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MONOTONICITY OF FIXED POINT AND NORMAL MAPPINGS ASSOCIATED WITH VARIATIONAL INEQUALITY AND ITS APPLICATION*

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Abstract. We prove sufficient conditions for the monotonicity and the strong monotonicity of fixed point and normal maps associated with variational inequality problems over a general closed convex set. Sufficient conditions for the strong monotonicity of their perturbed versions are also shown. These results include some well known in the literature as particular instances. Inspired by these results, we propose a modified Solodov and Svaiter iterative algorithm for the variational inequality problem whose fixed point map or normal map is monotone.

Key words. variational inequalities, cocoercive maps, (strongly) monotone maps, fixed point and normal maps, iterative algorithm

AMS subject classifications. 90C30, 90C33, 90C25

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1. Introduction.

Given a continuous function $f : R^n \to R^n$ and a closed convex set K in R^n, the well-known finite-dimensional variational inequality, denoted by $\mathrm{VI}(K, f)$, is to find an element $x^* \in K$ such that

$$(x - x^*)^T f(x^*) \geq 0 \quad \text{ for all } x \in K.$$

It is well known that the above problem can be reformulated as nonsmooth equations such as the fixed point and normal equations (see, e.g., [9, 18]). The fixed point equation is defined by

$$\pi_\alpha(x) = x - \Pi_K(x - \alpha f(x)) = 0, \tag{1}$$

and the normal equation is defined by

$$\Phi_\alpha(x) = f(\Pi_K(x)) + \alpha(x - \Pi_K(x)) = 0, \tag{2}$$

where $\alpha > 0$ is a positive scalar and $\Pi_K(\cdot)$ denotes the projection operator on the convex set K, i.e.,

$$\Pi_K(x) = \arg\min\{\|z - x\| : z \in K\}.$$

Throughout the paper, $\|\cdot\|$ denotes the 2-norm (Euclidean norm) of the vector in R^n. It turns out that x^* solves $\mathrm{VI}(K, f)$ if and only if $\pi_\alpha(x^*) = 0$ and that if x^* solves $\mathrm{VI}(K, f)$, then $x^* - \frac{1}{\alpha} f(x^*)$ is a solution to $\Phi_\alpha(x) = 0$; conversely, if $\Phi_\alpha(u^*) = 0$, then $\Pi_K(u^*)$ is a solution to $\mathrm{VI}(K, f)$.

Recently, several authors studied the P_0 property of fixed point and normal maps when K is a rectangular box in R^n, i.e., the Cartesian product of n one-dimensional

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intervals. For such a K, Ravindran and Gowda [17] (respectively, Gowda and Tawhid [8]) showed that $\pi_\alpha(x)$ (respectively, $\Phi_\alpha(x)$) is a P_0-function if f is. Notice that the monotone maps are very important special cases of the class of P_0-functions. It is worth considering the problem:

(P) *When are the mappings $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ monotone if K is a general closed convex set?*

Intuitively, we may conjecture that the fixed point map and the normal map are monotone if f is. However, this conjecture is not true. The following example shows that for a given $\alpha > 0$ the monotonicity of f, in general, does not imply the monotonicity of the fixed point map $\pi_\alpha(x)$ and the normal map $\Phi_\alpha(x)$.

Example 1.1. Let K be a closed convex set given by

$$K = \{x \in R^2 : \ x_1 \geq 0, \ x_2 = 0\}$$

and

$$f(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

For any $x, y \in R^2$, we have that $(x-y)^T(f(x) - f(y)) = 0$. Hence the function f is monotone on R^2. We now show that for an arbitrary scalar $\alpha > 0$ the fixed point mapping $\pi_\alpha(x) = x - \Pi_K(x - \alpha f(x))$ is not monotone in R^2. Indeed, let $u = (0,0)^T$ and $y = (1, \alpha/2)^T$. It is easy to verify that $\pi_\alpha(u) = (0,0)^T$ and $\pi_\alpha(y) = (-\alpha^2/2, \alpha/2)^T$. Thus, we have

$$(u-y)^T(\pi_\alpha(u) - \pi_\alpha(y)) = -\alpha^2/2 < 0,$$

which implies that $\pi_\alpha(\cdot)$ is not monotone on R^n.

Example 1.2. Let K be a closed convex set given by

$$K = \{x \in R^2 : \ x_1 \leq 0, \ x_2 = 0\}$$

and $f(x) : R^2 \to R^2$ be given as in Example 1.1. We now show that for an arbitrary $\alpha > 0$ the normal mapping $\Phi_\alpha(x) = f(\Pi_K(x)) + \alpha(x - \Pi_K(x))$ is not monotone in R^2. Indeed, let $u = (0,0)^T$ and $y = (-2\alpha^2, \alpha)^T$. We have that $\Phi_\alpha(u) = (0,0)^T$ and $\Phi_\alpha(y) = (0, -\alpha^2)^T$. Thus, we have

$$(u-y)^T(\Phi_\alpha(u) - \Phi_\alpha(y)) = -\alpha^3 < 0,$$

which implies that $\Phi_\alpha(\cdot)$ is not monotone on R^n.

From the above examples, we conclude that a certain condition stronger than the monotonicity of f is required to guarantee the monotonicity of $\pi_\alpha(x)$ and $\Phi_\alpha(x)$. One such condition is the so-called cocoercivity condition. We recall that f is said to be cocoercive with modulus $\beta > 0$ on a set $S \subset R^n$ if there exists a constant $\beta > 0$ such that

$$(x-y)^T(f(x) - f(y)) \geq \beta \|f(x) - f(y)\|^2 \quad \text{for all } x, y \in S.$$

The cocoercivity condition was used in several works, such as Bruck [1], Gabay [7] (in which this condition is used implicitly), Tseng [25], Marcotte and Wu [15], Magnanti and Perakis [13, 14], and Zhu and Marcotte [29, 30]. It is also used to study the strict feasibility of complementarity problems [27]. It is interesting to note that in an affine case the cocoercivity has a close relation to the property of positive semidefinite (psd)-plus matrices [12, 30]. A special case of the cocoercive map is the strongly monotone and Lipschitzian map. We recall that a mapping f is said to be strongly monotone with modulus $c > 0$ on the set S if there is a scalar $c > 0$ such that

$$(x-y)^T(f(x)-f(y)) \geq c\|x-y\|^2 \quad \text{for all } x, y \in S.$$

It is evident that any cocoercive map on the set S must be monotone and Lipschitz continuous (with constant $L = 1/\beta$), but not necessarily strongly monotone (for instance, the constant mapping) on the same set.

In fact, the aforementioned problem (P) is not completely unknown. By using the cocoercivity condition implicitly and using properties of nonexpansive maps, Gabay [7] actually showed (but did not explicitly state) that $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ are monotone if the scalar α is chosen such that the map $I - \alpha f$ is nonexpansive. Furthermore, for strongly monotone and Lipschitzian map f, Gabay [7] and Sibony [20] actually showed that $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ are strongly monotone if the scalar α is chosen such that the map $I - \alpha f$ is contractive. Throughout this paper, we use the standard concept "nonexpansive" map and "contractive" map in the literature to mean a Lipschitzian map with constant $L = 1$ and $L < 1$, respectively.

However, it is easy to give an example to show that $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ are still monotone (strongly monotone) even when α is chosen such that $I - \alpha f$ is not nonexpansive (contractive). For instance, let $K = R^n_+$ and $f(x) = x$. We see that the function f is cocoercive with modulus $\beta = 1$. While $I - \alpha f$ is not nonexpansive for $\alpha > 2$, the map $\pi_\alpha(x)$ remains monotone. As a result, the main purpose of this paper is to expand the results of Sibony [20] and Gabay [7]. We show that if f is cocoercive (strongly monotone and Lipschitz continuous, respectively), the monotonicity (strong monotonicity, respectively) of the maps $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ can be ensured when α lies in a larger interval in which the map $I - \alpha f$ may not be nonexpansive (contractive, respectively). The results derived in this paper are not obtainable by the proof based on the nonexpansiveness and contractiveness of maps.

The other purpose of the paper is to introduce an application of the monotonicity of $\pi_\alpha(x)$ and $\Phi_\alpha(x)$. This application (see section 3) is motivated by the globally convergent inexact Newton method for the system of monotone equations proposed by Solodov and Svaiter [21]. See also [22, 23, 24]. We propose a modified Solodov and Svaiter method to solve the monotone equations $\pi_\alpha(x) = 0$ or $\Phi_\alpha(x) = 0$. This modified algorithm requires no projection operations in the line-search step.

2. Monotonicity of $\boldsymbol{\pi_\alpha(x)}$ and $\boldsymbol{\Phi_\alpha(x)}$. It is known (see Sibony [20] and Gabay [7]) that if f is strongly monotone with modulus $c > 0$ and Lipschitz continuous with constant $L > 0$, then $I - \alpha f$ is contractive when $0 < \alpha < 2c/L^2$. Since Π_K is nonexpansive, this in turn implies that $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ are both strongly monotone for $0 < \alpha < 2c/L^2$. Similarly, it follows from Gabay [7] (see Theorem 6.1 therein) that if f is cocoercive with modulus $\beta > 0$, then $I - \alpha f$ is nonexpansive for $0 < \alpha \leq 2\beta$, and thus we can easily verify that $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ are monotone for $0 < \alpha \leq 2\beta$.

In this section, we prove an improved version of the above-mentioned results. We prove that (i) when α lies outside of the interval $(0, 2c/L^2)$, for instance, $2c/L^2 \leq \alpha \leq 4c/L^2$, $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ are still strongly monotone although $I - \alpha f$, in this case, is not contractive, and (ii) when α lies outside of the interval $(0, 2\beta]$, for instance, $2\beta < \alpha \leq 4\beta$, $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ remain monotone although $I - \alpha f$ is not nonexpansive. This new result on monotonicity (strong monotonicity) of $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ for $\alpha > 2\beta$ ($\alpha \geq 2c/L^2$) is not obtainable by using the nonexpansive (contractive) property of $I - \alpha f$. The reason is as follows: Let f be cocoercive with modulus $\beta > 0$ on the set $S \subseteq R^n$, where

$$\beta = \sup\{\gamma > 0 : (x-y)^T(f(x) - f(y)) \geq \gamma \|f(x) - f(y)\|^2 \quad \text{for all } x, y \in S\}.$$

Clearly, such a scalar β is unique and $0 < \beta < \infty$ provided that f is not a constant mapping. We now verify that $I - \alpha f$ is nonexpansive on S if and only if $0 < \alpha \leq 2\beta$. It is sufficient to show that if $\alpha > 0$ is chosen such that $I - \alpha f$ is nonexpansive on S, then we must have $\alpha \leq 2\beta$. In fact, if $I - \alpha f$ is nonexpansive, then for any x, y in S we have

$$\begin{aligned}\|x-y\|^2 &\geq \|(I-\alpha f)(x) - (I - \alpha f)(y)\|^2 \\ &= \|x-y\|^2 - 2\alpha(x-y)^T(f(x) - f(y)) + \alpha^2\|f(x) - f(y)\|^2,\end{aligned}$$

which implies that

$$(x-y)^T(f(x) - f(y)) \geq (\alpha/2)\|f(x) - f(y)\|^2.$$

By the definition of β, we deduce that $\alpha/2 \leq \beta$, the desired consequence. Similarly, let f be strongly monotone with modulus $c > 0$ and Lipschitz continuous with constant $L > 0$ on the set S, where

$$c = \sup\{\gamma > 0 : (x-y)^T(f(x) - f(y)) \geq \gamma\|x-y\|^2 \quad \text{for all } x, y \in S\}$$

and

$$L = \inf\{\gamma > 0 : \|f(x) - f(y)\| \leq \gamma\|x - y\| \quad \text{for all } x, y \in S\}.$$

We can easily see that $0 < c < \infty$ and $L > 0$ provided that S is not a single point set. It is also easy to show that $I - \alpha f$ is contractive if and only if $0 < \alpha < 2c/L^2$.

Since the map $I-\alpha f$ is not contractive (nonexpansive, respectively) for $\alpha \geq 2c/L^2$ ($\alpha > 2\beta$, respectively), our result established in this section cannot follow directly from the proof of Sibony [20] and Gabay [7].

We also study the strong monotonicity of the perturbed fixed point and normal maps defined by

$$\pi_{\alpha,\varepsilon}(x) := x - \Pi_K(x - \alpha(f(x) + \varepsilon x)),$$

and

$$\Phi_{\alpha,\varepsilon}(x) := f(\Pi_K(x)) + \varepsilon\Pi_K(x) + \alpha(x - \Pi_K(x)),$$

respectively. This is motivated by the well-known Tikhonov regularization method for complementarity problems and variational inequalities. See, for example, Isac [10, 11], Venkateswaran [26], Facchinei [3], Facchinei and Kanzow [4], Facchinei and Pang [5], Gowda and Tawhid [8], Qi [16], Ravindran and Gowda [17], Zhao and Li [28], etc. It is worth mentioning that Gowda and Tawhid [8] showed that when $\alpha = 1$ the perturbed mapping $\Phi_{1,\varepsilon}(x)$ is a P-function if f is a P_0-function and K is a rectangular set. We show in this paper a sufficient condition for the strong monotonicity of $\pi_{\alpha,\varepsilon}(x)$ and $\Phi_{\alpha,\varepsilon}(x)$. The following lemma is helpful.

Lemma 2.1. (i) *Denote*

$$u_z = z - \Pi_K(z) \quad \textit{for all } z \in R^n. \tag{3}$$

Then

$$(z-w)^T(u_z-u_w) \geq \|u_z-u_w\|^2.$$

(ii) *For any* $\alpha > 0$ *and vector* $b \in R^n$*, the following inequality holds for all* $v \in R^n$*:*

$$\alpha\|v\|^2 + v^T b \geq -\frac{\|b\|^2}{4\alpha}.$$

Proof. By the property of projection operator, we have

$$(\Pi_K(z)-\Pi_K(w))^T(\Pi_K(w))-w) \geq 0 \text{ for all } z, w \in R^n,$$

$$(\Pi_K(w)-\Pi_K(z))^T(\Pi_K(z))-z) \geq 0 \text{ for all } z, w \in R^n.$$

Adding the above two inequalities leads to

$$(\Pi_K(z)-\Pi_K(w))^T(z-\Pi_K(z)-(w-\Pi_K(w))) \geq 0 \text{ for all } z, w \in R^n,$$

i.e.,

$$[z-u_z-(w-u_w)]^T(u_z-u_w) \geq 0 \text{ for all } z, w \in R^n.$$

This proves the result (i).

Given $\alpha > 0$ and $b \in R^n$, it is easy to check that the minimum value of $\alpha\|v\|^2+v^T b$ is $-\|b\|^2/(4\alpha)$. This proves the result (ii). ∎

We are ready to prove the main result in this section.

THEOREM 2.1. *Let* K *be an arbitrary closed convex set in* R^n *and* $K \subseteq S \subseteq R^n$*. Let* $f : R^n \to R^n$ *be a function.*

(i) *If* f *is cocoercive with modulus* $\beta > 0$ *on the set* S*, then for any fixed scalar* α *satisfying* $0 < \alpha \leq 4\beta$*, the fixed point map* $\pi_\alpha(x)$ *defined by* (1) *is monotone on the set* S*.*

(ii) *If* f *is strongly monotone with modulus* $c > 0$ *on the set* S*, and* f *is Lipschitz continuous with constant* $L > 0$ *on* S*, then for any fixed scalar* α *satisfying* $0 < \alpha < 4c/L^2$*, the fixed point map* $\pi_\alpha(x)$ *is strongly monotone on the set* S*.*

(iii) *If* f *is cocoercive with modulus* $\beta > 0$ *on the set* S*, then for any* $0 < \alpha < 4\beta$ *and* $0 < \varepsilon \leq 2(\frac{1}{\alpha}-\frac{1}{4\beta})$ *the perturbed map* $\pi_{\alpha,\varepsilon}(x)$ *is strongly monotone in* x *on the set* S*.*

Proof. Let $\alpha > 0$ and $0 \leq \varepsilon \leq 2/\alpha$ be two scalars. For any vector x, y in S, denote

$$z = x-\alpha(f(x)+\varepsilon x), \; w = y-\alpha(f(y)+\varepsilon y).$$

By using the notation of (3) and Lemma 2.1, we have

$$\begin{aligned}
&(x-y)^T(\pi_{\alpha,\varepsilon}(x)-\pi_{\alpha,\varepsilon}(y)) \\
&= (x-y)^T[(z-\Pi_K(z)-(w-\Pi_K(w))+\alpha(f(x)+\varepsilon x) \\
&\qquad\qquad - \alpha(f(y)+\varepsilon y))] \\
&= (x-y)^T(u_z-u_w)+\alpha\varepsilon\|x-y\|^2+\alpha(x-y)^T(f(x)-f(y)) \\
&= [z+\alpha(f(x)+\varepsilon x)-(w+\alpha(f(y)+\varepsilon y))]^T(u_z-u_w) \\
&\quad + \alpha\varepsilon\|x-y\|^2+\alpha(x-y)^T(f(x)-f(y)) \\
&= (z-w)^T(u_z-u_w)+\alpha[f(x)+\varepsilon x-(f(y)+\varepsilon y)]^T(u_z-u_w)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \alpha\varepsilon\|x-y\|^2 + \alpha(x-y)^T(f(x)-f(y)) \\
&\geq \|u_z - u_w\|^2 + \alpha[f(x)+\varepsilon x - (f(y)+\varepsilon y)]^T(u_z-u_w) \\
&\quad + \alpha\varepsilon\|x-y\|^2 + \alpha(x-y)^T(f(x)-f(y)) \\
&\geq -(\alpha^2/4)\|f(x)+\varepsilon x-(f(y)+\varepsilon y)\|^2 + \alpha\varepsilon\|x-y\|^2 \\
&\quad + \alpha(x-y)^T(f(x)-f(y)) \\
&= \left(\alpha\varepsilon - \alpha^2\varepsilon^2/4\right)\|x-y\|^2 - (\alpha^2/4)\|f(x)-f(y)\|^2 \\
&\quad + (\alpha - \alpha^2\varepsilon/2)(x-y)^T(f(x)-f(y)). \qquad (4)
\end{aligned}
$$

If f is cocoercive with modulus $\beta > 0$, using $\varepsilon \leq 2/\alpha$ we see from the above that

$$
\begin{aligned}
&(x-y)^T(\pi_{\alpha,\varepsilon}(x) - \pi_{\alpha,\varepsilon}(y)) \\
&\geq \left(\alpha\varepsilon - \alpha^2\varepsilon^2/4\right)\|x-y\|^2 - (\alpha^2/4)\|f(x)-f(y)\|^2 \\
&\quad + (\alpha - \alpha^2\varepsilon/2)\beta\|f(x)-f(y)\|^2 \\
&= \alpha\varepsilon(1-\alpha\varepsilon/4)\|x-y\|^2 + \alpha^2\beta\left(\frac{1}{\alpha} - \frac{1}{4\beta} - \frac{\varepsilon}{2}\right)\|f(x)-f(y)\|^2.
\end{aligned}
$$

Setting $\varepsilon = 0$ in the above inequality, we see that for $0 < \alpha \leq 4\beta$ the right-hand side is nonnegative, showing that π_α is monotone on the set S. This proves the result (i). Also, if $\alpha < 4\beta$ and $0 < \varepsilon \leq 2(\frac{1}{\alpha} - \frac{1}{4\beta})$, the right-hand side of the above inequality is greater than or equal to $r\|x-y\|^2$, where $r = \alpha\varepsilon(1-\alpha\varepsilon/4) > 0$, showing that $\pi_{\alpha,\varepsilon}$ is strongly monotone on the set S. The proof of the result (iii) is complete.

Assume that f is strongly monotone with modulus $c > 0$ and Lipschitz continuous with constant $L > 0$. We now prove the result (ii). For this case, setting $\varepsilon = 0$ in (4), we have that

$$
\begin{aligned}
&(x-y)^T(\pi_\alpha(x) - \pi_\alpha(y)) \\
&\geq -(\alpha^2/4)\|f(x)-f(y)\|^2 + \alpha(x-y)^T(f(x)-f(y)) \\
&\geq -(\alpha^2 L^2/4)\|x-y\|^2 + \alpha c\|x-y\|^2 \\
&= (\alpha c - \alpha^2 L^2/4)\|x-y\|^2.
\end{aligned}
$$

For $\alpha < 4c/L^2$, it is evident that the scalar

$$
r = \alpha c - \frac{\alpha^2 L^2}{4} = \frac{\alpha L^2}{4}\left(\frac{4c}{L^2} - \alpha\right) > 0.
$$

Result (ii) is proved. $\square$

Similarly, we have the following result for $\Phi_\alpha(x)$.

THEOREM 2.2. *Let f be a function from R^n into itself and K be a closed convex set and $K \subseteq S \subseteq R^n$.*

(i) *If f is cocoercive with modulus $\beta > 0$ on the set S, then for any constant α such that $\alpha > 1/(4\beta)$, the normal map $\Phi_\alpha(x)$ given by* (2) *is monotone on the set S.*

(ii) *If f is strongly monotone with modulus $c > 0$ and Lipschitz continuous with constant $L > 0$ on the set S, then for any α satisfying $\alpha > L^2/(4c)$, the normal map $\Phi_\alpha(x)$ given by* (2) *is strongly monotone on the set S.*

(iii) *If f is cocoercive with modulus $\beta > 0$ on the set S, then for any constant $\alpha > 1/(4\beta)$, the perturbed normal map $\Phi_{\alpha,\varepsilon}(x)$, where $0 < \varepsilon < \alpha$, is strongly monotone in x on the set S.*

Proof. Let α, ε, r be given such that $\alpha > \varepsilon \geq r \geq 0$. For any vector x, y in S, let u_x and u_y be defined by (3) with $z = x$ and $z = y$, respectively. Then, by Lemma 2.1 we have

$$\begin{aligned}&(\alpha - r)\|u_x - u_y\|^2 + (u_x - u_y)^T(f(\Pi_K(x)) - f(\Pi_K(y)))\\ &\geq -\frac{1}{4(\alpha - r)}\|f(\Pi_K(x)) - f(\Pi_K(y))\|^2 \end{aligned}\tag{5}$$

and

$$(x - y)^T(u_x - u_y) \geq \|u_x - u_y\|^2,\tag{6}$$

which further implies

$$\|x - y\| \geq \|u_x - u_y\|.$$

By using the above three inequalities, we have

$$\begin{aligned}
&(x - y)^T(\Phi_{\alpha,\varepsilon}(x) - \Phi_{\alpha,\varepsilon}(y)) - r\|x - y\|^2\\
&= (x - y)^T[f(\Pi_K(x)) + \varepsilon\Pi_K(x) + \alpha u_x - f(\Pi_K(y)) - \varepsilon\Pi_K(y) - \alpha u_y]\\
&\quad -r\|x - y\|^2\\
&= \alpha(x - y)^T(u_x - u_y) + \varepsilon(x - y)^T(\Pi_K(x) - \Pi_K(y)) - r\|x - y\|^2\\
&\quad + (x - y)^T(f(\Pi_K(x)) - f(\Pi_K(y)))\\
&= (\alpha - \varepsilon)(x - y)^T(u_x - u_y) + (\varepsilon - r)\|x - y\|^2\\
&\quad + (x - y)^T(f(\Pi_K(x)) - f(\Pi_K(y))) \qquad (7)\\
&= (\alpha - \varepsilon)(x - y)^T(u_x - u_y) + (\varepsilon - r)\|x - y\|^2 + (u_x - u_y)^T(f(\Pi_K(x))\\
&\quad -f(\Pi_K(y))) + (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y)))\\
&\geq (\alpha - \varepsilon)\|u_x - u_y\|^2 + (\varepsilon - r)\|u_x - u_y\|^2 + (u_x - u_y)^T(f(\Pi_K(x))\\
&\quad -f(\Pi_K(y))) + (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y)))\\
&= (\alpha - r)\|u_x - u_y\|^2 + (u_x - u_y)^T(f(\Pi_K(x)) - f(\Pi_K(y)))\\
&\quad + (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y)))\\
&\geq -\frac{1}{4(\alpha - r)}\|f(\Pi_K(x)) - f(\Pi_K(y))\|^2\\
&\quad + (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y))). \qquad (8)
\end{aligned}$$

Let f be cocoercive with modulus $\beta > 0$ on the set S. Setting $\varepsilon = r = 0$ in the above inequality, and using the cocoercivity of f, we have

$$\begin{aligned}(x - y)^T(\Phi_\alpha(x) - \Phi_\alpha(y)) &\geq -\frac{1}{4\alpha}\|f(\Pi_K(x)) - f(\Pi_K(y))\|^2 +\\
&\quad (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y)))\\
&\geq \left(\beta - \frac{1}{4\alpha}\right)\|f(\Pi_K(x)) - f(\Pi_K(y))\|^2.\end{aligned}$$

For $\alpha > 1/(4\beta)$, the right-hand side is nonnegative, and hence the map Φ_α is monotone on the set S. This proves the result (i).

Let $\alpha > 1/(4\beta)$, $0 < \varepsilon < \alpha$, and $0 < r < \min\{\varepsilon,\ \alpha - 1/(4\beta)\}$. By the cocoercivity of f, the inequality (8) can be further written as

$$\begin{aligned}&(x - y)^T(\Phi_{\alpha,\varepsilon}(x) - \Phi_{\alpha,\varepsilon}(y)) - r\|x - y\|^2\\
&\geq \left(\beta - \frac{1}{4(\alpha - r)}\right)\|f(\Pi_K(x)) - f(\Pi_K(y)\|^2.\end{aligned}$$

Since $0 < r < \alpha - 1/(4\beta)$, the right-hand side of the above is nonnegative, and thus the map $\Phi_{\alpha,\varepsilon}$ is strongly monotone on the set S. Result (iii) is proved.

Finally, we prove result (ii). Assume that f is strongly monotone with modulus $c > 0$ and Lipschitz continuous with constant $L > 0$. For any vector x, y in S, we note that (7) holds for any $\alpha > 0, \varepsilon \geq 0$ and $r \geq 0$. Setting $\varepsilon = 0$, (7) reduces to

$$\begin{aligned}
&(x-y)^T(\Phi_\alpha(x) - \Phi_\alpha(y)) - r\|x-y\|^2 \\
&= \alpha(x-y)^T(u_x - u_y) - r\|x-y\|^2 \\
&\quad + (x-y)^T(f(\Pi_K(x)) - f(\Pi_K(y))).
\end{aligned} \tag{9}$$

Given $\alpha > L^2/(4c)$, let r be a scalar such that $0 < r < \alpha/2$ and $2r + \frac{L^2}{4(\alpha-2r)} < c$. Notice that

$$\begin{aligned}
\|x-y\|^2 &= \|\Pi_K(x) - \Pi_K(y) + u_x - u_y\|^2 \\
&\leq 2(\|\Pi_K(x) - \Pi_K(y)\|^2 + \|u_x - u_y\|^2).
\end{aligned}$$

Substituting the above into (9) and using inequalities (5) and (6), we have

$$\begin{aligned}
&(x-y)^T(\Phi_\alpha(x) - \Phi_\alpha(y)) - r\|x-y\|^2 \\
&\geq \alpha\|u_x - u_y\|^2 - 2r(\|\Pi_K(x) - \Pi_K(y)\|^2 + \|u_x - u_y\|^2) \\
&\quad + (x-y)^T(f(\Pi_K(x)) - f(\Pi_K(y)) \\
&= (\alpha - 2r)\|u_x - u_y\|^2 + (u_x - u_y)^T(f(\Pi_K(x)) - f(\Pi_K(y))) \\
&\quad + (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y))) - 2r\|\Pi_K(x) - \Pi_K(y)\|^2 \\
&\geq -\frac{1}{4(\alpha-2r)}\|f(\Pi_K(x)) - f(\Pi_K(y))\|^2 - 2r\|\Pi_K(x) - \Pi_K(y)\|^2 \\
&\quad + (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y))) \\
&\geq \left(-\frac{L^2}{4(\alpha-2r)} - 2r + c\right)\|\Pi_K(x) - \Pi_K(y)\|^2,
\end{aligned}$$

where the last inequality follows from the Lipschitz continuity and strong monotonicity of f. The right-hand side of the above is nonnegative. Thus, the map Φ_α is strongly monotone on the set S. This proves result (ii). ∎

The following result is an immediate consequence of Theorems 2.1 and 2.2.

COROLLARY 2.1. *Assume that f is monotone and Lipschitz continuous with constant $L > 0$ on a set $S \supseteq K$.*

(i) *If $0 < \varepsilon < \infty$ and $0 < \alpha < \frac{4\varepsilon}{(L+\varepsilon)^2}$, then the perturbed map $\pi_{\alpha,\varepsilon}(x)$ is strongly monotone in x on the set S.*

(ii) *If $0 < \varepsilon < \infty$ and $\alpha > \frac{(L+\varepsilon)^2}{4\varepsilon}$, then the perturbed normal map $\Phi_{\alpha,\varepsilon}(x)$ is strongly monotone in x on the set S.*

Proof. Let $\varepsilon \in (0, \infty)$ be a fixed scalar. It is evident that under the condition of the corollary, the function $F(x) = f(x) + \varepsilon x$ is strongly monotone with modulus $\varepsilon > 0$ and Lipschitz continuous with constant $L + \varepsilon$. Therefore, from Theorem 2.1(ii) we deduce that if $0 < \alpha < 4\varepsilon/(L+\varepsilon)^2$, the map $\pi_{\alpha,\varepsilon}(x)$ is strongly monotone on S. Similarly, the strong monotonicity of $\Phi_{\alpha,\varepsilon}(x)$ follows from Theorem 2.2(ii). ∎

Items (iii) in both Theorem 2.1 and Theorem 2.2 show that for any sufficiently small parameter ε, the perturbed fixed point and normal maps are strongly monotone. This result is quite different from Corollary 2.1. When α is a fixed constant, Corollary 2.1 does not cover the case where ε can be sufficiently small. Indeed, for a fixed $\alpha > 0$, the inequalities $0 < \alpha < \frac{4\varepsilon}{(L+\varepsilon)^2}$ and $\alpha > \frac{(L+\varepsilon)^2}{4\varepsilon}$ fail to hold when $\varepsilon \to 0$.

Up to now, we have shown that the fixed point map $\pi_\alpha(x)$ (respectively, the normal map $\Phi_\alpha(x)$) is monotone if f is cocoercive with modulus $\beta > 0$ and $\alpha \in (0, 4\beta]$ (respectively, $\alpha \in (1/(4\beta), \infty)$). This result includes those known from Sibony [20] and Gabay [7] as special cases. Under the same assumption on f and α, we deduce from items (iii) of Theorems 2.1 and 2.2 that the perturbed forms $\pi_{\alpha,\varepsilon}$ and $\Phi_{\alpha,\varepsilon}$ are strongly monotone provided that the scalar ε is sufficiently small. In the succeeding sections, we will introduce an application of the above results on globally convergent iterative algorithms for VI(K, f) whose fixed point map or normal map is monotone.

3. Application: Iterative algorithm for VI(K, f). Since $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ are monotone if the function f is cocoercive and α lies in a certain interval, we can solve the cocoercive variational inequity problems via solving the system of monotone equation $\pi_\alpha(x) = 0$ or $\Phi_\alpha(x) = 0$. Recently, Solodov and Svaiter [21] (see also [22, 23, 24]) proposed a class of inexact Newton methods for monotone equations. Let $\mathcal{F}(x)$ be a monotone mapping from R^n into R^n. The Solodov and Svaiter algorithm for the equation $\mathcal{F}(x) = 0$ proceeds as follows.

ALGORITHM SS (see [21]). Choose any $x^0 \in R^n, t \in (0, 1)$, and $\lambda \in (0, 1)$. Set $k := 0$.

Inexact Newton step. Choose a psd matrix G_k. Choose $\mu_k > 0$ and $\gamma_k \in [0, 1)$. Compute $d^k \in R^n$ such that

$$0 = \mathcal{F}(x^k) + (G_k + \mu_k I)d^k + e^k,$$

where $\|e^k\| \le \gamma_k \mu_k \|d^k\|$. Stop if $d^k = 0$. Otherwise,

Line-search step. Find $y^k = x^k + \alpha_k d^k$, where $\alpha_k = t^{m_k}$ with m_k being the smallest nonnegative integer m such that

$$-\mathcal{F}(x^k + t^m d^k)^T d^k \ge \lambda(1 - \gamma_k)\mu_k \|d^k\|^2.$$

Projection step. Compute

$$x^{k+1} = x^k - \frac{\mathcal{F}(y^k)^T(x^k - y^k)}{\|\mathcal{F}(y^k)\|^2}\mathcal{F}(y^k).$$

Set $k := k + 1$, and repeat.

As pointed out in [21], the above inexact Newton step is motivated by the idea of the proximal point algorithm [2, 6, 19]. Algorithm SS has an advantage over other Newton methods in that the whole iteration sequence is globally convergent to a solution of the system of equations, provided a solution exists, under no assumption on $\mathcal{F}$ other than continuity and monotonicity. Setting $\mathcal{F}(x) = \pi_\alpha(x)$ or $\Phi_\alpha(x)$, from Theorems 2.1 and 2.2 in this paper and Theorem 2.1 in [21], we have the following result.

THEOREM 3.1. *Let f be a cocoercive map with constant $\beta > 0$. Substitute $\mathcal{F}(x)$ in Algorithm SS by $\pi_\alpha(x)$ (respectively, $\Phi_\alpha(x)$) where $0 < \alpha \le 4\beta$ (respectively, $\alpha > 1/4\beta$). If μ_k is chosen such that $C_2 \ge \mu_k \ge C_1\|\mathcal{F}(x^k)\|$, where C_1 and C_2 are two constants, then Algorithm SS converges to a solution of the variational inequality provided that a solution exists.*

While Algorithm SS can be used to solve the monotone equations $\pi_\alpha(x) = 0$ and $\Phi_\alpha(x) = 0$, each line-search step needs to compute the values of $\pi_\alpha(x^k + \beta^m d^k)$ and $\Phi_\alpha(x^k + \beta^m d^k)$, which represents a major cost of the algorithm in calculating projection operations. Hence, in general cases, Algorithm SS has high computational cost per iteration when applied to solve $\Phi_\alpha(x) = 0$ or $\pi_\alpha(x) = 0$. To reduce this major computational burden, we propose the following algorithm which needs no projection operations other than the evaluation of the function f in line-search steps.

ALGORITHM 3.1. Choose $x^0 \in R^n, t \in (0,1)$, and $\gamma \in [0,1)$. Set $k := 0$.

Inexact Newton Step: Choose a positive semidefinite matrix G_k. Choose $\mu_k > 0$. Compute $d^k \in R^n$ such that

$$0 = \pi_\alpha(x^k) + (G_k + \mu_k I)d^k + e^k, \tag{10}$$

where $\|e^k\| \le \gamma\mu_k\|d^k\|$. Stop if $d^k = 0$. Otherwise,

Line-search step. Find $y^k = x^k + s_k d^k$, where $s_k = t^{m_k}$ with m_k being the smallest nonnegative integer m such that

$$\|f(x^k + t^m d^k) - f(x^k)\| < \frac{(1-\gamma)\mu_k - 4t^m}{2\alpha}\|d^k\|. \tag{11}$$

Projection step. Compute

$$x^{k+1} = x^k - \frac{\pi_\alpha(y^k)^T(x^k - y^k)}{\|\pi_\alpha(y^k)\|^2}\pi_\alpha(y^k).$$

Set $k := k+1$. Return.

The above algorithm has the following property.

LEMMA 3.1. *Let $\pi_\alpha(x)$ be given as* (1). *At kth iteration, if m_k is the smallest nonnegative integer such that* (11) *holds, then $y^k = x^k + t^{m_k}d^k$ satisfies the following estimation:*

$$-\pi_\alpha(y^k)^T d^k \ge \frac{1}{2}(1-\gamma)\mu_k\|d^k\|^2.$$

Proof. By the definition of $\pi_\alpha(x)$, the nonexpansiveness of the projection operator, and (11), we have

$$\begin{aligned}
&\|\pi_\alpha(x^k + t^{m_k}d^k) - \pi_\alpha(x^k)\| \\
&= \|x^k + t^{m_k}d^k - \Pi_K(x^k + t^{m_k}d^k - \alpha f(x^k + t^{m_k}d^k)) \\
&\quad -(x^k - \Pi_K(x^k - \alpha f(x^k)))\| \\
&\le t^{m_k}\|d^k\| + \|\Pi_K(x^k + t^{m_k}d^k - \alpha f(x^k + t^{m_k}d^k)) \\
&\quad -\Pi_K(x^k - \alpha f(x^k))\| \\
&\le t^{m_k}\|d^k\| + \|x^k + t^{m_k}d^k - \alpha f(x^k + t^{m_k}d^k) \\
&\quad -(x^k - \alpha f(x^k))\| \\
&\le 2t^{m_k}\|d^k\| + \alpha\|f(x^k + t^{m_k}d^k) - f(x^k)\| \\
&\le \frac{1}{2}(1-\gamma)\mu_k\|d^k\|. \qquad (12)
\end{aligned}$$

Also,

$$\begin{aligned}
&-\pi_\alpha(x^k + t^{m_k}d^k)^T d^k \\
&= -[\pi_\alpha(x^k + t^{m_k}d^k) - \pi_\alpha(x^k)]^T d^k - \pi_\alpha(x^k)^T d^k \\
&\ge -\|\pi_\alpha(x^k + t^{m_k}d^k) - \pi_\alpha(x^k)\|\|d^k\| - \pi_\alpha(x^k)^T d^k. \qquad (13)
\end{aligned}$$

By (10) and positive semidefiniteness of G_k, we have

$$\begin{aligned}
-\pi_\alpha(x^k)^T d^k &= (d^k)^T(G_k + \mu_k I)d^k + (e^k)^T d^k \\
&\ge \mu_k\|d^k\|^2 - \gamma\mu_k\|d^k\|^2 \\
&= (1-\gamma)\mu_k\|d^k\|^2. \qquad (14)
\end{aligned}$$

Combining (12), (13), and (14) yields

$$-\pi_\alpha(x^k + t^{m_k} d^k)^T d^k \geq \frac{1}{2}(1-\gamma)\mu_k \|d^k\|^2.$$

The proof is complete. □

Using Lemma 3.1 and following the line of the proof of Theorem 2.1 in [21], it is not difficult to prove the following convergence result.

THEOREM 3.2. *Let $f : R^n \to R^n$ be a continuous function such that there exists a constant $\alpha > 0$ such that $\pi_\alpha(x)$ defined by* (1) *is monotone. Choose G_k and μ_k such that $\|G_k\| \leq C'$ and $\mu_k = C\|\pi_\alpha(x^k)\|^p$, where C', C and p are three fixed positive numbers and $p \in (0, 1]$. Then the sequence $\{x^k\}$ generated by Algorithm* 3.1 *converges to a solution of the variational inequality provided that a solution exists.*

Algorithm 3.1 can solve the variational inequality whose fixed point mapping $\pi_\alpha(x)$ is monotone for some $\alpha > 0$. Since the cocoercivity of f implies the monotonicity of the functions $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ for suitable choices of the value of α, Algorithm 3.1 can locate a solution of any solvable cocoercive variational inequality problem. This algorithm has an advantage over Algorithm SS in that it does not carry out any projection operation in the line-search step and hence the computational cost is significantly reduced.

4. Conclusions. In this paper, we show some sufficient conditions for the monotonicity (strong monotonicity) of the fixed point and normal maps associated with the variational inequality problem. The results proved in the paper encompass some known results as particular cases. Based on these results, an iterative algorithm for a class of variational inequalities is proposed. This algorithm can be viewed as a modification of Solodov and Svaiter's method but has lower computational cost than the latter.

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