SPECIAL TOPICS IN GRAPH THEORY

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Abstract. This three part lecture series is based primarily on the book *Elementary Number Theory, Group Theory, and Ramanujan Graphs* by Davidoff, Sarnak, and Valette. I will discuss background information and the explicit construction of $(p+1)$-regular Ramanujan graphs by Lubotzky-Phillips-Sarnak and Margulis, where p is an odd prime.

1. Basic definitions and results

We start with some basic definitions and results in spectral graph theory. Let $X = (V, E)$ be a graph. The adjacency matrix A of a finite graph X on n vertices is an n by n symmetric matrix. Therefore, A has n real eigenvalues counting multiplicities

$$\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}.$$

Lemma 1. *Let X be a finite k-regular graph with n vertices. Then*

(1) $\lambda_0 = k$
(2) $|\lambda_i| \leq k$ *for* $1 \leq i \leq (n-1)$
(3) λ_0 *has multiplicity 1 if and only if X is connected. [Furthermore, the multiplicity is equal to the number of components.]*

Lemma 2. *Let X be a connected, k-regular graph on n vertices. Then the following are equivalent*

(1) $\lambda_{n-1} = -k$,
(2) *the spectrum of X is symmetrical about 0,*
(3) *X is bipartite.*

Let $X = (V, E)$ be a finite, connected graph on n vertices, and $F \subseteq V$.

Definition 1. The *boundary* of F, denoted δF, is the set of edges required to disconnect F from any vertex in $V - F$.

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Definition 2. The *isoperimetric* or *expanding constant* of X is

$$h(X) = \min\left\{\frac{|\delta F|}{|F|} : F \subseteq V, 0 < |F| \leq \frac{n}{2}\right\}$$
$$= \inf\left\{\frac{|\delta F|}{\min\{|F|, |V-F|\}} : F \subseteq V, 0 < |F| < +\infty\right\}.$$

Think of X as a network that is transmitting information from vertex to vertex; the expanding constant measures the "quality" of X as a network in some sense. A large expanding constant means that information is able to propagate well. Consider the complete graph K_n versus the cycle C_n. For $X = K_n$, if $|F| = k$, then $|\delta F| = k(n-k)$ so $\frac{|\delta F|}{|F|} = \frac{k(n-k)}{k} = n - k$ and $h(K_n) = n - \left[\frac{n}{2}\right] \sim \frac{n}{2}$; K_n is highly connected and has a large expanding constant which grows proportionately with the number of vertices. For $X = C_n$, if $|F|$ is a half cycle, $|\delta F| = 2$ and $h(C_n) = \frac{2}{\left[\frac{n}{2}\right]} \sim \frac{4}{n}$; C_n is not highly connected and has a small expanding constant that tends to 0 as the number of vertices increases.

2. Families of expanders

Definition 3. If $\{X_m\}_{m\geq 1}$ is a family of finite, connected, k-regular graphs with $|V_m| \to +\infty$ as $m \to +\infty$, then $\{X_m\}_{m\geq 1}$ is a *family of expanders* if there exists an $\epsilon > 0$ such that $h(X_m) \geq \epsilon$ for all $m \geq 1$.

Simply speaking, expander graphs are sparse yet highly connected k-regular graphs. Because of these nice properties, expander graphs have many applications in engineering and computer science from network design to cryptography. We would like to explicitly construct infinite family of expanders. A "good quality" expander has a large spectral gap

$$\lambda_0(X_m) - \lambda_1(X_m) = k - \lambda_1(X_m)$$

(this will motivate the definition of a Ramanujan graph) as it measures "high connectedness".

For an arbitrary graph $X = (V, E)$, consider the functions $f : V \to \mathbb{C}$ and define Hilbert spaces,

$$l^2(V) = \left\{f : V \to \mathbb{C} : \sum_{v\in V} |f(v)|^2 < +\infty\right\} \text{ and}$$
$$l^2(E) = \left\{f : E \to \mathbb{C} : \sum_{e\in E} |f(e)|^2 < +\infty\right\}.$$

Theorem 1 (1985 Alon-Milman; 1984 Dodziuk)**.** *Let X be a finite, connected k-regular graph. Then*

$$\frac{k-\lambda_1}{2} \leq h(X) \leq \sqrt{2k(k-\lambda_1)}.$$

Proof. Let $X = (V, E)$ be a finite, connected k-regular graph without loops. Randomly orient the edges; given an edge $e \in E$, let e^+ denote the head and e^- denote the tail. The *simplicial coboundary operator* is $d : l^2(V) \to l^2(E)$ if for $f \in l^2(V)$ and $e \in E$,

$$df(e) = f(e^+) - f(e^-).$$

Endow $l^2(V)$ and $l^2(E)$ with hermitian scalar product; for example,

$$\langle f, g\rangle = \sum_{x\in V} \overline{f(x)}g(x).$$

Then, there is a unique continuous operator, the adjoint operator $d^* : l^2(E) \to l^2(V)$ that is characterized by $\langle df, g\rangle = \langle f, d^*g\rangle$ for all $f \in l^2(V)$ and $g \in l^2(E)$.

Define a function $\delta : V \times E \to \{-1, 0, 1\}$ by

$$\delta(x,e) = \begin{cases} 1, & \text{if } x = e^+ \\ -1, & \text{if } x = e^- \\ 0, & \text{otherwise,} \end{cases}$$

then for $e \in E$ and $f \in l^2(V)$ $df(e) = \sum_{x\in V} \delta(x,e)f(x)$ and for $x \in V$ and $g \in l^2(E)$ $d^*g(x) = \sum_{e\in E} \delta(x,e)g(e)$. Let the *combinatorial Laplacian operator* be

$$\Delta = d^*d : l^2(V) \to l^2(V).$$

In other words, $\Delta = k \cdot \mathrm{Id} - A$. The combinatorial Laplacian operator has a number of nice properties. It does not depend on orientation. If f is a function on the vertex set, and $\sum_{x\in V} f(x) = 0$,

$$\|df\|_2^2 = \langle df, df\rangle = \langle \Delta f, f\rangle \geq (k-\lambda_1)\|f\|_2^2.$$

Consider the following special function

$$f(x) = \begin{cases} |V-F|, & \text{if } x \in F \\ -|F|, & \text{if } x \in V - F. \end{cases}$$

Then $\sum_{x\in V} f(x) = 0$ so

$$\|f\|_2^2 = |F||V-F|^2 + |V-F||F|^2 = |F||V-F|(|V-F|+|F|) = |F||V-F||V|,$$

and

$$df(x) = \begin{cases} 0, & \text{if } e \text{ is not a cross edge between } F \text{ and } V-F \\ \pm|V|, & \text{if } e \text{ is a cross edge.} \end{cases}$$

Because $\|df\|_2^2 = |\delta F||V|^2 + 0 = |\delta F||V|^2$,

$$|V|^2|\delta F| \geq (k-\lambda_1)|F||V-F||V|$$

and

$$\frac{|\delta F|}{|F|} \geq (k-\lambda_1)\frac{|V-F|}{|V|}.$$

If $|F| \leq \frac{|V|}{2}$, $\frac{|\delta F|}{F} \geq \frac{k-\lambda_1}{2}$, and then $h(X) \geq \frac{k-\lambda_1}{2}$. The second inequality is much more complicated see pages 20-23 [4]. □

A family of k-regular graphs is a family of expanders if and only if the spectral gap is bounded away from zero. The bigger the spectral gap, the better the "the quality" of the expander.

Theorem 2. *Let $\{X_m\}_{m\geq 1}$ be a family of finite, connected k-regular graphs without loops, such that $|V_m| \to +\infty$ as $m \to +\infty$. The family $\{X_m\}_{m\geq 1}$ is a family of expanders if and only if there exists $\epsilon > 0$ such that $k - \lambda_1(X_m) \geq \epsilon$ for every $m \geq 1$.*

For many years, constructing large families of expanders has been an important problem. Motivated by problems in network theory, in 1972-1973, Pinsker and Margulis worked on constructions. This work, however, gave no measure of the quality of the expanders. More recent work does (Gabber-Galil, Widgerson-Zuckerman, etc). Historically, isoperimetric inequalities have been studied in the Riemannian geometry setting and sometimes are called the Cheeger-Buser inequalities.

3. Ramanujan Graphs

Theorem 3 (Alon-Boppana)**.** *Let $\{X_m\}_{m\geq 1}$ be a family of finite, connected, k-regular graphs with $|V_m| \to +\infty$ as $m \to +\infty$. Then*

$$\liminf_{m\to+\infty} \lambda_1(X_m) \geq 2\sqrt{k-1}.$$

Proof. The inequality is actually from the fact that the number of paths of length m from a vertex v to v in a k-regular graph is at least number of paths from a vertex v to v in a k-regular tree.

Let $X = (V, E)$ be a k-regular simple graph with $|V|$ possibly infinite. A path of length r without backtracking is a sequence $\underline{e} = (x_0, x_1, \dots, x_r)$ of vertices in V such that x_i is adjacent to x_{i+1} for $i = 0, 1, \dots, r-1$ and $x_{i+1} \neq x_{i-1}$ for $i = 1, 2, \dots r-1$. The origin of $\underline{e}$ is x_0, and the extremity of $\underline{e}$ is x_r. For $r \in \mathbb{N}$, matrix A_r is an n by n matrix indexed by $V \times V$ with

$(A_r)_{xy} =$ the number of paths of length r with origin x and extremity y without backtracking.

Define $A_0 = \mathrm{Id}$ and note that $A_1 = A$, the adjacency matrix of X.

Lemma 3. *Both of the following equalities hold:*

(1) $A_1^2 = A_2 + k \cdot A_0$
(2) $A_1 A_r = A_r A_1 = A_{r+1} + (k-1)A_{r-1}$, *for* $r \geq 2$.

Proof.

(1) For $x \neq y \in V$, $(A_1^2)_{x,y}$ counts the number of paths of length 2; there can be no backtracking. Thus, $(A_1^2)_{x,y} = (A_2)_{x,y}$. If $x = y$, then $(A_1^2)_{x,y} = k$ as the degree of x is k; however, $(A_2)_{x,y} = 0$. Thus $(A_1^2)_{x,y} = (A_2)_{x,y} + k$.
(2) $(A_r A_1)_{x,y}$ is the number of paths $(x_0 = x, x_1, \dots, x_r, x_{r+1} = y)$ without backtracking except possibly on the last step. If $x_{r-1} \neq y$ then $(x_0 = x, x_1, \dots, x_r, x_{r+1} = y)$ has no backtracking and there are $(A_{r+1})_{x,y}$ such paths. If $x_{r-1} = y$ then there was backtracking at the last step; there are $(k-1)(A_{r-1})_{x,y}$ such paths. Thus, $A_r A_1 = A_{r+1} + (k-1)A_{r-1}$; $A_1 A_r = A_{r+1} + (k-1)A_{r-1}$ is left as an exercise.

□

Because

$$\left(\sum_{r=0}^{\infty} A_r t^r\right)(A_0 - A_1 t + (k-1)t^2 A_0) = (1-t^2)A_0,$$

the generating function for A_r is

$$\sum_{r=0}^{\infty} A_r t^r = \frac{(1-t^2)}{1 - A_1 t + (k-1)t^2}$$

We would like to eliminate $(1-t^2)$ on the Right Hand Side. Let

$$T_m = \sum_{0 \le r \le \frac{m}{2}} A_{m-2r}.$$

Then

$$\begin{aligned}
\sum_{m=0}^{\infty} T_m t^m &= \sum_{m=0}^{\infty} \sum_{0 \le r \le \frac{m}{2}} A_{m-2r} t^m = \sum_{r=0}^{\infty} \sum_{m \ge 2r} A_{m-2r} t^m \\
&= \sum_{r=0}^{\infty} t^{2r} \sum_{m \ge 2r} A_{m-2r} t^{m-2r} = \left(\sum_{r=0}^{\infty} t^{2r}\right)\left(\sum_{l=0}^{\infty} A_l t^l\right) \\
&= \frac{1}{1-t^2} \cdot \frac{1-t^2}{1 - A_1 t + (k+1)t^2} = \frac{1}{1 - A_1 t + (k+1)t^2},
\end{aligned}$$

and

$$\sum_{m=0}^{\infty} T_m t^m = \frac{1}{1 - A_1 t + (k-1)t^2}.$$

Definition 4. The *Chebyshev polynomials of the 2nd kind* are defined by expressing

$$\frac{\sin(m+1)\theta}{\sin\theta}$$

as a polynomial of degree m in $\cos\theta$,

$$U_m(\cos\theta) = \frac{\sin(m+1)\theta}{\sin\theta}.$$

Because $U_0(x) = 1, U_1(x) = 2x, U_2(x) = 4x^2 - 1, \ldots$, these polynomials satisfy the recurrence relation

$$U_{m+1}(x) = 2xU_m(x) - U_{m-1}(x)$$

and the generating function is

$$\sum_{m=0}^{\infty} U_m(x) t^m = \frac{1}{1 - 2xt + t^2}.$$

By a change of variables

$$\sum_{m=0}^{\infty}(k-1)^{\frac{m}{2}}U_m\left(\frac{x}{2\sqrt{k-1}}\right)t^m = \frac{1}{1-xt+(k-1)t^2},$$

and

$$T_m = (k-1)^{\frac{m}{2}}U_m\left(\frac{A_1}{2\sqrt{k-1}}\right).$$

For the rest of the proof of Theorem 3, see pages 28-34 [4]. □

Definition 5. A finite, connected k-regular graph X is *Ramanujan* if for every eigenvalue λ of A, except for $\pm k$, one has

$$|\lambda| \leq 2\sqrt{k-1}.$$

Some simple examples of Ramanujan graphs are K_n, $K_{n,n}$, and the Petersen graph. When constructing k-regular graphs with large spectral gaps, $2\sqrt{k-1}$ serves as the lower limit on $|\lambda_1|$. Explicitly constructing an infinite family of k-regular Ramanujan graphs gives us an infinite family of expanders. In fact, this family is "optimal" from a spectral point of view.

Theorem 4. *For the following values of k, there exist infinite families of k-regular Ramanujan graphs:*

(1) $k = p+1, p$ *an odd prime (1988 Lubotzky-Phillips-Sarnak* [6] *and 1988 Margulis* [7]*)*
(2) $k = 2+1 = 3$ *(1992 Chiu* [2]*),*
(3) $k = q+1, q$ *a prime power(1994 Morgenstern* [8]*).*

The main goal of the book is to describe the Ramanujan graphs of Lubotzky-Phillips-Sarnak and Margulis. While the construction of these Ramanujan graphs is fairly simple, proving they have the desired properties is not and relies heavily on group theory, modular forms (analytic functions on the upper half-plane satisfying a certain kind of functional equation and growth condition), and even the Riemann Hypothesis for curves over finite fields.

The name "Ramanujan" comes from the constructions' dependence on Ramanujan's conjecture (solved by Eichler in 1954) concerning coefficients of modular forms with weight 2, a consequence of Weil's proof of the Riemann hypothesis for curves over a finite field. Eichler related the eigenvalues of Hecke operators T_m acting on spaces of cusp forms to the zeros of zeta functions of modular curves over the fields $\mathbb{F}_p$. Hecke operators are relatives of A_r. For m prime, varying the space on which the Hecke operators T_m act, we obtain a large family of Ramanujan graphs. For T_m, m not prime, we are able to associate an "almost Ramanujan" graph. The Ramanujan property is actually equivalent to the Riemann Hypothesis for these zeta functions [10].

4. Independence number and chromatic number

Lemma 4. *Let X be a finite, connected, k-regular graph on n vertices. Then*

$$\alpha(X) \leq \frac{n}{k} \max\{|\lambda_1|, |\lambda_{n-1}|\}.$$

Proof. If $F \subseteq V$, $|F| = \alpha(X)$, and $A_{x,y} = 0$ for $x, y \in F$, then consider the function $f \in l^2(V)$ as before

$$f(x) = \begin{cases} |V - F|, & \text{if } x \in F \\ -|F|, & \text{if } x \in V - F. \end{cases}$$

Then $\sum_{x \in V} f(x) = 0$ so

$$\|f\|_2^2 = |F||V-F|^2 + |V-F||F|^2 = |F||V-F|(|V-F| + |F|) = |F||V-F||V| \leq \alpha(X)n^2.$$

For $x \in F$, let

$$(Af)(x) = \sum_{y \notin F} A_{x,y} f(y) = -|F| \sum_{y \notin F} A_{x,y} = -|F| \sum_{y \in V} A_{x,y} = -k\alpha(X).$$

$$\|Af\|_2^2 \geq \sum_{x \in F} (Af)(x)^2 = \alpha(X) \cdot (-k\alpha(X))^2 = k^2 \alpha(X)^3.$$

Claim 1.

$$\|Af\|_2 \leq \max\{|\lambda_1|, |\lambda_{n-1}|\} \|f\|_2$$

Proof. See page 37 [4]. □

Using the lower bound for $\|Af\|_2$ and upper bound for $\|f\|_2$

$$k\alpha(X)^{\frac{3}{2}} \leq \max\{|\lambda_1|, |\lambda_{n-1}|\} \cdot n \cdot \alpha(X)^{\frac{1}{2}} \text{ and}$$

$$\alpha(X) \leq \frac{n}{k} \max\{|\lambda_1|, |\lambda_{n-1}|\}.$$

□

Corollary 1. *Let X be a finite, connected, k-regular graph on n vertices, without loops. Then*

$$\chi(X) \geq \frac{k}{\max\{|\lambda_1|, |\lambda_{n-1}|\}}.$$

If X is a non-bipartite Ramanujan graph, then

$$\chi(X) \geq \frac{k}{2\sqrt{k-1}} \sim \frac{\sqrt{k}}{2}.$$

Proof. This is clear as $n \leq \alpha(X)\chi(X)$. □

5. Projective linear groups

The Ramanujan graphs $X^{p,q}$ are associated with finite groups $PGL_2(q)$ and $PSL_2(q)$. For K a field, the general linear group $GL_2(K)$ is the group of invertible 2 by 2 matrices with coefficients in K, and special linear group $SL_2(K)$ is the subgroup of matrices with determinate 1. The projective linear group $PGL_2(q)$, aka the projective general linear group is the quotient of $GL_2(K)$ by its center:

$$PGL_2(K) = GL_2(K)/Z(GL_2(K)) = GL_2(K)/K^{\times}.$$

Similarly, the projective special linear group $PSL_2(K)$ is the quotient of $SL_2(K)$ by its center:

$$PSL_2(K) = SL_2(K)/Z(SL_2(K)) = SL_2(K)/\left\{a \in K^{\times} : a^2 = 1\right\}.$$

If $K = \mathbb{F}_q$, then we write $GL_2(q), SL_2(q), PGL_2(q), PSL_2(q)$ for short. We are able to embed both $PGL_2(K)$ and $PSL_2(K)$ into the group of permutations of the projective line over K,

$$P^1(K) = K \cup \{\infty\}.$$

For every

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K),$$

we may associate the Möbius transformation $\varphi_M : P^1(K) \to P^1(K)$ defined by

$$\varphi_M(z) = \frac{az+b}{cz+d}.$$

Set $\varphi_M(-\frac{d}{c}) = \infty$ and

$$\varphi_M(\infty) = \begin{cases} \frac{a}{c}, & \text{if } c \neq 0 \\ \infty, & \text{if } c = 0. \end{cases}$$

This is a group homomorphism

$$\varphi : GL_2(K) \to \mathrm{Sym}P^1(K)$$

where $\varphi(M) = \varphi_M$.

6. Explicit Construction

Theorem 5 (Lagrange's Four Square Theorem)**.** *Any natural number n can be represented as the sum of four integer squares*

$$n = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

Jacobi gives an exact formula for the number of representations of n.

Definition 6. Let G be a group and S be a finite, nonempty subset of G. The *Cayley graph* $\mathcal{G}$(G, S) is the graph with vertex set $V = G$, and edge set

$$E = \{\{x, y\} : x, y \in G, \exists s \in S \text{ s.t. } y = x \cdot s\}.$$

If $q \equiv 1 \pmod 4$, then -1 is a square in $\mathbb{F}_q$. If $p \neq q$ are primes with $p, q \equiv 1 \pmod 4$ and $u \in \mathbb{Z}$ such that $u^2 \equiv -1 \pmod q$, then by Jacobi's Theorem there are $8(p+1)$ solutions $v = (a, b, c, d)$ with $a^2 + b^2 + c^2 + d^2 = p$. To each v, associate the matrix

$$V = \begin{pmatrix} a + ub & c + ud \\ -c + ud & a - ub \end{pmatrix}.$$

There are $p + 1$ solutions with $a > 0$ and odd and b, c, d even. We have S, the set of $p + 1$ corresponding matrices in $G = PGL_2(q)$. The Cayley graphs $\mathcal{G}(G, S)$ are the desired Ramanujan graphs. By varying q we are able to get an infinite family of $(p + 1)$-regular Ramanujan graphs $X^{p,q}$.

The original paper of Lubotzky, Phillips, and Sarnak gives two constructions of the Ramanujan graphs $X^{p,q}$. One is based on quaternion algebra and the other describes $X^{p,q}$ as a Cayley graph of $PGL_2(q)$ or $PSL_2(q)$. The isomorphism of these two constructions depends on the Hardy-Littlewood theory of quadratic forms. The first construction produces connected graphs by construction, and the second provides information about the number of vertices.

7. Open problems

- For p^n, p prime, there have been infinite families of $(p^n + 1)$-regular Ramanujan graphs constructed. Extend the results of Morgenstern by explicitly constructing an infinite family of k-regular Ramanujan graphs for all $k \geq 3$. The smallest open value is currently $k = 7$.
- The only known constructions of infinite families of Ramanujan graphs involve results from number theory or algebraic geometry. Find a combinatorial approach to explicitly constructing an infinite families of Ramanujan graphs instead.

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