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PROPERTIES OF A MULTIVALUED MAPPING ASSOCIATED WITH SOME NONMONOTONE COMPLEMENTARITY PROBLEMS*

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Abstract. Using the homotopy invariance property of the degree and a newly introduced concept of the interior-point-ε-exceptional family for continuous functions, we prove an alternative theorem concerning the existence of a certain interior-point of a continuous complementarity problem. Based on this result, we develop several sufficient conditions to assure some desirable properties (nonemptyness, boundedness, and upper-semicontinuity) of a multivalued mapping associated with continuous (nonmonotone) complementarity problems corresponding to semimonotone, $\mathrm{P}(\tau,\alpha,\beta)$-, quasi-$\mathrm{P}_*$-, and exceptionally regular maps. The results proved in this paper generalize well-known results on the existence of central paths in continuous P_0 complementarity problems.

Key words. nonlinear complementarity problems, central path, interior-point-ε-exceptional family, weakly univalent maps, generalized monotonicity

AMS subject classifications. 90C30, 90C33

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1. Introduction.

Consider the nonlinear complementarity problem (NCP)

$$f(x) \geq 0,\ x \geq 0,\ x^T f(x) = 0,$$

where f is a continuous function from R^n into itself. This problem has now gained much importance because of its many applications in optimization, economics, engineering, etc. (see [8, 12, 16, 18]).

There are several equivalent formulations of the NCP in the form of a nonlinear equation $F(x) = 0$, where F is a continuous function from R^n into R^n. Given such an equation $F(x) = 0$, the most used technique is to perturb F to a certain F_ε, where ε is a positive parameter, and then consider the equation $F_\varepsilon(x) = 0$. If $F_\varepsilon(x) = 0$ has a unique solution denoted by $x(\varepsilon)$ and $x(\varepsilon)$ is continuous in ε, then the solutions $\{x(\varepsilon)\}$ describe, depending on the nature of $F_\varepsilon(x)$, a short path denoted by $\{x(\varepsilon) : \varepsilon \in (0, \bar{\varepsilon}]\}$, or a long path $\{x(\varepsilon) : \varepsilon \in (0, \infty)\}$. If a short path $\{x(\varepsilon) : \varepsilon \in (0, \bar{\varepsilon}]\}$ is bounded, then for any subsequence $\{\varepsilon_k\}$ with $\varepsilon_k \to 0$, the sequence $\{x(\varepsilon_k)\}$ has at least one accumulation point, and by the continuity each of the accumulation points is a solution to the NCP. Thus, a path can be viewed as a certain continuous curve associated with the solution set of the NCP. Based on the path, we may construct various computational methods for solving the NCP, such as interior-point path-following methods (see, e.g., [15, 25, 26, 27, 28, 32, 39]), regularization methods (see [8, 10, 11, 41]), and noninterior path-following methods (see [1, 2, 3, 5, 7, 17, 21]). The most common interior-point path-following method is based on the central path. The curve $\{x(\varepsilon) : \varepsilon \in (0, \infty)\}$ is said to be the central path if for each $\varepsilon > 0$ the vector $x(\varepsilon)$ is the unique solution to the system

$$x(\varepsilon) > 0,\ f(x(\varepsilon)) > 0,\ X(\varepsilon) f(x(\varepsilon)) = \varepsilon e, \tag{1}$$

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where $X(\varepsilon) = \text{diag}(x(\varepsilon)), e = (1, \dots, 1)^T$, and $x(\cdot)$ is continuous on $(0, \infty)$.

In the case when f is a monotone function and the NCP is strictly feasible (i.e., there is a vector $u \in R^n$ such that $u > 0$ and $f(u) > 0$), the existence of the central path is well known (see, for example, [14, 25, 30, 31]). This existence result has been extended to some nonmonotone complementarity problems. Kojima, Mizuno, and Noma [27] proved that the central path exists if f is a uniform-P function. If f is a P_0-function satisfying a properness condition and the NCP is strictly feasible, Kojima, Megiddo, and Noma [25] showed that there exists a class of interior-point trajectories which includes the central path as a special case. If f is a P_0-function and NCP has a nonempty and bounded solution set, Chen, Chen, and Kanzow [4] and Gowda and Tawhid [13] proved that the NCP has a short central path $\{x(\varepsilon) : \varepsilon \in (0, \bar{\varepsilon})\}$. Under a certain properness condition, Gowda and Tawhid [13] showed that the NCP with a P_0-function has a long central path [13, Theorem 9]. It should be pointed out that noninterior-point trajectories have also been extensively studied in the recent literature (see [1, 2, 3, 5, 10, 11, 13, 17, 35, 37]).

However, for a general complementarity problem, the system (1) may have multiple solutions for a given $\varepsilon > 0$, and even if the solution is unique, it is not necessarily continuous in ε. As a result, the existence of the central path is not always guaranteed. We define the (multivalued) mapping $\mathcal{U} : (0, \infty) \to \mathcal{S}(R^n_{++})$ by

$$\mathcal{U}(\varepsilon) = \{x \in R^n_{++} : f(x) > 0, Xf(x) = \varepsilon e\}, \tag{2}$$

where $X = \text{diag}(x)$ and $\mathcal{S}(R^n_{++})$ is the set of all subsets of R^n_{++}, the positive orthant of R^n. The main contribution of this paper is to describe several sufficient conditions which ensure that the multivalued mapping $\mathcal{U}(\varepsilon)$ has the following desirable properties.

(a) $\mathcal{U}(\varepsilon) \neq \emptyset$ *for each* $\varepsilon \in (0, \infty)$.

(b) *For any fixed* $\bar{\varepsilon} > 0$, *the set* $\bigcup_{\varepsilon \in (0, \bar{\varepsilon}]} \mathcal{U}(\varepsilon)$ *is bounded.*

(c) *If* $\mathcal{U}(\varepsilon) \neq \emptyset$, *then* $\mathcal{U}(\varepsilon)$ *is upper-semicontinuous at* ε. *(That is, for any sufficiently small* $\delta > 0$, *we have that* $\emptyset \neq \mathcal{U}(\varepsilon') \subseteq \mathcal{U}(\varepsilon) + \delta B$ *for all* ε' *sufficiently close to* ε, *where* $B = \{x \in R^n : \|x\| < 1\}$ *is the Euclidean unit ball.)*

(d) *If* $\mathcal{U}(\cdot)$ *is single-valued, then* $\mathcal{U}(\varepsilon)$ *is continuous at* ε *provided that* $\mathcal{U}(\varepsilon) \neq \emptyset$.

If the mapping $\mathcal{U}(\cdot)$ satisfies properties (a), (b), and (c), then the set $\bigcup_{\varepsilon \in (0,\infty)} \mathcal{U}(\varepsilon)$ can be viewed as an "interior band" associated with the solution set of the NCP. The "interior band" can be viewed as a generalization of the concept of the central path. Indeed, if $\mathcal{U}(\cdot)$ satisfies properties (a), (b), and (d), then the set $\bigcup_{\varepsilon \in (0,\infty)} \mathcal{U}(\varepsilon)$ coincides with the central path of the NCP.

There exist several ways of generating the central path of the NCP, including maximal monotone methods [14, 30], minimization methods [31], homeomorphism techniques [6, 14, 15, 25, 33], the parameterized Sard theorem [42], and weakly univalent properties of continuous functions [13, 35, 37]. In this paper, we develop a different method for the analysis of the existence of the central path. By means of the homotopy invariance property of the degree and a newly introduced concept of interior-point-ε-exceptional family for continuous functions, we establish an alternative theorem for the nonemptyness of the mapping $\mathcal{U}(\varepsilon)$. For a given $\varepsilon > 0$, the result states that there exists either an interior-point-ε-exceptional family for f or $\mathcal{U}(\varepsilon) \neq \emptyset$. Consequently, to show the nonemptyness of the mapping $\mathcal{U}(\cdot)$, it is sufficient to verify conditions under which the function f possesses no interior-point-ε-exceptional family for any $\varepsilon > 0$. Along with this idea, we provide several sufficient conditions that guarantee the aforementioned desirable properties of the multivalued mapping $\mathcal{U}(\cdot)$.

These sufficient conditions are related to several classes of (nonmonotone) functions such as semimonotone, quasi-P$_*$-, P(τ, α, β)-, and exceptionally regular maps. The results proved in the paper include several known results on the central path as special instances.

This paper is organized as follows. In section 2, we introduce some definitions and some basic results that will be utilized in the paper. In section 3, we show an essential alternative theorem that is useful in later derivations. In section 4, we establish some sufficient conditions to guarantee the nonemptyness, boundedness, and upper-semicontinuity of the map $\mathcal{U}(\varepsilon)$, and the existence of the central path. Some concluding remarks are given in section 5.

Notations: R^n_+ (respectively, R^n_{++}) denotes the space of n-dimensional real vectors with nonnegative components (respectively, positive components), and $R^{n\times n}$ stands for the space of $n \times n$ matrices. For any $x \in R^n$, we denote by $\|x\|$ the Euclidean norm of x, by x_i the ith component of x for $i = 1, \dots, n$, and by $[x]_+$ the vector whose ith component is $\max\{0, x_i\}$. When $x \in R^n_+(R^n_{++})$, we also write it as $x \geq 0$ $(x > 0)$ for simplicity.

2. Preliminaries. We first introduce the concept of an E$_0$-function, which is a generalization of an E$_0$-matrix, i.e., semimonotone matrix, (see [8]). Recall that an $n \times n$ matrix M is said to be an E$_0$-matrix if for any $0 \neq x \geq 0$, there exists a component $x_i > 0$ such that $(Mx)_i \geq 0$. M is a strictly semimonotone matrix if for any $0 \neq x \geq 0$, there exists a component $x_i > 0$ such that $(Mx)_i > 0$.

DEFINITION 2.1. *A function $f : R^n \to R^n$ is said to be an E_0-function (i.e., semimonotone function) if for any $x \neq y$ and $x \geq y$ in R^n, there exists some i such that $x_i > y_i$ and $f_i(x) \geq f_i(y)$. f is a strictly semimonotone function if for any $x \neq y$ and $x \geq y$ in R^n, there exists some i such that $x_i > y_i$ and $f_i(x) > f_i(y)$.*

It is evident that $f = Mx + q$, where $M \in R^{n\times n}$ and $q \in R^n$, is an E$_0$-function if and only if M is an E$_0$-matrix. We recall that a function f is said to be a P$_0$(P)-function if for any $x \neq y$ in R^n

$$\max_{x_i \neq y_i} (x_i - y_i)(f_i(x) - f_i(y)) \geq 0 (> 0).$$

Clearly, a P$_0$-function is an E$_0$-function. However, the converse is not true (see [8, Example 3.9.2]). Thus the class of E$_0$-functions is larger than that of P$_0$-functions.

DEFINITION 2.2. (D1) [23, 24]. *A map $f : R^n \to R^n$ is said to be quasi monotone if for $x \neq y$ in R^n, $f(y)^T(x - y) > 0$ implies that $f(x)^T(x - y) \geq 0$.*

(D2) [26, 44]. *A map $f : R^n \to R^n$ is said to be a P_*-map if there exists a scalar $\kappa \geq 0$ such that for any $x \neq y$ in R^n we have*

$$(1 + \kappa) \sum_{i\in I_+(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) + \sum_{i\in I_-(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) \geq 0,$$

where

$$I_+(x, y) = \{i : (x_i - y_i)(f_i(x) - f_i(y)) > 0\}, \tag{3}$$

$$I_-(x, y) = \{i : (x_i - y_i)(f_i(x) - f_i(y)) < 0\}.$$

(D3) [26]. *M is said to be a P_*-matrix if there exists a scalar $\kappa \geq 0$ such that*

$$(1 + \kappa) \sum_{i\in I_+} x_i(Mx)_i + \sum_{i\in I_-} x_i(Mx)_i \geq 0,$$

where $I_+ = \{i : x_i(Mx)_i > 0\}$ *and* $I_- = \{i : x_i(Mx)_i < 0\}$.

Clearly, for a linear map $f(x) = Mx + q$, f is a P_*-map if and only if M is a P_*-matrix. Väliaho [40] showed that the class of P_*-matrices coincides with the class of sufficient matrices [8, 9]. A new equivalent definition of the P_*-matrix is given in [46]. The next concept is a generalization of the quasi monotone function and the P_*-map.

DEFINITION 2.3. [46] *A function* $f : R^n \to R^n$ *is said to be a quasi-*P_**-map if there exists a constant* $\tau \geq 0$ *such that the following implication holds for all* $x \neq y$ *in* R^n.

$$f(y)^T(x-y) - \tau \sum_{i \in I_+(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) > 0 \Rightarrow f(x)^T(x-y) \geq 0,$$

where $I_+(x,y)$ *is defined by* (3).

From the above definition, it is evident that the class of quasi-P_*-maps includes quasi monotone functions and P_*-maps. (see [46] for details). The following concept of a $P(\tau,\alpha,\beta)$-map is also a generalization of the P_*-map. In [46], it is pointed out that monotone functions and P_*-maps are special cases of $\mathrm{P}(\tau,\alpha,\beta)$-maps.

DEFINITION 2.4. [46] *A mapping* $f : R^n \to R^n$ *is said to be a* $P(\tau,\alpha,\beta)$*-map if there exist constants* $\tau \geq 0, \alpha \geq 0$, *and* $0 \leq \beta < 1$ *such that the following inequality holds for all* $x \neq y$ *in* R^n :

$$(1+\tau) \max_{1 \leq i \leq n} (x_i - y_i)(f_i(x) - f_i(y)) + \min_{1 \leq i \leq n} (x_i - y_i)(f_i(x) - f_i(y)) + \alpha \|x-y\|^\beta \geq 0.$$

The concept of exceptional regularity that we are going to define next has a close relation to such concepts as copositive, R_0-, P_0-, and E_0-functions. It is shown that the exceptional regularity is a weak sufficient condition for the nonemptyness and the boundedness of the mapping $\mathcal{U}(\varepsilon)$ (see section 4.4 for details).

DEFINITION 2.5. *Let* f *be a function from* R^n *into* R^n. f *is said to be exceptionally regular if, for each* $\beta \geq 0$, *the following complementarity problem has no solution of norm* 1:

$$G(x) + \beta x \geq 0, \;\; x \geq 0, \;\; x^T(G(x) + \beta x) = 0,$$

where $G(x) = f(x) - f(0)$.

The following two results are employed to prove the main result of the next section. Let S be an open bounded set of R^n. We denote by $\overline{S}$ and $\partial(S)$ the closure and boundary of S, respectively. Let F be a continuous function from $\overline{S}$ into R^n. For any $y \in R^n$ such that $y \notin F(\partial(S))$, the symbol $\deg(F, S, y)$ denotes the topological degree associated with F, S, and y (see [34]).

LEMMA 2.1. [34] *Let* $S \subset R^n$ *be an open bounded set and* F, G *be two continuous functions from* $\overline{S}$ *into* R^n.

(i) *Let the homotopy* $H(x,t)$ *be defined as*

$$H(x,t) = tG(x) + (1-t)F(x), \;\; 0 \leq t \leq 1,$$

and let y *be an arbitrary point in* R^n. *If* $y \notin \{H(x,t) : x \in \partial S$ *and* $t \in [0,1]\}$, *then* $deg(G,S,y) = deg(F,S,y)$.

(ii) *If* $deg(F,S,y) \neq 0$, *then the equation* $F(x) = y$ *has a solution in* S.

The following upper-semicontinuity theorem of weakly univalent maps is due to Ravindran and Gowda [35].

LEMMA 2.2. [35] *Let $g : R^n \to R^n$ be weakly univalent; that is, g is continuous and there exist one-to-one continuous functions $g_k : R^n \to R^n$ such that $g_k \to g$ uniformly on every bounded subset of R^n. Suppose that $q^* \in R^n$ such that $g^{-1}(q^*)$ is nonempty and compact. Then for any given scalar $\delta > 0$ there exists a scalar $\gamma > 0$ such that for any weakly univalent function $h : R^n \to R^n$ and for any $q \in R^n$ with*

$$\sup_{\bar{\Omega}} \|h(x) - g(x)\| < \gamma, \ \|q - q^*\| < \gamma,$$

we have

$$\emptyset \neq h^{-1}(q) \subseteq g^{-1}(q^*) + \delta B,$$

where B denotes the open unit ball in R^n and $\Omega = g^{-1}(q^*) + \delta B$.

3. Interior-point-ε-exceptional family and an alternative theorem. We now introduce the concept of the interior-point-ε-exceptional family for a continuous function, which brings us to a new idea, to investigate the properties of the mapping $\mathcal{U}(\varepsilon)$ defined by (2), especially the existence of the central path for NCPs. This concept can be viewed as a variant of the exceptional family of elements which was originally introduced to study the solvability of complementarity problems and variational inequalities [19, 20, 36, 43, 44, 45, 46].

DEFINITION 3.1. *Let $f : R^n \to R^n$ be a continuous function. Given a scalar $\varepsilon > 0$, we say that a sequence $\{x^r\}_{r>0} \subset R^n_{++}$ is an interior-point-ε-exceptional family for f if $\|x^r\| \to \infty$ as $r \to \infty$ and for each x^r there exists a positive number $0 < \mu_r < 1$ such that*

$$f_i(x^r) = \frac{1}{2}\left(\mu_r - \frac{1}{\mu_r}\right) x_i^r + \frac{\varepsilon \mu_r}{x_i^r} \quad \text{for all } i = 1, \ldots, n. \tag{4}$$

Based on the above concept, we can prove the following result which plays a key role in the analysis of the paper.

THEOREM 3.1. *Let f be a continuous function from R^n into R^n. Then for each $\varepsilon > 0$ there exists either a point $x(\varepsilon)$ such that*

$$x(\varepsilon) > 0, \ f(x(\varepsilon)) > 0, \ x_i(\varepsilon) f_i(x(\varepsilon)) = \varepsilon, \ i = 1, \ldots, n \tag{5}$$

or an interior-point-ε-exceptional family for f.

Proof. Let $F(x) = (F_1(x), \ldots, F_n(x))^T$ be the Fischer–Burmeister function of f defined by

$$F_i(x) = x_i + f_i(x) - \sqrt{x_i^2 + f_i^2(x)}, \ i = 1, \ldots, n.$$

It is well known that x solves the NCP if and only if x solves the equation $F(x) = 0$. Given $\varepsilon > 0$, we perturb $F(x)$ to $F_\varepsilon(x)$ given by

$$[F_\varepsilon(x)]_i = x_i + f_i(x) - \sqrt{x_i^2 + f_i^2(x) + 2\varepsilon}, \ i = 1, \ldots, n. \tag{6}$$

It is easy to see that $x(\varepsilon)$ solves the equation $F_\varepsilon(x) = 0$ if and only if $x(\varepsilon)$ satisfies the system (5). We now consider the convex homotopy between the mapping $F_\varepsilon(x)$ and the identity mapping, that is,

$$H(x, t) = tx + (1 - t) F_\varepsilon(x), \ 0 \le t \le 1.$$

Let $r > 0$ be an arbitrary positive scalar. Consider the open bounded set $S_r = \{x \in R^n : \|x\| < r\}$. The boundary of S_r is given by $\partial S_r = \{x \in R^n : \|x\| = r\}$. There are only two cases.

Case 1. There exists a number $r > 0$ such that $0 \notin \{H(x,t) : x \in \partial S_r \text{ and } t \in [0,1]\}$. In this case, by (i) of Lemma 2.1, we have that $\deg(I, S_r, 0) = \deg(F_\varepsilon(x), S_r, 0)$, where I is the identity mapping. Since $\deg(I, S_r, 0) = 1$, from the above equation and (ii) of Lemma 2.1, we deduce that the equation $F_\varepsilon(x) = 0$ has a solution, denoted by $x(\varepsilon)$, which satisfies the system (5).

Case 2. For each $r > 0$, there exists some point $x^r \in \partial S_r$ and $t_r \in [0,1]$ such that

$$0 = H(x^r, t_r) = t_r x^r + (1 - t_r) F_\varepsilon(x^r). \tag{7}$$

If $t_r = 0$ for some $r > 0$, then the above equation reduces to $F_\varepsilon(x^r) = 0$, which implies that $x(\varepsilon) := x^r$ satisfies the system (5).

We now verify that $t_r \neq 1$. In fact, if $t_r = 1$ for some $r > 0$, then from (7) we have that $x^r = 0$, which is impossible since $x^r \in \partial S_r$.

Therefore, it is sufficient to consider the case of $0 < t_r < 1$ for all $r > 0$. In this case, it is easy to show that f actually has an interior-point-ε-exceptional family. Indeed, in this case, (7) can be written as

$$x_i^r + (1 - t_r) f_i(x^r) = (1 - t_r)\sqrt{(x_i^r)^2 + f_i^2(x^r) + 2\varepsilon},\ i = 1, \ldots, n. \tag{8}$$

Squaring both sides of the above and simplifying, we have

$$x_i^r f_i(x^r) = \frac{1}{2}\left[(1 - t_r) - \frac{1}{1 - t_r}\right](x_i^r)^2 + (1 - t_r)\varepsilon,\ i = 1, \ldots, n.$$

Since $t_r \in (0,1)$, the above equation implies that $x_i^r \neq 0$ for all $i = 1, \ldots, n$. Denote $\mu_r = 1 - t_r$. We see from the above equation that

$$f_i(x^r) = \frac{1}{2}\left(\mu_r - \frac{1}{\mu_r}\right) x_i^r + \frac{\mu_r \varepsilon}{x_i^r},\ i = 1, \ldots, n. \tag{9}$$

We further show that $x^r \in R^n_{++}$. In fact, it follows from (8) that

$$x_i^r + \mu_r f_i(x^r) > \mu_r \sqrt{2\varepsilon} > 0,\ i = 1, \ldots, n. \tag{10}$$

On the other hand, by using (9) we obtain

$$x_i^r + \mu_r f_i(x^r) = \frac{1}{2}(\mu_r^2 + 1) x_i^r + \frac{\mu_r^2 \varepsilon}{x_i^r},\ i = 1, \ldots, n.$$

Combining (10) and the above equation yields $x^r \in R^n_{++}$. Since $\|x^r\| = r$, it is clear that $\|x^r\| \to \infty$ as $r \to \infty$. Consequently, the sequence $\{x^r\}$ is an interior-point-ε-exceptional family for f. $\square$

The above result shows that if f has no interior-point-ε-exceptional family for each $\varepsilon > 0$, then property (a) of the mapping $\mathcal{U}(\cdot)$ holds. From the result, it is interesting to study various practical conditions under which a continuous function does not possess an interior-point-ε-exceptional family for every $\varepsilon \in (0, \infty)$. In the next section, we provide several such conditions ensuring the aforementioned desirable properties of the mapping $\mathcal{U}(\cdot)$.

4. Sufficient conditions for properties of $\mathcal{U}(\cdot)$.

4.1. E_0-function. In this section, we prove that the multivalued mapping $\mathcal{U}(\cdot)$ has properties (a) and (b) if f is a continuous E_0-function satisfying a certain properness condition. Moreover, if $F_\varepsilon(x)$ given by (6) is weakly univalent, then property (c) also holds. Applied to P_0 complementarity problems, this existence result extends a recent result due to Gowda and Tawhid [13]. The following lemma is quite useful.

LEMMA 4.1. *Let $f : R^n \to R^n$ be an E_0-function. Then for any sequence $\{u^k\} \subset R^n_{++}$ with $\|u^k\| \to \infty$, there exist an index i and a subsequence of $\{u^k\}$, denoted by $\{u^{k_j}\}$, such that $u_i^{k_j} \to \infty$ and $f_i(u^{k_j})$ is bounded below.*

Proof. This proof has appeared in several works, see [11, 13, 35, 38]. Let $\{u^k\} \subset R^n_{++}$ be a sequence satisfying $\|u^k\| \to \infty$. Choosing a subsequence if necessary, we may suppose that there exists an index set $I \subseteq \{1, \ldots, n\}$ such that $u_i^k \to \infty$ for each $i \in I$, and $\{u_i^k\}$ is bounded for each $i \notin I$. Let $v^k \in R^n$ be a vector constructed as follows:

$$v_i^k = u_i^k \text{ for } i \notin I, \quad v_i^k = 0 \text{ for } i \in I.$$

Thus, $\{v^k\}$ is a bounded sequence. Clearly, $u^k \geq v^k$. Since f is an E_0-function, there exist an index $i \in I$ and a subsequence of $\{u^k\}$, denoted by $\{u^{k_j}\}$, such that $u_i^{k_j} > v_i^{k_j}$ and $f_i(u^{k_j}) \geq f_i(v^{k_j})$ for all j. Thus,

$$f_i(u^{k_j}) \geq \inf_j f_i(v^{k_j}).$$

Note that the right-hand side of the above inequality is bounded. The desired result follows. ∎

To show the main result of this subsection, we will make use of the following assumption which is weaker than several previously known conditions.

CONDITION 4.1. *For any sequence $\{x^k\}$ satisfying*

(i) *$\{x^k\} \subset R^n_{++}$, $\|x^k\| \to \infty$ and $[-f(x^k)]_+/\|x^k\| \to 0$, and*

(ii) *for each index i with $x_i^k \to \infty$, the corresponding sequence $\{f_i(x^k)\}$ is bounded above, and*

(iii) *there exists at least one index i_0 such that $x_{i_0}^k \to \infty$ and $\{f_{i_0}(x^k)\}$ is bounded, it holds that*

$$\max_{1 \leq i \leq n} x_i^{k_l} f_i(x^{k_l}) \to \infty$$

for some subsequence $\{x^{k_l}\}$.

As we see in the following result the above condition encompasses several particular cases; we omit the details.

PROPOSITION 4.1. *Condition* 4.1 *is satisfied if one of the following conditions holds.*

(C1) *For any positive sequence $\{x^k\} \subset R^n_{++}$ with $\|x^k\| \to \infty$ and $[-f(x^k)]_+/\|x^k\| \to 0$, it holds that $\max_{1 \leq i \leq n} x_i^{k_l} f_i(x^{k_l}) \to \infty$ for some subsequence $\{x^{k_l}\}$.*

(C2) *For any sequence $\{x^k\} \subset R^n_{++}$ with $\|x^k\| \to \infty$ and $\min_{1 \leq i \leq n} f_i(x^k)/\|x^k\| \to 0$, it holds that $\max_{1 \leq i \leq n} x_i^{k_l} f_i(x^{k_l}) \to \infty$ for some subsequence $\{x^{k_l}\}$.*

(C3) [22, 29] *For any sequence $\{x^k\}$ with $\|x^k\| \to \infty$, $[-x^k]_+/\|x^k\| \to 0$, and $[-f(x^k)]_+/\|x^k\| \to 0$, it holds that*

$$\liminf_{k \to \infty} (x^k)^T f(x^k)/\|x^k\| > 0.$$

(C4) [13] *For any sequence* $\{x^k\}$ *with* $\|x^k\| \to \infty$,

$$\liminf_{k\to\infty} \frac{\min_{1\le i\le n} x_i^k}{\|x^k\|} \ge 0, \quad \textit{and} \quad \liminf_{k\to\infty} \frac{\min_{1\le i\le n} f_i(x^k)}{\|x^k\|} \ge 0,$$

there exist an index j *and a subsequence* $\{x^{k_l}\}$ *such that* $x_j^{k_l} f_j(x^{k_l}) \to \infty$.

(C5) [6, 39] f *is a* R_0*-function.*

(C6) [14, 25, 30, 31] f *is monotone and the NCP is strictly feasible.*

(C7) [27] f *is a uniform P-function.*

Remark 4.1. The condition (C1) of the above proposition is weaker than each of the conditions (C2) through (C7). (C2) is weaker than each of the conditions (C4) through (C7). The concept of the R_0-function, a generalization of the R_0-matrix [8], was introduced in [39] and later modified in [6].

In what follows, we show under a properness condition that the short "interior band" $\bigcup_{\varepsilon\in(0,\bar\varepsilon]} \mathcal{U}(\varepsilon)$ is bounded for each given $\bar\varepsilon > 0$. The boundedness is important because it implies that the sequence $\{x(\varepsilon_k)\}$, where $x(\varepsilon_k) \in \mathcal{U}(\varepsilon_k)$ and $\varepsilon_k \to 0$, is bounded and each accumulation point of the sequence is a solution to the NCP provided that f is continuous. We impose the following condition on f.

CONDITION 4.2. *For any positive sequence* $\{x^k\} \subset R^n_{++}$ *such that* $\|x^k\| \to \infty$, $\lim_{k\to\infty}[-f(x^k)]_+ = 0$, *and the sequence* $\{f_i(x^k)\}$ *is bounded for each index* i *with* $x_i^k \to \infty$, *it holds that*

$$\max_{1\le i\le n} x_i^{k_l} f_i(x^{k_l}) \to \infty$$

for some subsequence $\{x^{k_l}\}$.

Clearly, Condition 4.2 is weaker than Condition 4.1 and thereby weaker than all conditions listed in Proposition 4.1. We now prove the boundedness of the short "interior band" under the above condition.

LEMMA 4.2. *Suppose that Condition* 4.2 *is satisfied. If* $\mathcal{U}(\varepsilon) \ne \emptyset$ *for each* $\varepsilon > 0$, *then for any* $\bar\varepsilon > 0$ *the set* $\bigcup_{\varepsilon\in(0,\bar\varepsilon]} \mathcal{U}(\varepsilon)$ *is bounded, i.e., property* (b) *holds. Particularly,* $\mathcal{U}(\varepsilon)$ *is bounded for each* $\varepsilon > 0$.

Proof. Suppose that there exists some $\bar\varepsilon > 0$ such that $\bigcup_{\varepsilon\in(0,\bar\varepsilon]} \mathcal{U}(\varepsilon)$ is unbounded. Then there exists a sequence $\{x(\varepsilon_k)\}$, where $\varepsilon_k \in (0, \bar\varepsilon]$, such that $\|x(\varepsilon_k)\| \to \infty$ as $k \to \infty$. Since $x(\varepsilon_k) \in \mathcal{U}(\varepsilon_k)$, we deduce that $[-f(x(\varepsilon_k))]_+ = 0$ for all k, and that

$$0 < f_i(x(\varepsilon_k)) = \frac{\varepsilon_k}{x_i(\varepsilon_k)} < \frac{\bar\varepsilon}{x_i(\varepsilon_k)} \quad \text{for all } i = 1, \dots, n.$$

Thus, for each i such that $x_i(\varepsilon_k) \to \infty$, the sequence $\{f_i(x(\varepsilon_k))\}$ is bounded. By Condition 4.2, we deduce that there exists a subsequence $\{x(\varepsilon_{k_l})\}$ such that

$$\max_{1\le i\le n} x_i(\varepsilon_{k_l}) f_i(x(\varepsilon_{k_l})) \to \infty.$$

This is a contradiction since $x_i(\varepsilon_{k_l}) f_i(x(\varepsilon_{k_l})) = \varepsilon_{k_l} < \bar\varepsilon$ for all $i = 1, \dots, n$. ∎

The main result on E_0-functions is given as follows. Even for P_0-functions, this result is new.

THEOREM 4.1. *Suppose that* f *is a continuous* E_0*-function and Condition* 4.1 *is satisfied. Then the properties* (a) *and* (b) *of the mapping* $\mathcal{U}(\varepsilon)$ *hold. Moreover, if* $F_\varepsilon(x)$ *defined by* (6) *is weakly univalent in* x, *then the mapping* $\mathcal{U}(\cdot)$ *is upper-semicontinuous, i.e., property* (c) *also holds.*

Proof. To prove property (a), by Theorem 3.1, it suffices to show that there exists no interior-point-ε-exceptional family of f for any $\varepsilon > 0$. Assume to the contrary that for certain $\varepsilon > 0$ the function f has an interior-point-ε-exceptional family $\{x^r\}$. Since $\|x^r\| \to \infty$, $\{x^r\} \subset R^n_{++}$, and f is an E_0-function, by Lemma 4.1 there exist some index m and a subsequence $\{x^{r_j}\}$, such that $x_m^{r_j} \to \infty$ and $f_m(x^{r_j})$ is bounded below. From (4), we have

$$0 > \frac{1}{2}\left(\mu_{r_j} - \frac{1}{\mu_{r_j}}\right) x_m^{r_j} = f_m(x^{r_j}) - \frac{\mu_{r_j}\varepsilon}{x_m^{r_j}}.$$

Since $x_m^{r_j} \to \infty$ and $f_m(x^{r_j})$ is bounded below, the right-hand side of the above equation is bounded below. It follows that $\lim_{j\to\infty} \mu_{r_j} = 1$.

On the other hand, we note that for any $0 < \mu < 1$ the function

$$\phi(t) = \frac{1}{2}\left(\mu - \frac{1}{\mu}\right) t + \frac{\mu\varepsilon}{t} \tag{11}$$

is monotonically decreasing with respect to the variable $t \in (0, \infty)$. Passing through a subsequence, we may suppose that there exists an index set $I \subseteq \{1, \ldots, n\}$ such that $x_i^{r_j} \to \infty$ for each $i \in I$, and $\{x_i^{r_j}\}$ is bounded for each $i \notin I$.

If $i \notin I$, then there exists some scalar $C > 0$ such that $x_i^{r_j} \le C$ for all j. Since $\phi(t)$ is decreasing and $\mu_{r_j} \to 1$, we have

$$f_i(x^{r_j}) = \frac{1}{2}\left(\mu_{r_j} - \frac{1}{\mu_{r_j}}\right) x_i^{r_j} + \frac{\mu_{r_j}\varepsilon}{x_i^{r_j}} \ge \frac{1}{2}\left(\mu_{r_j} - \frac{1}{\mu_{r_j}}\right) C + \frac{\mu_{r_j}\varepsilon}{C} \to \frac{\varepsilon}{C} > 0.$$

Thus, for all sufficiently large j, we have

$$[-f_i(x^{r_j})]_+ = 0 \quad \text{for all } i \notin I.$$

If $i \in I$, by using (4) and the facts $\mu_{r_j} \to 1$ and $x_i^{r_j} \to \infty$, we have

$$\frac{f_i(x^{r_j})}{\|x^{r_j}\|} = \frac{1}{2}\left(\mu_{r_j} - \frac{1}{\mu_{r_j}}\right) \frac{x_i^{r_j}}{\|x^{r_j}\|} + \frac{\mu_{r_j}\varepsilon}{x_i^{r_j}\|x^{r_j}\|} \to 0,$$

which implies that

$$[-f_i(x^{r_j})]_+/\|x^{r_j}\| \to 0 \quad \text{for all } i \in I.$$

Therefore, $[-f(x^{r_j})]_+/\|x^{r_j}\| \to 0$. Moreover, it follows from (4) that

$$f_i(x^{r_j}) \le \frac{\mu_{r_j}\varepsilon}{x_i^{r_j}} \le \frac{\varepsilon}{x_i^{r_j}} \to 0 \text{ for all } i \in I,$$

which implies that $\{f_i(x^{r_j})\}$ is bounded above for all $i \in I$. Since $m \in I$ and $\{f_m(x^{r_j})\}$ is bounded below, the sequence $\{f_m(x^{r_j})\}$ is indeed bounded. From Condition 4.1, there is a subsequence of $\{x^{r_j}\}$, denoted also by $\{x^{r_j}\}$, such that

$$\max_{1\le i\le n} x_i^{r_j} f_i(x^{r_j}) \to \infty.$$

However, from (4) we have

$$x_i^{r_j} f_i(x^{r_j}) = \frac{1}{2}\left(\mu_{r_j} - \frac{1}{\mu_{r_j}}\right) (x_i^{r_j})^2 + \mu_{r_j}\varepsilon \le \mu_{r_j}\varepsilon < \varepsilon \tag{12}$$

for all $i \in \{1, \ldots, n\}$. This is a contradiction. Property (a) of $\mathcal{U}(\varepsilon)$ follows.

Since Condition 4.1 implies Condition 4.2, the boundedness of the set $\bigcup_{\varepsilon\in(0,\bar{\varepsilon}]}\mathcal{U}(\varepsilon)$ follows immediately from Lemma 4.2. It is known that $x(\varepsilon) \in \mathcal{U}(\varepsilon)$ if and only if $x(\varepsilon)$ is a solution to the equation $F_\varepsilon(x) = 0$, i.e., $\mathcal{U}(\varepsilon) = F_\varepsilon^{-1}(0)$. Since $\mathcal{U}(\varepsilon)$ is bounded, the set $F_\varepsilon^{-1}(0)$ is bounded (in fact, compact, since f is continuous). If $F_\varepsilon(x)$ is weakly univalent in x, by Lemma 2.2, for each scalar $\delta > 0$ there is a $\gamma > 0$ such that for any weakly univalent function $h : R^n \to R^n$ with

$$\sup_{x\in\bar{\Omega}} \|h(x) - F_\varepsilon(x)\| < \gamma, \text{ where } \Omega = F_\varepsilon^{-1}(0) + \delta B, \tag{13}$$

we have

$$\emptyset \neq h^{-1}(0) \subseteq F_\varepsilon^{-1}(0) + \delta B. \tag{14}$$

It is easy to see that for the given $\gamma > 0$ there exists a scalar $\beta > 0$ such that

$$\sup_{x\in\bar{\Omega}} \|F_{\varepsilon'}(x) - F_\varepsilon(x)\| < \gamma \text{ for all } |\varepsilon' - \varepsilon| < \beta.$$

Setting $h(x) := F_{\varepsilon'}(x)$ in (13) and (14), we obtain that $\emptyset \neq F_{\varepsilon'}^{-1}(0) \subseteq F_\varepsilon^{-1}(0) + \delta B$ for all $|\varepsilon' - \varepsilon| < \beta$, i.e., $\mathcal{U}(\varepsilon') \subseteq \mathcal{U}(\varepsilon) + \delta B$ for all ε' sufficiently close to ε. Thus, $\mathcal{U}(\varepsilon)$ is upper-semicontinuous. □

Ravindran and Gowda [35] showed that if f is a P_0-function, then $F_\varepsilon(x)$ given by (6) is a P-function in x, and hence the equation $F_\varepsilon(x) = 0$ has at most one solution $x(\varepsilon)$. In this case, the upper-semicontinuity of $\mathcal{U}(\cdot)$ reduces to the continuity of $x(\varepsilon)$. By the fact that every P_0-function is an E_0-function and is weakly univalent, we have the following result from Theorem 4.1.

Corollary 4.1. *Suppose that $f : R^n \to R^n$ is a continuous P_0-function and Condition* 4.1 *is satisfied. Then the central path exists and any slice of it is bounded, i.e., for each $\varepsilon > 0$ there exists a unique $x(\varepsilon)$ satisfying the system* (1)*, $x(\varepsilon)$ is continuous on* $(0, \infty)$*, and the set $\{x(\varepsilon) : \varepsilon \in (0, \bar{\varepsilon}]\}$ is bounded for each $\bar{\varepsilon} > 0$.*

When f is a P_0-function, Gowda and Tawhid [13, Theorem 9] showed that the (long) central path exists if condition (C4) of Proposition 4.1 is satisfied. Corollary 4.1 can serve as a generalization of the Gowda and Tawhid result. It is worth noting that the consequences of Corollary 4.1 remain valid if condition (C1) or (C2) of Proposition 4.1 holds.

4.2. Quasi-P_*-maps. The concept of the quasi-P_*-map that is a generalization of the quasi monotone function and the P_*-map was first introduced in [46] to study the solvability of the NCP. Under the strictly feasible assumption as well as the following condition, we can show the nonemptyness and the boundedness of $\mathcal{U}(\cdot)$ if f is a continuous quasi-P_*-map .

Condition 4.3. *For any sequence $\{x^k\} \subset R^n_{++}$ such that*

$$\|x^k\| \to \infty, \ \lim_{k\to\infty} [-f(x^k)]_+ = 0,$$

and $\{f(x^k)\}$ is bounded, it holds that

$$\max_{1\le i\le n} x_i^{k_l} f_i(x^{k_l}) \to \infty$$

for some subsequence $\{x^{k_l}\}$.

Clearly, the above condition is weaker than Conditions 4.1 and 4.2. It is also weaker than Condition 3.8 in [4] and Condition 1.5(iii) in [25]. The following is the main result of this subsection.

THEOREM 4.2. *Let f be a continuous quasi-P_*-map with the constant $\tau \geq 0$ (see Definition* 2.3*). Suppose that Condition* 4.3 *is satisfied. If the NCP is strictly feasible, then property* (a) *of $\mathcal{U}(\varepsilon)$ holds. Moreover, if Condition* 4.2 *is satisfied, then property* (b) *holds, and if $F_\varepsilon(x)$ is weakly univalent in x, then property* (c) *also holds.*

While the nonemptyness of $\mathcal{U}(\varepsilon)$ is ensured under Condition 4.3, it is not clear if the boundedness of $\mathcal{U}(\varepsilon)$ can follow from this condition. However, from the implications Condition 4.1$\Rightarrow$ Condition 4.2 $\Rightarrow$ Condition 4.3, we have the next consequence.

COROLLARY 4.2. *Suppose that f is a continuous quasi-P_*-map and $F_\varepsilon(x)$ is weakly univalent in x. If the NCP is strictly feasible and Condition* 4.1 *or* 4.2 *is satisfied, then the mapping $\mathcal{U}(\cdot)$ has properties* (a), (b), *and* (c)*.*

The proof of Theorem 4.2 is postponed until we have proved two technical lemmas.

LEMMA 4.3. *Let f satisfy Condition* 4.3*. Assume that $\{x^r\}_{r>0}$ is an interior-point-ε-exceptional family for f. If there exists a subsequence of $\{x^r\}$, denoted by $\{x^{r_k}\}$, such that for some $0 < \gamma < 1$,*

$$\lim_{k\to\infty}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)\|x^{r_k}\|^{1+\gamma} = 0, \tag{15}$$

then we have

$$\lim_{k\to\infty}\left(\min_{1\leq i\leq n} x_i^{r_k}\right) = 0.$$

Proof. Suppose that $\{x^{r_k}\}$ is an arbitrary subsequence of $\{x^r\}$ such that (15) holds. Since $\phi(t)$ defined by (11) is decreasing on $(0,\infty)$, for each $i \in \{1,\ldots,n\}$ we have

$$f_i(x^{r_k}) \leq \frac{1}{2}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)\min_{1\leq i\leq n} x_i^{r_k} + \frac{\mu_{r_k}\varepsilon}{\min_{1\leq i\leq n} x_i^{r_k}} \tag{16}$$

and

$$f_i(x^{r_k}) \geq \frac{1}{2}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)\max_{1\leq i\leq n} x_i^{r_k} + \frac{\mu_{r_k}\varepsilon}{\max_{1\leq i\leq n} x_i^{r_k}}. \tag{17}$$

Suppose to the contrary that there exists a subsequence of $\{x^{r_k}\}$, denoted also by $\{x^{r_k}\}$, such that $\min_{1\leq i\leq n} x_i^{r_k} \geq \alpha > 0$ for all $k > 0$, where α is a constant. We derive a contradiction. Indeed, since $\mu_{r_k} - \frac{1}{\mu_{r_k}} < 0$, from (16) we have

$$f_i(x^{r_k}) \leq \frac{\mu_{r_k}\varepsilon}{\min_{1\leq i\leq n} x_i^{r_k}} \leq \frac{\varepsilon}{\alpha} \quad \text{for all } i = 1,\ldots,n.$$

From (17) and the above relation, we obtain

$$\frac{\varepsilon}{\alpha} \geq f_i(x^{r_k}) \geq \frac{1}{2}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)\max_{1\leq i\leq n} x_i^{r_k} \quad \text{for all } i = 1,\ldots,n. \tag{18}$$

Since $\|x^{r_k}\| \to \infty$, we deduce from (15) that

$$\lim_{k\to\infty}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)\max_{1\leq i\leq n} x_i^{r_k} = 0.$$

Therefore, it follows from (18) that there exists a scalar c such that $c \le f_i(x^{r_k}) \le \varepsilon/\alpha$ for all $i = 1, \dots, n$ and $\lim_{k\to\infty}[-f_i(x^{r_k})]_+ = 0$. By Condition 4.3, there exists a subsequence of $\{x^{r_k}\}$, denoted still by $\{x^{r_k}\}$, such that $\max_{1\le i\le n} x_i^{r_k} f_i(x^{r_k}) \to \infty$. However, from (12) we have that $x_i^{r_k} f(x_i^{r_k}) \le \mu_{r_k}\varepsilon < \varepsilon$ for all $i = 1, \dots, n$. This is a contradiction. □

LEMMA 4.4. *Let f satisfy Condition 4.3. Assume that $\{x^r\}$ is an interior-point-ε-exceptional family for f. Let $u > 0$ be an arbitrary vector in R^n. Then for any subsequence $\{x^{r_k}\}$ (where $r_k \to \infty$ as $k \to \infty$) there exists a subsequence of $\{x^{r_k}\}$, denoted still by $\{x^{r_k}\}$, such that $f(x^{r_k})^T(x^{r_k} - u) < 0$ for all sufficiently large k.*

Proof. Let $\{x^{r_k}\}$ be an arbitrary subsequence of $\{x^r\}$ (where $r_k \to \infty$ as $k \to \infty$). By using (4) we have

$$\begin{aligned} & f(x^{r_k})^T(x^{r_k} - u) \\ &= \frac{1}{2}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)\|x^{r_k}\|^2 + n\varepsilon\mu_{r_k} - \frac{1}{2}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)(x^{r_k})^T u - \sum_{i=1}^n \frac{\mu_{r_k}\varepsilon u_i}{x_i^{r_k}} \\ &= \frac{1}{2}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)(\|x^{r_k}\|^2 - (x^{r_k})^T u) + \mu_{r_k}\varepsilon\left(n - \sum_{i=1}^n \frac{u_i}{x_i^{r_k}}\right). \end{aligned} \tag{19}$$

We suppose that $f(x^{r_k})^T(x^{r_k} - u) \ge 0$ for all sufficiently large k. We derive a contradiction. From (19), we have

$$0 \le f(x^{r_k})^T(x^{r_k} - u) \le \frac{1}{2}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)(\|x^{r_k}\|^2 - (x^{r_k})^T u) + \mu_{r_k}\varepsilon n.$$

Since $\|x^{r_k}\| \to \infty$, for all sufficiently large k we have

$$0 \le \frac{1}{2}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)(\|x^{r_k}\|^2 - (x^{r_k})^T u) + \mu_{r_k}\varepsilon n \le \mu_{r_k}\varepsilon n,$$

which implies that

$$\begin{aligned} & \lim_{k\to\infty}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)\|x^{r_k}\|^{1+\gamma} \\ &= \lim_{k\to\infty}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)(\|x^{r_k}\|^2 - (x^{r_k})^T u)\frac{\|x^{r_k}\|^{1+\gamma}}{\|x^{r_k}\|^2 - (x^{r_k})^T u} \\ &= 0 \end{aligned}$$

for any scalar $0 < \gamma < 1$. Thus, we see from Lemma 4.3 that

$$\min_{1\le i\le n} x_i^{r_k} \to 0. \tag{20}$$

Notice that

$$\frac{1}{2}\left(\mu_{r_k} - \frac{1}{\mu_{r_k}}\right)(\|x^{r_k}\|^2 - (x^{r_k})^T u) < 0$$

for all sufficiently large k. From (19), (20), and the above inequality, we have

$$f(x^{r_k})^T(x^{r_k} - u) \le \mu_{r_k}\varepsilon\left(n - \sum_{i=1}^n \frac{u_i}{x_i^{r_k}}\right) \le \mu_{r_k}\varepsilon\left(n - \frac{\min_{1\le i\le n} u_i}{\min_{1\le i\le n} x_i^{r_k}}\right) < 0$$

for all sufficiently large k. This is a contradiction. ∎

We are now ready to prove the results of Theorem 4.2.

Proof of Theorem 4.2. To show property (a) of the mapping $\mathcal{U}(\varepsilon)$, by Theorem 3.1, it suffices to show that f has no interior-point-ε-exceptional family for any $\varepsilon > 0$. Assume to the contrary that there exists an interior-point-ε-exceptional family for f, denoted by $\{x^r\}$. By the strict feasibility of the NCP, there is a vector $u > 0$ such that $f(u) > 0$. Consider two possible cases.

Case (A). There exists a number $r_0 > 0$ such that

$$\max_{1\le i\le n}(x_i^r - u_i)(f_i(x^r) - f_i(u)) < 0 \quad \text{for all } r \ge r_0.$$

In this case, the index set $I_+(x^r, u)$ is empty. Since $f(u) > 0$, $x^r > 0$, and $\|x^r\| \to \infty$, it is easy to see that

$$f(u)^T(x^r - u) > 0$$

for all sufficiently large r. Since f is a quasi-P$_*$-map and $I_+(x^r, u)$ is empty, the above inequality implies that $f(x^r)^T(x^r - u) \ge 0$ for all sufficiently large r. However, by Lemma 4.4 there exists a subsequence of $\{x^r\}$, denoted by $\{x^{r_k}\}$, such that $f(x^{r_k})^T(x^{r_k} - u) < 0$ for all sufficiently large k. This is a contradiction.

Case (B). There exists a subsequence of $\{x^r\}$ denoted by $\{x^{r_j}\}$, where $r_j \to \infty$ as $j \to \infty$, such that

$$\max_{1\le i\le n}(x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) \ge 0 \quad \text{for all } j.$$

By using (4), for each i we have

$$\begin{aligned} A_i^{(r_j)} &:= (x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) \\ &= (x_i^{r_j} - u_i)\left(-\frac{1}{2}\left(\frac{1}{\mu_{r_j}} - \mu_{r_j}\right)x_i^{r_j} - f_i(u) + \frac{\mu_{r_j}\varepsilon}{x_i^{r_j}}\right). \end{aligned} \tag{21}$$

There exist a subsequence of $\{x^{r_j}\}$, denoted also by $\{x^{r_j}\}$, and a fixed index m such that

$$A_m^{(r_j)} := (x_m^{r_j} - u_m)(f_m(x^{r_j}) - f_m(u)) = \max_{1\le i\le n}(x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)).$$

For each i such that $x_i^{r_j} \to \infty$, (21) implies that $A_i^{(r_j)} \to -\infty$. Since $A_m^{(r_j)} \ge 0$ for all j, we deduce that $\{x_m^{r_j}\}$ is bounded, i.e., there is a constant $\bar{\delta}$ such that $0 < x_m^{r_j} \le \bar{\delta}$ for all j.

If $x_m^{r_j} \le u_m$, setting $i = m$ in (21), we have

$$\begin{aligned} A_m^{(r_j)} &\le (u_m - x_m^{r_j})\left(\frac{1}{2}\left(\frac{1}{\mu_{r_j}} - \mu_{r_j}\right)x_m^{r_j} + f_m(u)\right) \\ &\le u_m\left(\frac{1}{2}\left(\frac{1}{\mu_{r_j}} - \mu_{r_j}\right)u_m + f_m(u)\right). \end{aligned} \tag{22}$$

If $u_m < x_m^{r_j} \le \bar{\delta}$, setting $i = m$ in (21), we obtain

$$A_m^{(r_j)} \le (x_m^{r_j} - u_m)\frac{\mu_{r_j}\varepsilon}{x_m^{r_j}} \le \mu_{r_j}\varepsilon < \varepsilon. \tag{23}$$

We consider two subcases, choosing a subsequence whenever it is necessary.

Subcase 1. $\mu_{r_j} \to 1$. From (22) and (23), for all sufficiently large j we have

$$A_m^{(r_j)} \le \max\left\{\varepsilon, u_m\left(f_m(u) + \frac{u_m}{2}\right)\right\}.$$

Thus, for all sufficiently large j, we obtain

$$\begin{aligned} &f(u)^T(x^{r_j} - u) - \tau \max_{1\le i\le n}(x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) \\ &\ge f(u)^T(x^{r_j} - u) - \tau \max\{\varepsilon, u_m(f_m(u) + u_m/2)\} \\ &> 0. \end{aligned}$$

The last inequality above follows from the fact that $f(u) > 0, \{x^{r_j}\} \subset R^n_{++}$, and $\|x^{r_j}\| \to \infty$. Since f is a quasi-P$_*$-map, the above inequality implies that $f(x^{r_j})^T(x^{r_j} - u) \ge 0$ for all sufficiently large j, which is impossible according to Lemma 4.4.

Subcase 2. There exists a subsequence of $\{\mu_{r_j}\}$, denoted also by $\{\mu_{r_j}\}$, such that $\mu_{r_j} \le \delta^*$ for all j, where $0 < \delta^* < 1$. In this case, from (22) and (23), we have

$$A_m^{(r_j)} \le \max\left\{\varepsilon, u_m f_m(u) + \frac{u_m^2}{2}\left(\frac{1}{\mu_{r_j}} - \mu_{r_j}\right)\right\}.$$

It follows from (4) that

$$\begin{aligned} T^{(r_j)} :&= f(x^{r_j})^T(u - x^{r_j}) - \tau \max_{1\le i\le n}(x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) \\ &= \frac{1}{2}\left(\frac{1}{\mu_{r_j}} - \mu_{r_j}\right)(\|x^{r_j}\|^2 - (x^{r_j})^T u) + \mu_{r_j}\varepsilon\left(\sum_{i=1}^n \frac{u_i}{x_i^{r_j}} - n\right) - \tau A_m^{(r_j)}. \end{aligned}$$

We now show that $T^{(r_j)} > 0$ for all sufficiently large j.

If $\varepsilon \le u_m f_m(u) + \frac{u_m^2}{2}(\frac{1}{\mu_{r_j}} - \mu_{r_j})$, noting that $\mu_{r_j} \le \delta^*$ and $\|x^{r_j}\|^2 - (x^{r_j})^T u - \tau u_m^2 \to \infty$ as $j \to \infty$, we obtain

$$\begin{aligned} T^{(r_j)} &\ge \frac{1}{2}\left(\frac{1}{\mu_{r_j}} - \mu_{r_j}\right)(\|x^{r_j}\|^2 - (x^{r_j})^T u - \tau u_m^2) - \tau u_m f_m(u) - \mu_{r_j}\varepsilon n \\ &\ge \frac{1}{2}\left(\frac{1}{\delta^*} - \delta^*\right)(\|x^{r_j}\|^2 - (x^{r_j})^T u - \tau u_m^2) - \tau u_m f_m(u) - \delta^*\varepsilon n > 0. \end{aligned}$$

If $\varepsilon > u_m f_m(u) + \frac{u_m^2}{2}(\frac{1}{\mu_{r_j}} - \mu_{r_j})$, by the same argument as the above, we can show that

$$T^{(r_j)} \ge \frac{1}{2}\left(\frac{1}{\delta^*} - \delta^*\right)(\|x^{r_j}\|^2 - (x^{r_j})^T u) - \delta\varepsilon n - \tau\varepsilon > 0$$

for all sufficiently large j. Thus, by the quasi-P$_*$-property of f, we deduce from $T^{(r_j)} > 0$ that $f(u)^T(u - x^{r_j}) \ge 0$ for all sufficiently large j. It is a contradiction since $\{x^{r_j}\} \subset R^n_{++}, \|x^{r_j}\| \to \infty$, and $f(u) > 0$.

The above contradictions show that f has no interior-point-ε-exceptional family for each $\varepsilon > 0$. By Theorem 3.1, the set $\mathcal{U}(\varepsilon) \ne \emptyset$ for any $\varepsilon > 0$. The boundedness of the short "interior band " follows from Lemma 4.2, and the upper-semicontinuity of $\mathcal{U}(\varepsilon)$ follows easily from Lemma 2.2. □

The class of quasi-P_*-maps includes the quasi monotone functions as particular cases. The following result is an immediate consequence of Theorem 4.2.

COROLLARY 4.3. *Suppose that f is a continuous quasi monotone (in particular, pseudomonotone) function, and the NCP is strictly feasible.*

(i) *If Condition* 4.3 *is satisfied, then property* (a) *of $\mathcal{U}(\varepsilon)$ holds.*

(ii) *If Condition* 4.2 *is satisfied, then properties* (a) *and* (b) *of $\mathcal{U}(\varepsilon)$ hold.*

In the case when $F_\varepsilon(x)$ is univalent (continuous and one-to-one) in x, the equation $F_\varepsilon(x) = 0$ has at most one solution. Combining this fact and Theorem 4.2, we have the following result concerning the existence of the central path of the NCP. To our knowledge, this result can be viewed as the first existence result on the central path for the NCP with a (generalized) quasi monotone function. Up to now, there is no interior-point type algorithms designed for solving (generalized) quasi monotone complementarity problems.

COROLLARY 4.4. *Let f be a quasi-P_*-map, and $F_\varepsilon(x)$ is univalent in x. If the NCP is strictly feasible and Condition* 4.2 *is satisfied, then the central path exists and the set $\{x(\varepsilon) : \varepsilon \in (0, \bar{\varepsilon}]\}$ is bounded for any given $\bar{\varepsilon} > 0$.*

Particularly, if f is a P_0-function, then $F_\varepsilon(x)$ is univalent in x (see [35]). We have the following result.

COROLLARY 4.5. *Let f be a continuous P_0 and quasi-P_*-map. If the NCP is strictly feasible and Condition* 4.2 *is satisfied, then the conclusions of Corollary* 4.4 *are valid.*

4.3. $\boldsymbol{P(\tau,\alpha,\beta)}$-maps. It is well known (see [14, 25, 30, 31]) that the monotonicity combined with strict feasibility implies the existence of the central path. In this section, we extend the result to a class of nonmonotone complementarity problems. Our result states that if f is a $P(\tau,\alpha,\beta)$ and P_0-map (see Definition 2.4), the central path exists provided that the NCP is strictly feasible. This result gives an answer to the question "What class of nonlinear functions beyond P_*-maps can ensure the existence of the central path if the NCP is strictly feasible?" We first show properties of the mapping $\mathcal{U}(\cdot)$ when f is a $P(\tau,\alpha,\beta)$-map.

THEOREM 4.3. *Let f be a continuous $P(\tau,\alpha,\beta)$-map. If the NCP is strictly feasible, then properties* (a) *and* (b) *of $\mathcal{U}(\varepsilon)$ hold. Moreover, if $F_\varepsilon(x)$ is weakly univalent in x, property* (c) *also holds.*

Proof. Suppose that there exists a scalar $\varepsilon > 0$ such that f has an interior-point-ε-exceptional family denoted by $\{x^r\}$. Since $\{x^r\} \subset R^n_{++}$ and $\|x^r\| \to \infty$ as $r \to \infty$, there exist some p and a subsequence denoted by $\{x^{r_j}\}$, where $r_k \to \infty$ as $j \to \infty$, such that $\|x^{r_j}\| \to \infty$ and

$$x_p^{r_j} - u_p = \max_{1 \le i \le n} (x_i^{r_j} - u_i).$$

Clearly, $x_p^{r_j} \to \infty$ as $j \to \infty$. On the other hand, there exists a subsequence of $\{x^{r_j}\}$, denoted also by $\{x^{r_j}\}$, such that for some fixed index m and for all j we have

$$(x_m^{r_j} - u_m)(f_m(x^{r_j}) - f_m(u)) = \max_{1 \le i \le n} (x^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)).$$

By the definition of the $P(\tau,\alpha,\beta)$-map, we have

$$\begin{aligned}(x_p^{r_j}-u_p)&(f_p(x^{r_j})-f_p(u))\\ &\geq \min_{1\leq i\leq n}(x_i^{r_j}-u_i)(f_i(x^{r_j})-f_i(u))\\ &\geq -(1+\tau)\max_{1\leq i\leq n}(x_i^{r_j}-u_i)(f_i(x^{r_j})-f_i(u))-\alpha\|x^{r_j}-u\|^\beta\\ &=-(1+\tau)(x_m^{r_j}-u_m)(f_m(x^{r_j})-f_m(u))-\alpha\|x^{r_j}-u\|^\beta. \qquad (24)\end{aligned}$$

From (4), we have that $f_p(x^{r_j})<\varepsilon/x_p^{r_j}$, and hence

$$B_p^{(r_j)}:=\frac{(x_p^{r_j}-u_p)(f_p(x^{r_j})-f_p(u))}{\|x^{r_j}-u\|^\beta}\leq\frac{(x_p^{r_j}-u_p)}{\|x^{r_j}-u\|^\beta}\left(\frac{\varepsilon}{x_p^{r_j}}-f_p(u)\right). \qquad (25)$$

It is easy to see that

$$\frac{\|x^{r_j}-u\|^\beta}{x_p^{r_j}-u_p}=\left(\frac{\|x^{r_j}-u\|}{x_p^{r_j}-u_p}\right)^\beta\cdot\frac{1}{(x_p^{r_j}-u_p)^{(1-\beta)}}\leq\frac{n^{\beta/2}}{(x_p^{r_j}-u_p)^{1-\beta}}. \qquad (26)$$

Combining (25) and (26) leads to

$$B_p^{(r_j)}\to-\infty \text{ as } j\to\infty.$$

From

$$B_p^{(r_j)}\geq B_{min}^{r_j}:=\min_{1\leq i\leq n}\frac{(x_i^{r_j}-u_i)(f_i(x^{r_j})-f_i(u))}{\|x^{r_j}-u\|^\beta},$$

we deduce that

$$B_{min}^{r_j}\to-\infty \text{ as } j\to\infty. \qquad (27)$$

We now show that $\{x_m^{r_j}\}$ is bounded. Assume that there exists a subsequence of $\{x_m^{r_j}\}$, denoted still by $\{x_m^{r_j}\}$, such that $x_m^{r_j}\to\infty$. Then, from (21), we have

$$(x_m^{r_j}-u_m)(f_m(x^{r_j})-f_m(u))\to-\infty,$$

and hence for all sufficiently large j we have

$$B_m^{(r_j)}:=\frac{(x_m^{r_j}-u_m)(f_m(x^{r_j})-f_m(u))}{\|x^{r_j}-u\|^\beta}=\max_{1\leq i\leq n}\frac{(x_i^{r_j}-u_i)(f_i(x^{r_j})-f_i(u))}{\|x^{r_j}-u\|^\beta}<0.$$

By (27) and the above relation, we obtain

$$(1+\tau)B_m^{(r_j)}+B_{min}^{r_j}\to-\infty \text{ as } j\to\infty. \qquad (28)$$

However, since f is a P(τ,α,β)-map, we have

$$(1+\tau)B_m^{(r_j)}+B_{min}^{r_j}\geq-\alpha,$$

which contradicts (28). This contradiction shows that the sequence $\{x_m^{r_j}\}$ is bounded.

By using (4) and (24), we have

$$\begin{aligned}-(x_p^{r_j}-u_p)&\left(\frac{1}{2}\left(\frac{1}{\mu_{r_j}}-\mu_{r_j}\right)x_p^{r_j}+f_p(u)-\frac{\mu_{r_j}\varepsilon}{x_p^{r_j}}\right)\\ &\geq(1+\tau)(x_m^{r_j}-u_m)\left(\frac{1}{2}\left(\frac{1}{\mu_{r_j}}-\mu_{r_j}\right)x_m^{r_j}+f_m(u)-\frac{\mu_{r_j}\varepsilon}{x_m^{r_j}}\right)-\alpha\|x^{r_j}-u\|^\beta.\end{aligned}$$

Multiplying both sides of the above inequality by $1/(x_p^{r_j} - u_p)$, rearranging terms, and using (26), we have

$$-\frac{1}{2}\left(\frac{1}{\mu_{r_j}} - \mu_{r_j}\right)\left(x_p^{r_j} + \frac{(1+\tau)x_m^{r_j}(x_m^{r_j} - u_m)}{x_p^{r_j} - u_p}\right)$$
$$\geq f_p(u) - \frac{\mu_{r_j}\varepsilon}{x_p^{r_j}} + (1+\tau)\left(\frac{f_m(u)(x_m^{r_j} - u_m)}{x_p^{r_j} - u_p} - \frac{\mu_{r_j}\varepsilon(x_m^{r_j} - u_m)}{x_m^{r_j}(x_p^{r_j} - u_p)}\right) - \frac{\alpha\|x^{r_j} - u\|^\beta}{x_p^{r_j} - u_p}$$
$$\geq f_p(u) - \frac{\varepsilon}{x_p^{r_j}} - \frac{(1+\tau)f_m(u)u_m}{x_p^{r_j} - u_p} - \frac{(1+\tau)\varepsilon}{x_p^{r_j} - u_p} - \frac{\alpha n^{\beta/2}}{(x_p^{r_j} - u_p)^{1-\beta}}.$$

For all sufficiently large j, the left-hand side of the above inequality is negative, but the right-hand side tends to $f_p(u) > 0$ as $j \to \infty$. This is a contradiction. The contradiction shows that f has no interior-point-ε-exceptional family for every $\varepsilon > 0$. By Theorem 3.1, property (a) of $\mathcal{U}(\varepsilon)$ follows. The proof of the boundedness of the set $\bigcup_{\varepsilon\in(0,\bar{\varepsilon}]}\mathcal{U}(\varepsilon)$ is not straightforward. It can be proved by the same argument as the above. Indeed, we suppose that $\{x(\varepsilon_k)\}_{0<\varepsilon_k<\bar{\varepsilon}} \subseteq \bigcup_{\varepsilon\in(0,\bar{\varepsilon}]}\mathcal{U}(\varepsilon)$ is an unbounded sequence. Replacing $\{x^{r_j}\}$ by $\{x(\varepsilon_k)\}$, using

$$f(x(\varepsilon_k)) = \frac{\varepsilon_k}{x(\varepsilon_k)} < \frac{\bar{\varepsilon}}{x(\varepsilon_k)}$$

instead of (4), and repeating the aforementioned proof, we can derive a contradiction. The upper-semicontinuity of $\mathcal{U}(\cdot)$ can be obtained by Lemma 2.2. The proof is complete. ∎

The class of P(τ,α,β)-maps includes several particular cases such as $P(\tau,\alpha,0)$-, P$(\tau,0,0)$-, and P$(0,\alpha,\beta)$-maps. It is shown in [46] that the class of P$(\tau,0,0)$-maps coincides with the class of P_*-maps. Therefore, f is said to be a P_*-map if and only if there exists a nonnegative scalar $\kappa \geq 0$ such that

$$(1+\kappa)\max_{1\leq i\leq n}(x_i - y_i)(f_i(x) - f_i(y)) + \min_{1\leq i\leq n}(x_i - y_i)(f_i(x) - f_i(y)) \geq 0.$$

Particularly, a matrix $M \in R^{n\times n}$ is a P_*-matrix if and only if there is a constant $\kappa \geq 0$ such that

$$(1+\kappa)\max_{1\leq i\leq n} x_i(Mx)_i + \min_{1\leq i\leq n} x_i(Mx)_i \geq 0.$$

This is an equivalent definition of the concept of a P_*-matrix (sufficient matrix) introduced by Kojima et al. [26] and Cottle, Pang, and Venkateswaran [9]. The following result follows immediately from Theorem 4.3.

COROLLARY 4.6. *Let f be a continuous P_0 and $P(\tau,\alpha,\beta)$-map. If the NCP is strictly feasible, then the central path exists and any slice of it is bounded.*

It is worth noting that each P_*-map is a P_0 and a P(τ,α,β)-function. The following result is a straightforward consequence of the above corollary.

COROLLARY 4.7. *Let f be a continuous P_*-map. If the NCP is strictly feasible, then the central path exists and any slice of it is bounded.*

It should be pointed out that P_*-maps are also special instances of quasi-P_*-maps. A result similar to Corollary 4.3 can be stated for P_*-maps. However, as we have shown in Corollary 4.7, the additional conditions such as Conditions 4.1, 4.2, and 4.3 are not necessary for a P_*-map to guarantee the existence of the central path. While P_*-maps and quasi monotone functions are contained in the class of quasi-P_*-maps, Zhao and Isac [46] gave examples to show that a P_*-map, in general, is not a quasi monotone function, and vice versa.

4.4. Exceptionally regular functions. In section 4.1, we study the properties of the mapping $\mathcal{U}(\varepsilon)$ for E_0-functions satisfying a properness condition, i.e., Condition 4.1. In sections 4.2, we show properties of $\mathcal{U}(\varepsilon)$ for quasi-P_*-maps under the strictly feasible condition as well as some properness conditions. In the above section, properness assumptions are removed, and properties of $\mathcal{U}(\varepsilon)$ for $P(\tau,\alpha,\beta)$-maps are proved under the strictly feasible condition only. In this section, removing both the strictly feasible condition and properness conditions, we prove that properties of $\mathcal{U}(\varepsilon)$ hold if f is an exceptionally regular function. The exceptional regularity of a function (see Definition 2.5) was originally introduced in [46] to investigate the existence of a solution to the NCP.

DEFINITION 4.1. [16] *A map $v : R^n \to R^n$ is said to be positively homogeneous of degree $\alpha > 0$ if $v(tx) = t^\alpha v(x)$ for all $x \in R^n$.*

When $\alpha = 1$, the above concept reduces to the standard concept of positive homogeneity. Under the assumption of positively homogeneous of degree $\alpha > 0$, we can show that properties (a) and (b) of $\mathcal{U}(\varepsilon)$ hold if f is exceptionally regular. See the following result.

THEOREM 4.4. *Let f be a continuous and exceptionally regular function from R^n into R^n. If $G(x) = f(x) - f(0)$ is positively homogeneous of degree $\alpha > 0$, then properties* (a) *and* (b) *of $\mathcal{U}(\varepsilon)$ hold. Moreover, if $F_\varepsilon(x)$ is weakly univalent, property* (c) *also holds.*

Proof. Suppose that there is a scalar $\varepsilon > 0$ such that f has an interior-point-ε-exceptional family $\{x^r\}$. We derive a contradiction. Indeed, since $G(x)$ is positively homogeneous of degree $\alpha > 0$, we have

$$f(x^r) = f(0) + \|x^r\|^\alpha (f(x^r/\|x^r\|) - f(0)).$$

Without loss of generality, assume that $x^r/\|x^r\| \to \hat{x}$. From the above relation, we have

$$\lim_{r\to\infty} f(x^r)/\|x^r\|^\alpha = f(\hat{x}) - f(0) = G(\hat{x}). \tag{29}$$

From (4), we have

$$\frac{1}{2}\left(\frac{1}{\mu_r} - \mu_r\right) = -\frac{f_i(x^r)}{x_i^r} + \frac{\mu_r \varepsilon}{(x_i^r)^2} \quad \text{for all } i = 1, \ldots, n. \tag{30}$$

Let $I_+(\hat{x}) = \{i : \hat{x}_i > 0\}$. Since $\|x^r\| \to \infty$ and $x_i^r/\|x^r\| \to \hat{x}_i$, we deduce that $x_i^r \to \infty$ for each $i \in I_+(\hat{x})$. We now show that

$$\lim_{r\to\infty} \frac{1}{2}\left(\frac{1}{\mu_r} - \mu_r\right) \frac{\|x^r\|}{\|x^r\|^\alpha} = \hat{\mu} \tag{31}$$

for some $\hat{\mu} \geq 0$. It is sufficient to show the existence of the above limit. Indeed, for each $i \in I_+(\hat{x})$, by using (30) and (29) we have

$$\lim_{r\to\infty} \frac{1}{2}\left(\frac{1}{\mu_r} - \mu_r\right) \frac{\|x^r\|}{\|x^r\|^\alpha} = \lim_{r\to\infty} \frac{\|x^r\|}{x_i^r}\left(-\frac{f_i(x^r)}{\|x^r\|^\alpha} + \frac{\mu_r \varepsilon}{x_i^r \|x^r\|^\alpha}\right) = -\frac{G_i(\hat{x})}{\hat{x}_i}.$$

Thus, (31) holds, with

$$\frac{G_i(\hat{x})}{\hat{x}_i} = -\hat{\mu} \quad \text{for all } i \in I_+(\hat{x}). \tag{32}$$

Now, we consider the case of $i \notin I_+(\hat{x})$. In this case, $\hat{x}_i = 0$. By using (4), (31), and (29), we see from $x_i^r/\|x^r\| \to 0$ that

$$\begin{aligned} 0 \le \lim_{r\to\infty} \frac{\mu_r \varepsilon}{x_i^r \|x^r\|^\alpha} &= \lim_{r\to\infty} \left(\frac{f_i(x^r)}{\|x^r\|^\alpha} + \frac{(1/\mu_r - \mu_r)x_i^r}{2\|x^r\|^\alpha} \right) \\ &= \lim_{r\to\infty} \left(\frac{f_i(x^r)}{\|x^r\|^\alpha} + \frac{(1/\mu_r - \mu_r)\|x^r\|}{2\|x^r\|^\alpha} \cdot \frac{x_i^r}{\|x^r\|} \right) \\ &= G_i(\hat{x}), \end{aligned}$$

i.e.,

$$G_i(\hat{x}) \ge 0 \quad \text{for all } i \notin I_+(\hat{x}).$$

Combining (32) and the above relation implies that f is not exceptionally regular. This is a contradiction. The contradiction shows that f has no interior-point-ε-exceptional family for each $\varepsilon > 0$, and hence property (a) of $\mathcal{U}(\varepsilon)$ follows from Theorem 3.1. Property (b) of $\mathcal{U}(\varepsilon)$ can be easily proved. Actually, suppose that there exists a sequence $\{x(\varepsilon_k)\}_{0<\varepsilon_k<\bar{\varepsilon}}$ with $\|x(\varepsilon_k)\| \to \infty$, where $x(\varepsilon_k) \in \mathcal{U}(\varepsilon_k)$. Without loss of generality, let $x(\varepsilon_k)/\|x(\varepsilon_k)\| \to \bar{x}$, where $\|\bar{x}\| = 1$. As in the proof of (29) we have

$$0 \le \lim_{k\to\infty} f(x(\varepsilon_k))/\|x(\varepsilon_k)\|^\alpha = G(\bar{x}).$$

Since $x(\varepsilon_k) \in \mathcal{U}(\varepsilon_k)$, we have that $x_i(\varepsilon_k) f_i(x(\varepsilon_k)) = \varepsilon_k$ for all $i = 1, \ldots, n$. Thus,

$$0 = \lim_{k\to\infty} \frac{x_i(\varepsilon_k) f_i(x(\varepsilon_k))}{\|x(\varepsilon_k)\|^{1+\alpha}} = \bar{x}_i G_i(\bar{x}) \quad \text{for all } i = 1, \ldots, n.$$

Therefore,

$$G_i(\bar{x}) = 0 \quad \text{whenever } \bar{x}_i > 0, \text{ and } G_i(\bar{x}) \ge 0 \quad \text{whenever } \bar{x}_i = 0,$$

which contradicts the exceptional regularity of $f(x)$. □

It is not difficult to see that a strictly copositive map and a strictly semimonotone function are special cases of exceptionally regular maps. Hence, we have the following result.

COROLLARY 4.8. *Suppose that $G(x) = f(x) - f(0)$ is positively homogeneous of degree $\alpha > 0$. Then conclusions of Theorem* 4.4 *are valid if one of the following conditions holds.*

(i) *f is an E_0-function, and for each $0 \ne x \ge 0$ there exists an index i such that $x_i > 0$ and $f_i(x) \ne f_i(0)$.*
(ii) *f is strictly copositive, that is, $x^T(f(x) - f(0)) > 0$ for all $0 \ne x \ge 0$.*
(iii) *f is a strictly semimonotone function.*

Proof. Since each of the above conditions implies that $f(x)$ is exceptionally regular, the result follows immediately from Theorem 4.4. □

Motivated by Definition 2.5, we introduce the following concept.

DEFINITION 4.2. *$M \in R^{n\times n}$ is said to be an exceptionally regular matrix if for all $\beta \ge 0, M + \beta I$ is an R_0-matrix.*

It is evident that an exceptionally regular matrix is an R_0-matrix, but the converse is not true. The following result is an immediate consequence of Theorem 4.4 and its corollary.

COROLLARY 4.9. *Let $f = Mx + q$, where $M \in R^{n\times n}$, and q is an arbitrary vector in R^n. If one of the following conditions is satisfied, then properties* (a) *and* (b) *of the mapping $\mathcal{U}(\varepsilon)$ hold:*

(i) *$M \in R^{n\times n}$ is an exceptionally regular matrix.*
(ii) *M is a strictly copositive matrix.*
(iii) *M is a strictly semimonotone matrix.*
(iv) *M is an E_0-matrix, and for each $0 \neq x \geq 0$ there exists an index i such that $x_i > 0$ and $(Mx)_i \neq 0$ (possibly, $(Mx)_i < 0$).*

Furthermore, if M is also a P_0-matrix, then the central path of a linear complementarity problem exists and any slice of it is bounded.

The R_0-property of f has played an important role in the complementarity theory. We close this section by considering this situation. The concept of a nonlinear R_0-function was first introduced by Tseng [38] and later modified by Chen and Harker [6]. We now give a definition of the R_0-function that is different from those in [38] and [6].

DEFINITION 4.3. *$f : R^n \to R^n$ is said to be an R_0-function if $x = 0$ is the unique solution to the following complementarity problem:*

$$G(x) = f(x) - f(0) \geq 0, \ x \geq 0, \ x^T G(x) = 0.$$

This concept is a natural generalization of the R_0-matrix [8]. In fact, for the linear function $f(x) = Mx + q$, it is easy to see that f is an R_0-function if and only if M is an R_0-matrix. In the case when f is an E_0-function, we have shown in Theorem 4.1 that there exists a subsequence $\{\mu_{r_k}\}$ such that $\mu_{r_k} \to 1$. Moreover, if G is positively homogeneous, then from (31) we deduce that $\hat{\mu} = 0$. By using these facts and the above R_0-property and repeating the proof of Theorem 4.4, we have the following result.

THEOREM 4.5. *Suppose that $G(tx) = tG(x)$ for each scalar $t \geq 0$ and $x \in R^n$, and that f is an E_0 and R_0-function. Then the conclusions of Theorem* 4.4 *remain valid. Moreover, if f is a P_0 and R_0 -function, the central path exists and any slice of it is bounded.*

5. Conclusions.

We introduced the concept of the interior-point-ε-exceptional family for continuous functions, which is important since it strongly pertains to the existence of an interior-point $x(\varepsilon) \in \mathcal{U}(\varepsilon)$ and the central path, even to the solvability of NCPs. By means of this concept, we proved that for every continuous NCP the set $\mathcal{U}(\varepsilon)$ is nonempty for each scalar $\varepsilon > 0$ if there exists no interior-point-ε-exceptional family for f. Based on the result, we established some sufficient conditions for the assurance of some desirable properties of the multivalued mapping $\mathcal{U}(\varepsilon)$ associated with certain nonmonotone complementarity problems. Since properties (a) and (b) of $\mathcal{U}(\varepsilon)$ imply that the NCP has a solution, the argument of this paper based on the interior-point-ε-exceptional family can serve as a new analysis method for the existence of a solution to the NCP.

It is worth noting that any point in $\mathcal{U}(\varepsilon)$ is strictly feasible, i.e., $x(\varepsilon) > 0$ and $f(x(\varepsilon)) > 0$. Therefore, the analysis method in this paper can also be viewed as a tool for investigating the strict feasibility of a complementarity problem. In fact, from Theorems 3.1, 4.1, 4.4, and 4.5, we have the following result.

THEOREM 5.1. *Let f be a continuous function. Then the complementarity problem is strictly feasible whenever one of the following conditions holds.*
(i) *There exists a scalar $\varepsilon^* > 0$ such that f has no interior-point-ε^*-exceptional family.*
(ii) *f is an E_0-function and Condition* 4.1 *is satisfied.*
(iii) *$G(x) = f(x) - f(0)$ is positively homogeneous of degree $\alpha > 0$ and f is exceptionally regular.*

(iv) $f(x) = Mx + q$, *where* M *is an* E_0 *and* R_0*-matrix.*

It should be pointed out that the results and the argument of this paper can be easily extended to other interior-point paths. For instance, we can consider the existence of the path

$$\{(x(\varepsilon), y(\varepsilon)) > 0 : \varepsilon > 0, y(\varepsilon) = f(x(\varepsilon)) + \varepsilon b, x_i(\varepsilon)y_i(\varepsilon) = \varepsilon a_i \text{ for all } i\} \tag{33}$$

(where b and $a > 0$ are fixed vectors in R^n) first studied by Kojima, Megiddo, and Noma [25]. (When $a = \varepsilon e, b = 0$, the above path reduces to the central path). This path can be studied by the concept of interior-point-$\varepsilon(a, b)$-exceptional family. For a continuous function $f : R^n \to R^n$, we say that a sequence $\{x^r\} \subset R^n_{++}$ is an interior-point-$\varepsilon(a, b)$-exceptional family for f if $\|x^r\| \to \infty$ as $r \to \infty$, and for each x^r there exists a positive number $\mu_r \in (0, 1)$ such that for each i

$$f_i(x^r) = -\varepsilon b_i + \frac{1}{2}\left[\mu_r - \frac{1}{\mu_r}\right] x_i^r + \frac{\mu_r \varepsilon a_i}{x_i^r}.$$

Using

$$F_i(x, \varepsilon) = x_i + (f_i(x) + \varepsilon b_i) - \sqrt{x_i^2 + (f_i(x) + \varepsilon b_i)^2 + 2\varepsilon a_i}$$

and arguing as in the same proof of Theorem 3.1, we can show that for any $\varepsilon > 0$ there exists either a point $x(\varepsilon)$ satisfying (33) or an interior-point-$\varepsilon(a, b)$-exceptional family for f. This result enables us to develop some sufficient conditions for the existence of the path (33).

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