Theorem 4.1 (Cauchy–Schwarz inequality). *If X is an inner product space then*

$$|\langle x, y\rangle| \leqslant \|x\| \cdot \|y\|. \tag{4.2}$$

Proof. First note that

$$0 \leqslant \Big\| \|u\|v - \|v\|u \Big\|^2 = 2\|u\|^2\|v\|^2 - 2\|u\|\ \|v\|\mathrm{Re}\langle u, v\rangle. \tag{4.3}$$

Therefore,

$$\mathrm{Re}\langle u, v\rangle \leqslant \|u\|\|v\| \tag{4.4}$$

for all $u, v \in X$. Choose now $\theta \in [0, 2\pi]$ such that $e^{i\theta}\langle x, y\rangle = |\langle x, y\rangle|$ then (4.2) follows immediately from (4.4) with $u = e^{i\theta}x$ and $v = y$. □

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Remark. Equality in $|\langle x, y\rangle| = \|x\| \cdot \|y\|$ holds if and only if x and y are linearly dependent. This follows from (4.3) using the nondegeneracy of the norm.

Lemma 4.2. *If X is an inner product space and $\|\cdot\|$ is given by* (4.1) *then $(X, \|\cdot\|)$ is a normed vector space.*

Proof. As we said, only the triangle inequality remains to be verified. If $x, y \in X$ then by Cauchy–Schwarz,

$$\begin{aligned}\|x + y\|^2 &= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\langle x, y\rangle \\ &\leqslant \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2.\end{aligned}$$

□

Another consequence of the Cauchy–Schwarz inequality is the continuity of the inner product.

Lemma 4.3. *Let (x_n) and (y_n) be two sequences in the inner product space X converging to x and y respectively. Then the sequence $(\langle x_n, y_n\rangle)$ converges to $\langle x, y\rangle$.*

Proof.

$$\begin{aligned}|\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle| \\ &\leqslant |\langle x_n - x, y_n\rangle| + |\langle x, y_n - y\rangle| \\ &\leqslant \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| \to 0\end{aligned}$$

as $n \to \infty$ since $\|y_n\| \to \|y\|$ by Proposition 2.3. □

Theorem 4.4. *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F}$ and let $\|\cdot\|$ denote the induced norm. Then*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{4.5}$$

for all $x, y \in X$. We call this the Parallelogram law.

Conversely, if $(X, \|\cdot\|)$ is a normed vector space such that the Parallelogram law (4.5) holds then there exists an inner product $\langle\cdot,\cdot\rangle : X\times X \to \mathbb{F}$ which induces the norm $\|\cdot\|$; i.e. $\langle x, x\rangle^{1/2} = \|x\|$ for each $x \in X$.

Remark. One can use Theorem 4.4 to show that the Banach space $(\ell^\infty, \|\cdot\|_\infty)$ is not an inner product space:

Take $x = (1,0,0,\dots)$, $y = (0,1,0,0,\dots)$. Then $\|x\|_\infty = 1$, $\|y\|_\infty = 1$, $\|x+y\|_\infty = 1$, $\|x-y\|_\infty = 1$. The parallelogram law is not satisfied as $1^2 + 1^2 \neq 2(1^2+1^2)$.

Proof of Theorem 4.4. If $(X, \langle\cdot,\cdot\rangle)$ is an inner product space then one can show (4.5) holds by expanding the left-hand side using the linearity of the inner product.

For the converse statement, we include the ideas for the case $\mathbb{F} = \mathbb{R}$. Assume the parallelogram law (4.5) is satisfied for all $x, y \in X$; define $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}$ by[1]

$$\langle x, y\rangle := \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2). \tag{4.6}$$

It is clear that $\langle x, x\rangle = \|x\|^2$ for each $x \in X$ by the 2^{nd} property of $\|\cdot\|$. Hence the 1^{st} property of $\langle\cdot,\cdot\rangle$ is satisfied. It is also clear that $\langle\cdot,\cdot\rangle$ is symmetric using the 2^{nd} property of $\|\cdot\|$: $\langle y, x\rangle = \langle x, y\rangle$. It remains to verify the linearity condition. To this end, the proof follows the steps:

1. $2\left\langle \frac{x+y}{2}, z\right\rangle = \langle x, z\rangle + \langle y, z\rangle$,
2. $\langle (x+y), z\rangle = \langle x, z\rangle + \langle y, z\rangle$,
3. $\langle \alpha x, z\rangle = \alpha\langle x, z\rangle$ for all $\alpha \in \mathbb{Q}$,
4. $\langle \alpha x, z\rangle = \alpha\langle x, z\rangle$ for all $\alpha \in \mathbb{R}$.

Details of the proof (**not covered in lectures**) are below.

[1] The proof for $\mathbb{F} = \mathbb{C}$ proceeds in a similar way under the definition, for all $x, y \in X$

$$\langle x, y\rangle := \frac{1}{4}\Big[(\|x+y\|^2 - \|x-y\|^2) + i(\|x+iy\|^2 - \|x-iy\|^2)\Big].$$

Note first that

$$
\begin{aligned}
2\left\langle \frac{x+y}{2}, z \right\rangle &= \frac{1}{2}\left(\left\| \frac{x+y}{2} + z \right\|^2 - \left\| \frac{x+y}{2} - z \right\|^2 \right) \\
&= \frac{1}{2}\left(\left\| \frac{x+y}{2} + z \right\|^2 + \left\| \frac{x-y}{2} \right\|^2 \right) - \frac{1}{2}\left(\left\| \frac{x+y}{2} - z \right\|^2 + \left\| \frac{x-y}{2} \right\|^2 \right) \\
&\underset{*}{=} \frac{1}{4}\left(\left\| \frac{x+y}{2} + z + \frac{x-y}{2} \right\|^2 + \left\| \frac{x+y}{2} + z - \frac{x-y}{2} \right\|^2 \right. \\
&\qquad \left. - \left\| \frac{x+y}{2} - z + \frac{x-y}{2} \right\|^2 - \left\| \frac{x+y}{2} - z - \frac{x-y}{2} \right\|^2 \right) \\
&= \frac{1}{4}(\|x+z\|^2 + \|y+z\|^2 - \|x-z\|^2 - \|y-z\|^2),
\end{aligned}
$$

where equality marked * follows from the parallelogram law (4.5) which we assumed holds for all pairs of vectors from X. Hence

$$2\left\langle \frac{x+y}{2}, z \right\rangle = \langle x, z\rangle + \langle y, z\rangle \tag{4.7}$$

for all $x, y, z \in X$. Taking $y = 0$ in (4.7) gives

$$2\langle x/2, z\rangle = \langle x, z\rangle$$

for all $x \in X$ and hence, using (4.7) again,

$$\langle x, z\rangle + \langle y, z\rangle = 2\langle (x+y)/2, z\rangle = \langle x+y, z\rangle \tag{4.8}$$

for all $x, y, z \in X$. Note that (4.8) immediately implies

$$\langle x, z\rangle - \langle y, z\rangle = \langle x-y, z\rangle \tag{4.9}$$

for all $x, y, z \in X$.

It remains to verify that $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for all $\alpha \in \mathbb{R}$ and all $x, y \in X$. To see this, first note that (4.8) with $x = y$ implies that $2\langle x, z\rangle = \langle 2x, z\rangle$ for all $x, z \in X$. Hence, by induction,

$$m\langle x, z\rangle = \langle mx, z\rangle \tag{4.10}$$

for all $m \in \mathbb{N}$ and all $x, z \in X$. However, (4.10) implies $n\langle x/n, z\rangle = \langle x, z\rangle$ for all $n \in \mathbb{N}$ and all $x, z \in X$. Hence,

$$\frac{m}{n}\langle x, z\rangle = \left\langle \frac{m}{n}x, z \right\rangle \tag{4.11}$$

for all $m, n \in \mathbb{N}$ and all $x, z \in X$.

We now claim that, for fixed $x, z \in X$ and $\alpha_n \to \alpha$ we have

$$\langle \alpha_n x, z\rangle \to \langle \alpha x, z\rangle.$$

Granted, it follows from (4.11) and the density of $\mathbb{Q}$ in $\mathbb{R}$ that

$$\alpha\langle x, z\rangle = \langle \alpha x, z\rangle \tag{4.12}$$

for all $\alpha \geqslant 0$ and all $x, z \in X$.

To see the claim, note that by (4.6) and continuity of the norm and linear combinations (Proposition 2.3)

$$\langle \alpha_n x, z\rangle = \frac{1}{4}\big(\|\alpha_n x + z\|^2 - \|\alpha_n x - z\|^2\big) \to \frac{1}{4}\big(\|\alpha x + z\|^2 - \|\alpha x - z\|^2\big) = \langle \alpha x, z\rangle.$$

Finally, for $\alpha < 0$, note first that (4.9) implies

$$\langle -x, z\rangle = \langle 0 - x, z\rangle = \langle 0, z\rangle - \langle x, z\rangle = 0 - \langle x, z\rangle = -\langle x, z\rangle$$

and therefore, by (4.12)

$$\alpha\langle x, z\rangle = -(-\alpha)\langle x, z\rangle = -\langle -\alpha x, z\rangle = \langle \alpha x, z\rangle$$

as required. This completes the proof of Theorem 4.4. □

Definition (Orthogonal vectors, orthogonal sets)

Elements y and z in the inner product space X are said to be *orthogonal* if $\langle y, z\rangle = 0$.

Example (Collections of orthogonal vectors)

Let $X = \ell^2$ and $(e^{(n)})_{n\in\mathbb{N}}$ be the standard basis of ℓ^2 (see Remark before Theorem 3.9). Then $e^{(n)}$ and $e^{(m)}$ are orthogonal if and only if $n \neq m$.

Theorem 4.5 (Pythagoras' theorem)**.** *Let X be an inner product space and suppose $x, y \in X$ are orthogonal. Then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.*

§5 Hilbert spaces

Definition (Hilbert space)

An inner product space that is a Banach space with respect to the norm associated to the inner product is called a *Hilbert space*.

Example (Hilbert spaces)

1. If $X = \mathbb{F}^d$ then we have seen that X is an inner product space. Since X is a finite-dimensional vector space it is a Banach space by Corollary 3.7. Thus $\mathbb{F}^d$ is a Hilbert space.

2. The space ℓ^2 is a Hilbert space (the space ℓ^2 is a Banach space).
3. The inner product space $C[a,b]$ with the product

$$\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}dt$$

as in the Example in the beginning of §4, is not a Hilbert space. This follows from the fact that $(C[a,b], \|\cdot\|_2)$:

$$\|f\|_2 = \Big(\int_a^b |f(t)|^2 dt\Big)^{1/2}, \quad f \in C[a,b]$$

is not a Banach space. The proof of this fact is similar to the proof of the fact that $(C[a,b], \|\cdot\|_1)$ is not a Banach space (see Example before definition of equivalent norms).

END OF LECTURE 11

Distances and minimising vectors.

We address the question of whether the distance from an element x of a Hilbert space H to a given subset M,

$$\text{dist}(x, M) := \inf\{\|x - y\| : y \in M\},$$

is attained or not.

Theorem 5.1. *Let M be a closed convex subset of the Hilbert space H. Then for every $x \in H$ there exists a unique $y_0 \in M$ such that dist$(x, M) = \|x - y_0\|$.*

Remark. Of course, if $x \in M$ then $y_0 = x$.

Proof. If $\delta :=$ dist(x, M) then there exists a sequence $(y_n) \in M$ such that

$$\delta_n := \|x - y_n\| \to \delta. \tag{5.1}$$

We claim that (y_n) is a Cauchy sequence. Indeed, due to convexity of M,

$$\|(y_n + y_m)/2 - x\| \geqslant \delta.$$

Thus using the Parallelogram Law (4.5) we get

$$\begin{aligned} 0 \leqslant \|y_n - y_m\|^2 &= -\|y_n + y_m - 2x\|^2 + 2(\|y_n - x\|^2 + \|y_m - x\|^2) \\ &\leqslant -(2\delta)^2 + 2(\delta_n^2 + \delta_m^2) \to 0 \end{aligned}$$

as $m, n \to \infty$ because of (5.1). We therefore conclude that (y_n) is a Cauchy sequence. Since H is a Banach space, it converges to some point $y_0 \in H$. As M is closed and $y_n \in M$ for all n we get $y_0 \in M$. Moreover, $\|x - y_0\| = \lim \|x - y_n\| = \lim \delta_n = \delta$.