Two Interior-Point Methods for Nonlinear $P_*(\tau)$-Complementarity Problems[1]

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Abstract. Two interior-point algorithms using a wide neighborhood of the central path are proposed to solve nonlinear P_*-complementarity problems. The proof of the polynomial complexity of the first method requires that the problem satisfies a scaled Lipschitz condition. When specialized to the monotone complementarity problems, the results of the first method are similar to the ones in Ref. 1. The second method is quite different from the first in that the proof of its global convergence does not require the scaled Lipschitz assumption. At each step of this algorithm, however, one has to compute an approximate solution of a nonlinear system such that a certain accuracy requirement is satisfied.

Key Words. Interior-point algorithms, nonlinear P_*-complementarity problems, polynomial complexity, scaled Lipschitz condition.

1. Introduction

Consider the complementarity problem (CP), that is, finding a pair $(x, u) \in R^n \times R^n$ such that

$$u = F(x), \quad (x, u) \geq 0 \text{ and } x^T u = 0,$$

where $F : R^n \to R^n$ is continuously differentiable on a neighborhood of R^n_+ (the nonnegative orthant). This problem has been extensively studied for several decades. It is a link between such concepts as fixed point theory, variational inequalities, linear and nonlinear analysis and other areas of applied mathematics like optimization, game theory, economics, classical mechanics and stochastic optimal control (see, Refs. 2−4). For a good introduction to CPs and the traditional solution methods we refer the reader to the book of Cottle et al. (Ref. 2).

The fundamental work of McLinden (Ref. 5) brought us a lot of ideas to develop interior-point algorithms for CPs (Refs. 6–26). A major advantage of interior-point methods, compared with other methods for CPs, is their polynomial worst-case complexity, that is, the total number of iterations necessary for finding an approximate solution is bounded by a polynomial of n multiplied by $|\log \varepsilon|$, where ε is the tolerance. However, many of the interior-point methods have been developed for solving monotone CPs. There are only a few papers investigating the interior-point methods for CPs which are not necessarily monotone (Refs. 9, 12, 15, 20, 21, 23–25, etc.). For example, Ye (Ref. 20) showed that the potential reduction algorithm in Ref. 11 solves the linear P-CPs (i.e., the CPs with P-finctions) in polynomial time. Later, Ye and Pardalos (Ref. 21) extended the algorithm to a class of linear CPs including the monotone and P-CPs. Kojima et al. (Ref. 10) introduced the class of P_*-matrices and proposed a potential reduction method to solve a linear P_*-CP in polynomial time. Miao (Ref. 15) generalized Mizuno-Todd-Ye's algorithm (Ref. 16) to linear P_*-CPs and showed that the algorithm is polynomially and quadratically convergent. Several infeasible-interior-point methods are also proposed for solving linear P_*-CPs by Potra et al. (Refs. 23–25). Recently, Jansen et al. (Ref. 9) extended the Jansen-Roos-Terlaky's method (Ref. 8) for monotone linear CPs to solving a class of nonmonotone nonlinear CPs. Their analysis needs a modified scaled Lipschitz condition. In fact, the similar conditions have been widely used in the analysis of interior-point algorithm for nonlinear problems (Refs. 1, 6, 17–19, 26–28).

In Section 3 of this paper, we develop a long-step interior-point method for a class of nonmonotone nonlinear CPs, where the Jacobian $F'(x)$ is a P_*-matrix (i.e., the class of CPs discussed in Ref. 9). In fact, long-step methods have been proved to be successful for solving linear CPs. Recently, Sun et al. (Refs. 1, 18, 26) extended a long-step method to monotone nonlinear complementarity and variational inequality problems. One purpose of this paper is to extend the algorithm of Sun et al. to the aforementioned class of nonmonotone nonlinear

CPs. The proof of polynomial complexity also requires a scaled Lipschitz condition.

While the scaled Lipschitz condition is indeed satisfied in some special cases, it seems to be restrictive for many nonlinear maps. An interesting problem in present research is then how this assumption can be relaxed so as to make the algorithm applicable to a wide class of problems. In Section 4, we will provide an interior-point algorithm for nonlinear complementarity problems, whose global convergence does not require the scaled Lipschitz condition. A nonlinear equation, however, needs to be solved approximately in each iteration of the algorithm.

Throughout the paper, we will adopt the standard notations used in literature. For example, for a generic vector $x \in R^n$, x^T will denote the vector transpose, and the corresponding capital X will denote the diagonal matrix whose (i, i)th entry is given by the ith component of x, designated by x_i. The Euclidean-norm and the infinity norm of x will be represented by $\|x\|$ and $\|x\|_\infty$, respectively. The vector e denotes the vector of all ones whose dimension is defined by the context of its use. For any index set $I \subseteq \{1, 2, ..., n\}$, x_I denotes the vector made up of components x_i for $i \in I$ (with the x_i's arranged in the same order as in x). If $D \in R^{n\times n}$ is diagonal, then $D_I \in R^{I\times I}$ denotes the diagonal matrix constructed from D_{ii} for $i \in I$.

2. Problem Statement

We denote the sets of feasible and interior-feasible points of a CP as follows:

$$S = \{(u, x) \in R^{2n} : u = F(x), (u, x) \geq 0\},$$

$$S^0 = \{(u, x) \in R^{2n} : u = F(x), (u, x) > 0\}.$$

We assume that S^0 is not empty. The following is the definition of the P_*-matrix.

Definition 2.1 See Ref. 10. The matrix $H \in R^{n\times n}$ is a $P_*(\tau)$-matrix, if there exists a constant $\tau \geq 0$ such that

$$(1+\tau) \sum_{i\in I_+(x)} x_i(Hx)_i + \sum_{i\in I_-(x)} x_i(Hx)_i \geq 0 \quad \text{for all } x \in R^n, \tag{1}$$

where

$$I_+(x) = \{i : x_i(Hx)_i > 0\}, \quad I_-(x) = \{i : x_i(Hx)_i < 0\}.$$

The matrix H is a P_*-matrix if it belongs to the union of all $P_*(\tau)$-matrices ($\tau \geq 0$).

Clearly, (1) can be written as

$$x^T Hx \geq -\tau \sum_{i\in I_+(x)} x_i(Hx)_i.$$

The class of P_*-matrices includes positive semidefinite matrices and P-matrices as the special cases. It is known that a matrix is a P_*-matrix if and only if it is sufficient (Refs. 30, 31).

In the next section of this paper, we impose the following two conditions on the mapping F to show the polynomial complexity of a long-step interior-point algorithm for nonlinear complementarity problems.

Condition 2.1. There exists a constant $\tau \geq 0$ such that the Jacobian $F'(x)$ of the mapping F is a $P_*(\tau)$-matrix for all $x \geq 0$.

Condition 2.2 Scaled Lipschitz Condition. There exists a scalar $M > 0$ such that the inequality below

$$\|X(F(x+\Delta x) - F(x) - F'(x)\Delta x)\|_\infty \leq M|\Delta x^T F'(x)\Delta x|$$

holds for any $x > 0$ and Δx satisfing $\|X^{-1}\Delta x\| \leq 1$.

The above Scaled Lipschitz Condition (SLC for short) can be viewed as a modified version of the Zhu's SLC (Ref. 28). The advantage of our SLC is that it does not require the monotonicity of the mapping. It is also quite different from that used by Jansen et al. (Ref. 9). In fact, scaled Lipschitz assumptions have been widely used in the analysis of the interior-point algorithms for convex programming (Ref.28), monotone CPs (Refs. 17–19, 26) and monotone variational inequalities (Refs. 1, 27).

For many of the nonlinear mappings, Condition 2.2 seems to be restrictive. In Section 4 we will provide an interior-point method for a class of nonmonotone CPs, whose global convergence does not require the scaled Lipschitz assumption. We will make use of the following concept of nonlinear P_*-mapping.

Definition 2.2 See Ref. 32. A mapping $F : R^n \to R^n$ is said to be a $P_*(\tau)$-mapping if there exists a scalar $\tau \geq 0$ such that

$$(1+\tau) \sum_{i \in I_+(x,y)} (x_i - y_i)(F_i(x) - F_i(y)) + \sum_{i \in I_-(x,y)} (x_i - y_i)(F_i(x) - F_i(y)) \geq 0 \tag{2}$$

for all x, y in R^n, where $I_+(x,y) = \{i : (x_i - y_i)(F_i(x) - F_i(y)) > 0\}$ and $I_-(x,y) = \{1, 2, ..., n\} \setminus I_+$. The class of P_*-mappings is the union of all $P_*(\tau)$-mappings ($\tau \geq 0$).

The following condition is used in Section 4.

Condition 2.3. F is a $P_*(\tau)$-mapping.

It is worth noting that (2) can be written as follows

$$(x-y)^T(F(x) - F(y)) \geq -\tau \sum_{I_+(x,y)} (x_i - y_i)(F_i(x) - F_i(y)). \tag{3}$$

Clearly, each monotone mapping (corresponding to the case $\tau = 0$) must be a P_*-mapping. The class of P_*-mappings also includes as a special case a subclass of nonlinear uniform P-functions (Ref. 32). If $F = Hx + q$, where $H \in R^{n\times n}$ and $q \in R^n$, it is evident that F is a P_*-mapping if and only if H is a P_*-matrix. The following lemma shows that the Jacobian of a nonlinear P_*-mapping is a P_*-matrix, but the converse is not clear yet.

Proposition 2.1. If F is a continuously differentiable P_*-mapping on R^n, then for each $u \in R^n$, the Jacobian $F'(u)$ is a P_*-matrix.

Proof. Let $\tau \geq 0$ be a constant such that (2) (equally, (3)) holds. For an arbitrary $u \in R^n$ we show that

$$v^T F'(u)v \geq -\tau \sum_{i\in I_+(v)} v_i(F'(u)v)_i \quad \text{for all } v \in R^n,$$

where

$$I_+(v) = \{i : v_i(F'(u)v)_i > 0\} = \{i : v_i F_i'(u)v > 0\},$$

where $F_i'(u)$ is the ith row vector of $F'(u)$. We denote $I_-(v) = \{1, 2, ..., n\} \setminus I_+(v)$. In fact, for all $i \in I_+(v)$, we have

$$\lim_{t\to 0^+} [v_i(F_i(u + tv) - F_i(u))]/t = v_i F_i'(u)v > 0.$$

Similarly,

$$\lim_{t\to 0^+} [v_i(F_i(u + tv) - F_i(u))]/t \leq 0 \quad \text{for all } i \in I_-(v). \tag{4}$$

Thus for sufficiently small $t > 0$

$$I_+(v) \subset I_+(u + tv, v) = \{i : v_i(F_i(u + tv) - F_i(u)) > 0\},$$

and hence

$$I_+(u + tv, v) \setminus I_+(v) \subset I_-(v). \tag{5}$$

By using (3), (4) and (5) we have

$$\begin{aligned}
v^T F'(u)v &= \lim_{t\to 0^+} v^T(F(u + tv) - F(u))/t \\
&\geq \lim_{t\to 0^+} \left[-\tau \sum_{i\in I_+(u+tv,v)} v_i(F_i(u + tv) - F_i(u))/t\right] \\
&= -\tau \lim_{t\to 0^+} \left(\sum_{i\in I_+(v)} + \sum_{i\in I_+(u+tv,v)\setminus I_+(v)}\right) v_i(F_i(u + tv) - F_i(u))/t \\
&\geq -\tau \sum_{i\in I_+(v)} v_i(F'(u)v)_i
\end{aligned}$$

which implies the desired result. □

In what follows, we employ the following set as a neighborhood of the central path, which plays a key role in our analysis:

$$\mathcal{N}_\infty^-(\beta) = \{(u,x) > 0 : Ux \geq \beta\mu e, \text{where } u = F(x), \mu = u^T x/n\},$$

where β is a fixed positive scalar.

3. Long-Step Algorithm and Its Complexity under SLC

In this section we propose a long-setp method to solve the nonlinear CPs. We show that the algorithm converges globally with $O[n(\tau+2)\max\{1,M\}\log(\mu^0/\varepsilon)]$ iteration complexity under Conditions 2.1 and 2.2, where, $\mu^0 = (x^0)^T u^0/n$ and (u^0, x^0) is the initial feasible point.

Suppose that the initial feasible point $(u^0, x^0) \in \mathcal{N}_\infty^-(\beta)$ is available. We now formally specify the algorithm as follows.

Algorithm 3.1:

Step 0. Let $\varepsilon > 0$ be a given tolerance and let $\beta \in (0,1)$ and $\lambda \in (0,1)$ be two constants. Given an initial solution $(u^0, x^0) \in \mathcal{N}_\infty^-(\beta)$. Set $k := 0$.

Step 1. If $\mu^k = (x^k)^T u^k/n \leq \varepsilon$, stop; Otherwise, go to Step 2.

Step 2. At the present point (u^k, x^k), solving the following linear system

$$U^k \Delta x + X^k \Delta u = -U^k x^k + \lambda \mu^k e, \tag{6}$$

$$\Delta u = F'(x^k)\Delta x, \tag{7}$$

where $\mu^k = (x^k)^T u^k/n$.

Step 3. Denote

$$x(\theta) = x^k + \theta\Delta x, \quad u(\theta) = F(x(\theta)), \quad \mu(\theta) = x(\theta)^T u(\theta)/n.$$

Define θ_k by minimizing $u(\theta)^T x(\theta)$ for all $\theta \in [0,1]$ such that $(u(\theta_k), x(\theta_k)) \in \mathcal{N}_\infty^-(\beta)$.

Step 4. Let $x^{k+1} = x(\theta_k), u^{k+1} = u(\theta_k)$ and $\mu^{k+1} = \mu(\theta_k)$. Set $k := k+1$ and go to Step 1.

Remark 3.1. When $\lambda = 0$, the direction generated by the system (6) and (7) is called the primal-dual affine scaling direction. If $\lambda = 1$, the direction is called the centering direction. When $0 < \lambda < 1$, the direction can be viewed as the combination of the two directions above. We perform a linear search to reduce μ^k more significantly, thus the algorithm allows long steps and fast convergence. For the convenience of notations, in what follows, we suppress the iteration index k. For example, we denote x^k by x.

Lemma 3.1. Assume that Condition 2.1 is satisfied. Let $(u,x) \in \mathcal{N}_\infty^-(\beta)$. Then the linear system (6) and (7) has a unique solution.

Proof. This result is immediate from Lemma 4.1 of Ref. 10. □

The following three lemmas are very useful for our complexity analysis.

Lemma 3.2. Suppose that Condition 2.1 is satisfied. Let $(u,x) > 0$ and D be the primal-dual scaling, i.e., $D = (XU^{-1})^{1/2}$. If $(\Delta u, \Delta x)$ is the solution to the system (6) and (7), then we have

$$(-\tau/4)\|z\|^2 \le \Delta u^T \Delta x \le \|z\|^2/4, \tag{8}$$

$$\|\Delta U \Delta x\| \le (\tau+2)\|z\|^2/4, \tag{9}$$

$$\max\{\|D^{-1}\Delta x\|, \|D\Delta u\|\} \le [(2+\tau)/2]^{1/2}\|z\|, \tag{10}$$

where

$$z = (XU)^{-1/2}(-Ux + \lambda\mu e). \tag{11}$$

Proof. Multiplying both sides of (6) by $(XU)^{-1/2}$, we have

$$D^{-1}\Delta x + D\Delta u = z$$

where z is given by (11). Let J be an arbitrary subset of $\{1, ..., n\}$. Then from the above

$$D_J^{-1}\Delta x_J + D_J \Delta u_J = z_J,$$

whence

$$\begin{aligned} \Delta x_J^T \Delta u_J &= \Delta x_J^T D_J^{-1} z_J - \Delta x_J^T D_J^{-2} \Delta x_J \\ &= \Delta x_J^T D_J^{-1} z_J - \|D_J^{-1}\Delta x_J\|^2 \le \|z_J\|^2/4, \end{aligned} \tag{12}$$

where the last inequality follows from the relation: $v_1^T v_2 - \|v_2\|^2 \le \|v_1\|^2/4$. Particularly, if $J = \{1, 2, ..., n\}$, we have

$$\Delta x^T \Delta u \le \|z\|^2/4.$$

On the other hand, setting $J = I_+ = \{i : \Delta x_i \Delta u_i > 0\}$, from (12) we obtain

$$\Delta x_{I_+}^T \Delta u_{I_+} \le \|z_{I_+}\|^2/4 \le \|z\|^2/4.$$

Since $(\Delta x, \Delta u)$ is the solution to the system (6) and (7) and $F'(x)$ is a $P_*(\tau)$ -matrix, we have

$$\begin{aligned} \Delta x^T \Delta u = \Delta x^T F'(x) \Delta x &\ge -\tau \sum_{i \in I_+} \Delta x_i (F(x)\Delta x)_i \\ &= -\tau \Delta x_{I_+}^T \Delta u_{I_+} \ge -\tau \|z\|^2/4. \end{aligned} \tag{13}$$

Thus the inequality (8) holds. Since

$$\|\Delta X \Delta u\| \leq \|D^{-1}\Delta x\| \|D\Delta u\| \leq (\|D^{-1}\Delta x\|^2 + \|D\Delta u\|^2)/2$$

and $\|D^{-1}\Delta x + D\Delta u\|^2 = \|z\|^2$, we have

$$\begin{aligned} 2\|\Delta X \Delta u\| &\leq \|D^{-1}\Delta x\|^2 + \|D\Delta u\|^2 \\ &= \|z\|^2 - 2\Delta x^T \Delta u \leq (\tau + 2)\|z\|^2/2 \end{aligned}$$

which implies the inequalities (9) and (10). $\square$

Lemma 3.3. Suppose that Conditions 2.1 and 2.2 are satisfied. Let $(u, x) \in \mathcal{N}_\infty^-(\beta)$ and $\lambda \leq 2\beta$, and let $(\Delta u, \Delta x)$ be the solution to the system (6) and (7). Denote

$$T(t) = t^2 \Delta U \Delta x + (X + t\Delta X)[F(x + t\Delta x) - F(x) - tF'(x)\Delta x]. \tag{14}$$

If $\theta\|X^{-1}\Delta x\| \leq 1$, then

$$\|T(\theta)\| \leq (2 + \tau + 2M \max\{1, \tau\})\theta^2 n\mu/4 = n\mu\theta^2 C^*, \tag{15}$$

where M is the constant defined in Condition 2.2, and

$$C^* = (2 + \tau + 2M \max\{1, \tau\})/4. \tag{16}$$

Proof. Notice that $\theta\|X^{-1}\Delta x\| \leq 1$ and by using Condition 2.2 we have

$$\begin{aligned} \|T(\theta)\|_\infty &= \|\theta^2 \Delta U \Delta x\|_\infty + \|(I + \theta X^{-1}\Delta X)X(F(x + \theta\Delta x) \\ &\quad -F(x) - \theta F'(x)\Delta x)\|_\infty \\ &\leq \theta^2\|\Delta U \Delta x\|_\infty + 2\|X(F(x + \theta\Delta x) - F(x) - \theta F'(x)\Delta x)\|_\infty \\ &\leq \theta^2\|\Delta U \Delta x\|_\infty + 2M\theta^2|\Delta x^T F'(x)\Delta x| \\ &\leq \theta^2\|\Delta U \Delta x\|_\infty + 2\theta^2 M|\Delta x^T \Delta u|. \end{aligned}$$

We apply Lemma 3.2 to reduce

$$|\Delta u^T \Delta x| \leq \max\{1, \tau\}\|z\|^2/4$$

and

$$\|\Delta U \Delta x\|_\infty \leq \|\Delta U \Delta x\| \leq (\tau + 2)\|z\|^2/4,$$

where z is given by (11). Thus we have

$$\|T(\theta)\|_\infty \leq (\tau + 2 + 2M \max\{1, \tau\})\theta^2\|z\|^2/4. \tag{17}$$

Notice that $\lambda \le 2\beta$ we have

$$\begin{aligned}
\|z\|^2 &= \|(UX)^{-1/2}(-Ux + \lambda\mu e)\|^2 \\
&= \|-(UX)^{1/2}e + \lambda\mu(UX)^{-1/2}e\|^2 \\
&= \|(UX)^{1/2}e\|^2 + \lambda^2\mu^2\|(UX)^{-1/2}e\|^2 - 2\lambda\mu e^T e \\
&\le u^T x + \lambda^2\mu^2 n/(\beta\mu) - 2\lambda\mu n \\
&= (1 + \lambda^2/\beta - 2\lambda)\mu n \\
&\le \mu n. \qquad (18)
\end{aligned}$$

Combining (17) and (18) yields (15). □

Lemma 3.4. Let $(u, x) \in \mathcal{N}_\infty^-(\beta)$ and $(\Delta u, \Delta x)$ be the solution to (6) and (7) and $0 < \lambda \le 2\beta$. Let

$$\theta^* = \min\{1/\|X^{-1}\Delta x\|, (1-\beta)\lambda/[(1+\beta)nC^*]\} \qquad (19)$$

where C^* is the constant given by (16). Then for all $\theta \in [0, \theta^*]$, we have that $(u(\theta), x(\theta)) \in \mathcal{N}_\infty^-(\beta)$ and

$$\mu(\theta) \le [1 - (1 + \beta - 2\lambda)\theta/(1+\beta)]\mu. \qquad (20)$$

Proof. Let $T(\theta)$ be defined by (14). By using the same proof of Lemma 3.2 of Ref. 1, it is easy to prove that for any $\theta \in [0, 1]$

$$X(\theta)u(\theta) \ge [(1-\theta)\beta\mu + \lambda\mu\theta]e - \|T(\theta)\|_\infty e,$$

$$\mu(\theta) = e^T X(\theta)u(\theta)/n \le (1 - \theta + \theta\lambda)\mu + \|T(\theta)\|_\infty, \qquad (21)$$

and hence

$$X(\theta)u(\theta) - \beta\mu(\theta)e \ge (1-\beta)\lambda\mu\theta e - (1+\beta)\|T(\theta)\|_\infty e. \qquad (22)$$

If $\theta \in [0, \theta^*]$, then $\theta\|X^{-1}\Delta x\| \le 1$, thus by Lemma 3.3 we have

$$\|T(\theta)\| \le C^* n\mu\theta^2. \qquad (23)$$

Substituting the above into (22), we have

$$\begin{aligned}
X(\theta)u(\theta) - \beta\mu(\theta)e &\ge (1-\beta)\lambda\mu\theta e - (1+\beta)C^* n\mu\theta^2 e \\
&= [(1-\beta)\lambda - (1+\beta)C^* n\theta]\mu\theta e \\
&\ge 0
\end{aligned}$$

for all $\theta \le (1-\beta)\lambda/[(1+\beta)nC^*]$. By the continuity of $u(\theta)$ and $x(\theta)$ and notice that $u > 0$ and $x > 0$, we deduce that $u(\theta) > 0$ and $x(\theta) > 0$ for all $\theta \in [0, \theta^*]$, and hence

$(u(\theta), x(\theta)) \in \mathcal{N}_\infty^-(\beta)$ for all $\theta \in [0, \theta^*]$. Moreover, for such θ, by (21) and (23) we have

$$\begin{aligned}
\mu(\theta) &\le (1-\theta+\theta\lambda)\mu + C^* n\mu\theta^2 \\
&\le (1-\theta+\theta\lambda)\mu + C^* n\mu\theta(1-\beta)\lambda/[(1+\beta)nC^*] \\
&= [1-(1+\beta-2\lambda)\theta/(1+\beta)]\mu
\end{aligned}$$

which is the desired inequality (20). $\square$

We are now ready to prove the main complexity result.

Theorem 3.1. Assume that Conditions 2.1 and 2.2 are satisfied. Let $0 < \lambda \le 2\beta$ and $0 < \beta \le 1/16$. Then the sequence $\{(u^k, x^k)\}$ generated by Algorithm 3.1 satisfies $(u^k, x^k) \in \mathcal{N}_\infty^-(\beta)$ and

$$\mu^{k+1} \le [1 - \delta/(n(\tau+2)\max\{1, M\})]\mu^k, \tag{24}$$

where δ is a constant given as follows

$$\delta = 4(1+\beta-2\lambda)(1-\beta)\lambda/(3(1+\beta)^2) < 1/6.$$

Thus the algorithm is globally convergent. Moreover, for any $\varepsilon > 0$ the algorithm will create an approximate solution (x^k, u^k) satisfying $\mu^k \le \varepsilon$ in at most $O[n(\tau+2)\max\{1, M\}\log(\mu^0/\varepsilon)]$ iterations.

Proof. We again suppress the iteration index k. Let $(u, x) \in \mathcal{N}_\infty^-(\beta)$. We will show, along the lines of the proof of Lemma 3.6 of Ref. 1, that the relation (24) holds. By using (10) and (18) we have

$$\begin{aligned}
\|X^{-1}\Delta x\| &= \|(UX)^{-1/2}D^{-1}\Delta x\| \\
&\le \|(UX)^{-1/2}\|\|D^{-1}\Delta x\| \\
&\le \left(1/(\beta\mu)^{1/2}\right)((\tau+2)/2)^{1/2}(n\mu)^{1/2} \\
&= (n(\tau+2)/(2\beta))^{1/2}.
\end{aligned} \tag{25}$$

Since

$$C^* = (2+\tau+2M\max\{1,\tau\})/4 \ge (\tau+2)/4$$

and $\lambda \le 2\beta$, $n(2+\tau) \ge 2$, $(1-\beta)/(1+\beta) \le 1$ and $0 < \beta \le 1/16$, it follows that

$$\begin{aligned}
\theta^c : &= (1-\beta)\lambda/[(1+\beta)nC^*] \\
&\le (1-\beta)\lambda/[(1+\beta)n(\tau+2)/4] \\
&\le 4(1-\beta)(2\beta)/[(1+\beta)n(\tau+2)] \\
&= \left(4(1-\beta)/\left((1+\beta)(n(\tau+2)/(2\beta))^{1/2}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
& \cdot \left(1/(n(\tau+2)/(2\beta))^{1/2}\right) \\
\leq\ & (4\sqrt{\beta})(1/\|X^{-1}\Delta x\|) \\
\leq\ & 1/\|X^{-1}\Delta x\|.
\end{aligned}
$$

Thus it follows from (19) that $\theta^c = \theta^*$. By Lemma 3.4, we deduce that $(x(\theta^c), u(\theta^c)) \in \mathcal{N}_\infty^-(\beta)$ and

$$
\begin{aligned}
\mu(\theta^c) &\leq [1 - (1+\beta-2\lambda)\theta^c/(1+\beta)]\mu \\
&= [1 - (1+\beta-2\lambda)(1-\beta)\lambda/((1+\beta)^2 nC^*)]\mu.
\end{aligned}
$$

Since $\max\{1,\tau\} \leq \tau + 2$, we have

$$
C^* \leq (\tau+2)(1+2M)/4 \leq 3(\tau+2)\max\{1,M\}/4.
$$

Hence

$$
\mu(\theta^c) \leq [1 - \delta/(n(\tau+2)\max\{1,M\})]\mu, \tag{26}
$$

where

$$
0 < \delta = 4(1+\beta-2\lambda)(1-\beta)\lambda/3(1+\beta)^2 < 1/6.
$$

By the definition of θ_k, we have that $(u^{k+1}, x^{k+1}) = (u(\theta_k), x(\theta_k)) \in \mathcal{N}_\infty^-(\beta)$ and $\mu^{k+1} = \mu(\theta_k) \leq \mu(\theta^c)$, which together with (26) imply the desired inequality (24). The estimation $O[n(\tau+2)\max\{1,M\}\log(\mu^0/\varepsilon)]$ is then immediate from (24). The proof is complete. $\square$

Remark 3.2. It is clear to see from the above proof that when the parameter τ and M can be determined in advance we can take

$$
\theta_k = (1-\beta)\lambda/(1+\beta)nC^*
$$

to assure the same result of Theorem 3.1. However in general, τ and M is not known in advance, thus we choose to perform the line search at the Step 3 of Algorithm 3.1.

4. Interior-Point Method and Global Convergence without SLC

In this section, we will provide an interior-point method to solve the nonlinear CPs where the mappings satisfy Condition 2.3. We show that for any $\varepsilon > 0$ the proposed algorithm will produce an approximate solution (x^k, y^k) satisfying $\mu^k \leq \varepsilon$ in at most $O[n(\tau+2)\log(\mu^0/\varepsilon)]$ iterations. Our proof does not require a scaled Lipschitz assumption. The algorithm proceeds as follows.

Algorithm 4.1:

Step 0. Let $\varepsilon > 0$ be a given tolerance. Let $\beta \in (0,1)$, $0 < \lambda \leq 1/(2(2-\beta))$ and $\alpha \in (0, 1-\beta)$ be three constants. Given an initial solution $(u^0, x^0) \in \mathcal{N}_\infty^-(\beta)$. Let

$$\bar{\theta} = (1 - \beta - \alpha)\lambda/\hat{C}, \tag{27}$$

where $\hat{C}$ is constant given by

$$\hat{C} = n(\tau + 2)\left(1 + (1 + \alpha^2)\lambda/\beta^{1/2}\right)^2/4. \tag{28}$$

Step 1. If $\mu^k \leq \varepsilon$, stop; Otherwise, go to Step 2.

Step 2. At the present point (u^k, x^k), solving the following nonlinear equation

$$\phi^k(\Delta x) := X^k[F(x^k + \bar{\theta}\Delta x) - F(x^k)]/\bar{\theta} + U^k(\Delta x + x^k) - \lambda\mu^k e = 0. \tag{29}$$

It is required only that an approximate solution, denoted also by Δx, is found such that

$$\|\phi^k(\Delta x)\|_\infty \leq \alpha^2\lambda\mu^k, \tag{30}$$

and set

$$\Delta u := [F(x^k + \bar{\theta}\Delta x) - F(x^k)]/\bar{\theta}.$$

Step 3. Let

$$x^{k+1} = x^k + \bar{\theta}\Delta x, \quad u^{k+1} = u^k + \bar{\theta}\Delta u.$$

Set $k := k + 1$ and go to Step 1.

Remark 4.1. We first note that the steplength $\bar{\theta}$ is fixed in all iterations of the algorithm and $\bar{\theta} < 1$. From Step 2 of the above algorithm, the approximate solution $(\Delta x, \Delta u)$ satisfies the following system

$$\Delta u = [F(x^k + \bar{\theta}\Delta x) - F(x^k)]/\bar{\theta}, \tag{31}$$

$$X^k\Delta u + U^k\Delta x = -U^k x^k + \lambda\mu^k e + \phi^k(\Delta x). \tag{32}$$

It is also worth noting that the point (u^k, x^k) generated by the algorithm is feasible in the sense that $u^k = F(x^k)$ for all $k \geq 0$.

Throughout the remainder of the section we denote

$$x(t) = x^k + t\bar{\theta}\Delta x, \quad u(t) = u^k + t\bar{\theta}\Delta u, \quad \mu(t) = x(t)^T u(t)/n.$$

Clearly, $(u(1), x(1)) = (x^{k+1}, u^{k+1})$ and $(u(0), x(0)) = (x^k, u^k)$. In what follows, the iteration index is dropped for the sake of convenience. The following result is very similar to Lemma 3.2.

Lemma 4.1. Assume that Condition 2.3 is satisfied. Let $(u, x) > 0$ and $D = (XU^{-1})^{1/2}$. η is an arbitrary positive constant. Given a vector $w \in R^n$, if $(\Delta x, \Delta u)$ satisfies the relation below

$$\Delta u = (F(x + \eta\Delta x) - F(x))/\eta,$$
$$D^{-1}\Delta x + D\Delta u = w,$$

then (8), (9) and (10) remain valid with $z = w$.

Proof. Let

$$I_+ = \{i : \Delta x_i \Delta u_i > 0\} = \{i : \Delta x_i (F_i(x + \eta\Delta x) - F_i(x))/\eta > 0\}.$$

Using Condition 2.3, we have

$$\begin{aligned} \Delta x^T \Delta u &= \Delta x^T (F(x + \eta\Delta x) - F(x))/\eta \\ &\geq -\tau \sum_{i \in I_+} \Delta x_i (F_i(x + \eta\Delta x) - F_i(x))/\eta \\ &= -\tau \sum_{i \in I_+} \Delta x_i \Delta u_i. \end{aligned}$$

Replacing (13) by the inequality above and repeating the proof of Lemma 3.2, the desired results follow. □

The next technical result is very useful in our analysis.

Lemma 4.2. Assume that Condition 2.3 is satisfied. Let $(u, x) \in \mathcal{N}_\infty^-(\beta)$ and $(\Delta x, \Delta u)$ be a direction generated at the Step 2 of Algorithm 4.1. Then

$$\begin{aligned} \|\Delta U \Delta x\|_\infty &\leq \|\Delta U \Delta x\| \\ &\leq n\mu(\tau + 2)\left(1 + (1 + \alpha^2)\lambda/\beta^{1/2}\right)^2/4 = \hat{C}\mu \end{aligned} \tag{33}$$

where $\hat{C}$ is given by (28).

Proof. Since the approximate solution $(\Delta x, \Delta u)$ satisfies the system (31) and (32), multiplying both sides of (32) by $(XU)^{-1/2}$ we have

$$D^{-1}\Delta x + D\Delta u = w,$$

where

$$w = (XU)^{-1/2}(-Ux + \lambda\mu e + \phi(\Delta x)). \tag{34}$$

Let $\eta = \bar{\theta}$ and w be given by (34), by Lemma 4.1 we have

$$\|\Delta U \Delta x\| \leq (\tau + 2)\|w\|^2/4. \tag{35}$$

Notice that $(u,x) \in \mathcal{N}_\infty^-(\beta)$ and by using (30), we have

$$\begin{aligned}
\|w\| &= \|(UX)^{-1/2}(-Ux+\lambda\mu e+\phi(\Delta x))\| \\
&\le \|(UX)^{-1/2}(-Ux+\lambda\mu e)\| + \|(UX)^{-1/2}\phi(\Delta x)\| \\
&\le \|-(UX)^{1/2}e\| + \lambda\mu\|(UX)^{-1/2}e\| \\
&\quad +\|(UX)^{-1/2}\|(n^{1/2}\|\phi(\Delta x)\|_\infty) \\
&\le (n\mu)^{1/2} + \lambda\mu n^{1/2}/(\beta\mu)^{1/2} + \alpha^2\lambda\mu n^{1/2}/(\beta\mu)^{1/2} \\
&\le (n\mu)^{1/2}\left(1+\lambda/\beta^{1/2}+\lambda\alpha^2/\beta^{1/2}\right).
\end{aligned}$$

Combining (35) and the relation above yields (33). □

The main result of this section is as follows.

Theorem 4.1. Assume that Condition 2.3 is satisfied and for each iteration, an approximate solution of (29) satisfing (30) can be obtained by a certain method in a constant number steps. If $(u^k,x^k) \in \mathcal{N}_\infty^-(\beta)$, then the next point generated by Algortihm 4.1 satisfies that $(u^{k+1},x^{k+1}) \in \mathcal{N}_\infty^-(\beta)$ and

$$\mu^{k+1} \le [1-\delta^*/(n(\tau+2))]\mu^k, \tag{36}$$

where

$$\delta^* = 2\lambda(1-\beta-\alpha)/\left(1+(1+\alpha^2)\lambda/\beta^{1/2}\right)^2.$$

Thus Algorithm 4.1 is globally convergent, and for any $\varepsilon > 0$ the algorithm will generate an approximate solution (u^k,x^k) such that $\mu^k \le \varepsilon$ in at most $O[n(\tau+2)\log(\mu^0/\varepsilon)]$ iterations.

Proof. We again suppress the superscripts and denote by (u^+,x^+) the (u^{k+1},x^{k+1}), and by μ^+ the μ^{k+1}. Notice that $(u,x) \in \mathcal{N}_\infty^-(\beta)$ and $\bar\theta < 1$, and by using (30), (33) and (27), for all $0 \le t \le 1$ we have

$$\begin{aligned}
X(t)u(t) &= (X+t\bar\theta\Delta X)(u+t\bar\theta\Delta u) \\
&= Xu + t\bar\theta(U\Delta x + X\Delta u) + (t\bar\theta)^2\Delta X\Delta u \\
&= Ux + t\bar\theta(\lambda\mu e - Ux + \phi(\Delta x)) + (t\bar\theta)^2\Delta U\Delta x \qquad (37)\\
&\ge (1-t\bar\theta)\beta\mu e + t\bar\theta\lambda\mu e - t\bar\theta\|\phi(\Delta x)\|_\infty e \\
&\quad -(t\bar\theta)^2\|\Delta U\Delta x\|_\infty e \\
&\ge (1-t\bar\theta)\beta\mu e + t\bar\theta\lambda\mu e - t\bar\theta\alpha^2\lambda\mu e - (t\bar\theta)^2\hat{C}\mu e \\
&= (1-t\bar\theta)\beta\mu e + (1-\alpha)t\bar\theta\lambda\mu e - t\bar\theta(1-\beta-\alpha^2)\lambda\mu e \\
&\ge \beta\mu(1-(1-\lambda)t\bar\theta)e. \qquad (38)
\end{aligned}$$

Since $\bar\theta < 1$ and $\lambda < 1$, from (38) we have that $X(t)u(t) > 0$ for all $t \in [0,1]$. By the continuity of $X(t)$ and $u(t)$ and notice that $(u,x) > 0$, we deduce that $(u^+,x^+) = (x(1),u(1)) > 0$.

To show $(u^+, x^+) \in \mathcal{N}_\infty^-(\beta)$, it suffices to prove that $X^+u^+ - \beta\mu^+ e \geq 0$. In fact, by (37) with $t = 1$, we have

$$\begin{aligned} \mu^+ &= e^T X^+ u^+/n \\ &= (1-\bar{\theta})\mu + \bar{\theta}\lambda\mu + \bar{\theta}e^T\phi(\Delta x)/n + \bar{\theta}^2 e^T \Delta U \Delta x/n. \end{aligned} \tag{39}$$

It is easy to show that

$$\|Z_1 z_2 - \eta(z_1^T z_2/n)e\| \leq \|Z_1 z_2\| \tag{40}$$

holds for all $z_1, z_2 \in R^n$ and $0 \leq \eta \leq 2$. Thus by using (37) with $t = 1$, and by using (39), (40), (33) and (27), we have

$$\begin{aligned} & X^+u^+ - \beta\mu^+ e \\ = \ & [(1-\bar{\theta})Ux + \lambda\bar{\theta}\mu e + \bar{\theta}\phi(\Delta x) + \bar{\theta}^2 \Delta U \Delta x] \\ & -\beta[(1-\bar{\theta})\mu + \bar{\theta}\lambda\mu + \bar{\theta}e^T\phi(\Delta x)/n + \bar{\theta}^2 e^T \Delta U \Delta x/n]e \\ \geq \ & (1-\beta)\lambda\bar{\theta}\mu e + \bar{\theta}[\phi(\Delta x) - \beta(e^T\phi(\Delta x)/n)e] \\ & +\bar{\theta}^2[\Delta U \Delta x - \beta(\Delta x^T \Delta u/n)e] \\ \geq \ & (1-\beta)\lambda\hat{\theta}\mu e - \bar{\theta}(\|\phi(\Delta x)\|_\infty + \beta\|\phi(\Delta x)\|/n^{1/2})e \\ & -\bar{\theta}^2\|\Delta U \Delta x - \beta(\Delta x^T \Delta u/n)\|e \\ \geq \ & (1-\beta)\lambda\bar{\theta}\mu e - \bar{\theta}(1+\beta)\|\phi(\Delta x)\|_\infty e - \bar{\theta}^2\|\Delta U \Delta x\|e \\ \geq \ & (1-\beta)\lambda\bar{\theta}\mu e - \bar{\theta}\alpha^2(1+\beta)\lambda\mu e - \bar{\theta}^2\hat{C}\mu e \\ = \ & (1-\beta)\lambda\bar{\theta}\mu e - \bar{\theta}\alpha^2(1+\beta)\lambda\mu e - \bar{\theta}(1-\beta-\alpha)\lambda\mu e \\ = \ & \bar{\theta}\alpha(1-(1+\beta)\alpha)\lambda\mu e \\ \geq \ & \bar{\theta}\alpha\beta^2\lambda\mu e. \end{aligned}$$

The last inequality above follows from that $\alpha < 1 - \beta$. Therefore $(u^+, x^+) \in \mathcal{N}_\infty^-(\beta)$. In the rest of the proof, we show the estimation (36) holding. Notice that $\lambda \leq 1/[2(2-\beta)]$, from (39), (30), (33) and (27) we have

$$\begin{aligned} \mu^+ &\leq (1-\bar{\theta}+\bar{\theta}\lambda)\mu + \bar{\theta}\|\phi(\Delta x)\|_\infty + \bar{\theta}^2\|\Delta X \Delta u\|_\infty \\ &\leq \left[1-\bar{\theta}+\bar{\theta}\lambda+\bar{\theta}\alpha^2\lambda+\bar{\theta}(1-\beta-\alpha)\lambda\right]\mu \\ &= [1-\bar{\theta}(1-\lambda-(1-\beta)\lambda)]\mu - \bar{\theta}\lambda\alpha(1-\alpha)\mu \\ &\leq [1-\bar{\theta}(1-\lambda(2-\beta))]\mu \\ &\leq (1-\bar{\theta}/2)\mu \\ &= [1-(1-\beta-\alpha)\lambda/(2\hat{C})]\mu \\ &= [1-\delta^*/(n(\tau+2))]\mu, \end{aligned}$$

where

$$0 < \delta^* = 2(1-\beta-\alpha)\lambda/\left(1+(1+\alpha^2)\lambda/\beta^{1/2}\right)^2 < 1.$$

Therefore, Algorithm 4.1 is globally convergent, and for any precision $\varepsilon > 0$ the algorithm will create an approximate solution (u^k, x^k) satisfying $\mu^k \leq \varepsilon$ in at most $O[n(\tau+2)\log(\mu^0/\varepsilon)]$ iterations. $\square$

Remark 4.2. Let $(u, x) > 0$. We consider the nonlinear equation

$$\phi(t) = X(F(x + \bar{\theta}t) - F(x))/\bar{\theta} + Ut - \lambda\mu e + Ux = 0, \tag{41}$$

where $\bar{\theta}$ is given by (27). Clearly, the Jacobian $\phi'(t) = XF'(x + \bar{\theta}t) + U$ is nonsingular since $(u, x) > 0$ and $F'(x + \bar{\theta}t)$ is a $P_*(\tau)$-matrix (Lemma 4.1 of Ref. 10). Thus the Newton method can be used to solve this nonlinear equation. It is not necessary to solve the equation precisely. Algorithm 4.1 is globally convergent provided that at each step the approximate solution of (41) satisfies the following condition:

$$\|\phi(\Delta x)\|_\infty \leq \alpha^2\lambda\mu. \tag{42}$$

We end this paper by showing that if the scaled Lipschtiz condition holds, the accuracy requirement (42) can be satisfied by applying one Newton step to the equation (41) with starting point $t = 0$. Therefore, the assumption of Theorem 4.1 is weaker than the scaled Lipschitz condtion.

Indeed, suppose that Condition 2.2 holds. Let the scalar $0 < \lambda < \min\{2\beta, 1/(2(2-\beta))\}$, $0 < \beta < 1/16$ and

$$2(1-\beta)/\left(1 + (1 + 4(1-\beta)/M)^{1/2}\right) \leq \alpha < 1 - \beta, \tag{43}$$

where M is the constant defined in Condition 2.2. Let Δx be the solution to the following Newton equation

$$\phi'(0)\Delta x + \phi(0) = 0.$$

That is,

$$XF'(x)\Delta x + U\Delta x = \lambda\mu e - Ux. \tag{44}$$

We now show that $\|\phi(\Delta x)\|_\infty \leq \alpha^2\lambda\mu$ holds. Indeed, the equation (44) coincides with the system (6) and (7), thus by Lemma 3.2, (18) and (25) we deduce that

$$|\Delta x^T F'(x)\Delta x| \leq \max\{1, \tau\}\|z\|^2/4 \leq \max\{1, \tau\}n\mu/4, \tag{45}$$

and

$$\|X^{-1}\Delta x\| \leq [n(\tau + 2)/(2\beta)]^{1/2}.$$

Hence

$$\begin{aligned}
\bar{\theta}\|X^{-1}\Delta x\| &\leq 4(1-\beta-\alpha)\lambda \\
&\quad \cdot [n(\tau+2)/(2\beta)]^{1/2}/[n(\tau+2)(1 + (1+\alpha^2)\lambda/\beta^{1/2})^2] \\
&\leq 4(1-\beta-\alpha)(2\beta)^{1/2}/(n(\tau+2))^{1/2} \\
&\leq 4(1-\beta-\alpha)\beta^{1/2} < 1.
\end{aligned}$$

Therefore, by (44), (45) and Condition 2.2 we have

$$\begin{aligned}
\|\phi(\Delta x)\|_\infty &= \|X[F'(x+\bar\theta\Delta x)-F'(x)]/\bar\theta + Ux - \lambda\mu e + Ux\|_\infty \\
&= \|X(F'(x+\bar\theta\Delta x)-F'(x)-\bar\theta F'(x)\Delta x)\|_\infty/\bar\theta \\
&\le M\bar\theta^2|\Delta x^T F'(x)\Delta x|/\bar\theta \\
&\le M\bar\theta\max\{1,\tau\}n\mu/4 \\
&= (1-\beta-\alpha)\lambda M \\
&\quad \cdot\max\{1,\tau\}n\mu/[n(\tau+2)(1+(1+\alpha^2)\lambda/\beta^{1/2})^2] \\
&\le (1-\beta-\alpha)M\lambda\mu. \qquad (46)
\end{aligned}$$

By using (43), it is easy to show that

$$(1-\beta-\alpha)M \le \alpha^2. \qquad (47)$$

Combining (46) and (47) yields $\|\phi(\Delta x)\|_\infty \le \alpha^2\lambda\mu$. □

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