_p^p = 1 \quad \text{and}$$
$$\sum_{k\in\mathbb{N}} B_k^p = \Big(\sum_{k\in\mathbb{N}} |b_k|^p\Big)/\|y\|_p^p = 1.$$

Fix any $k \in \mathbb{N}$; we have:

$$\begin{aligned}
|a_k + b_k|^p &\leqslant (|a_k| + |b_k|)^p \\
&= (\|x\|_p A_k + \|y\|_p B_k)^p \\
&= (\|x\|_p + \|y\|_p)^p \Big(\frac{\|x\|_p}{\|x\|_p + \|y\|_p} A_k + \frac{\|y\|_p}{\|x\|_p + \|y\|_p} B_k\Big)^p \\
&= (\|x\|_p + \|y\|_p)^p (\alpha A_k + (1-\alpha) B_k)^p,
\end{aligned}$$

where $\alpha = \|x\|_p/(\|x\|_p + \|y\|_p) \in (0,1)$. Since the function $h(t) = t^p : [0,\infty) \to [0,\infty)$ is convex it follows that

$$h(\alpha A_k + (1-\alpha)B_k) \leqslant \alpha h(A_k) + (1-\alpha) h(B_k)$$

and so

$$|a_k + b_k|^p \leqslant (\|x\|_p + \|y\|_p)^p (\alpha A_k^p + (1-\alpha) B_k^p).$$

Taking the sum over all $k \in \mathbb{N}$ it follows that

$$\sum_{k\in\mathbb{N}} |a_k + b_k|^p \leqslant (\|x\|_p + \|y\|_p)^p(\alpha \sum_{k\in\mathbb{N}} A_k^p + (1-\alpha)\sum_{k\in\mathbb{N}} B_k^p)$$
$$= (\|x\|_p + \|y\|_p)^p(\alpha \cdot 1 + (1-\alpha)\cdot 1) = (\|x\|_p + \|y\|_p)^p.$$

The latter implies that $\sum_{k\in\mathbb{N}} |a_k + b_k|^p$ converges and $\left(\sum_{k\in\mathbb{N}} |a_k + b_k|^p\right)^{1/p} \leqslant \|x\|_p + \|y\|_p$ as required. □

END OF LECTURE 3

Remark. Observe that Proposition 2.1 implies that

1. ℓ^p is closed under addition: $x, y \in \ell^p \Rightarrow x + y \in \ell^p$. It is easy to check that for $\lambda \in \mathbb{R}$ and $x \in \ell^p$ the vector λx also belongs to ℓ^p. Therefore, ℓ^p is a vector space.

2. (i) $\|x\|_p \geqslant 0$ for all $x \in \ell^p$, and if $\|x\|_p = 0$ then $x = 0$;

   (ii) $\|\lambda x\|_p = |\lambda|\|x\|_p$ for all $\lambda \in \mathbb{R}$ and $x \in \ell^p$.

   (iii) the triangle inequality for $\|\cdot\|_p$ holds – from Minkowski's inequality.

   Thus $\|\cdot\|_p$ is a norm on ℓ^p.

Proposition 2.2 (Hölder's inequality)**.** *Let* $p, q \geqslant 1$ *be such that* $1/p + 1/q = 1$. *Suppose* $x = (a_k) \in \ell^p$ *and* $y = (b_k) \in \ell^q$. *Then* $xy = (a_k b_k) \in \ell^1$ *and*

$$\|xy\|_1 \leqslant \|x\|_p\|y\|_q. \tag{2.2}$$

Remark. If $p = 1$ then condition $1/p + 1/q = 1$ implies $1/q = 0$, i.e., $q = \infty$.

Proof. We can assume $\|x\|_p$ and $\|y\|_q$ are both nonzero, otherwise both sides of (2.2) are zero and there is nothing to do.

Case 1: $p = 1$ and $q = \infty$. Then $|a_k b_k| \leqslant |a_k| \cdot \|y\|_\infty$ for every $k \geqslant 1$, therefore, since the series $\sum_{k\geqslant 1} |a_k|$ converges, the series $\sum_{k\geqslant 1} |a_k b_k|$ converges too (Comparison Test for series) and

$$\|xy\|_1 = \sum_{k\geqslant 1} |a_k b_k| \leqslant \|y\|_\infty \sum_{k\geqslant 1} |a_k| = \|y\|_\infty\|x\|_1.$$

Case 2: $p, q \in (1, \infty)$. We claim that for all $A, B \in (0, \infty)$

$$AB \leqslant \frac{A^p}{p} + \frac{B^q}{q}. \tag{2.3}$$

Inequality (2.3) is often referred to as Young's inequality. If we assume that it is true for the moment, then we let

$$A_k = \frac{|a_k|}{\|x\|_p}, \quad B_k = \frac{|b_k|}{\|y\|_q}$$