# CONSTRUCTING GENERALIZED MEAN FUNCTIONS USING CONVEX FUNCTIONS WITH REGULARITY CONDITIONS

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**Abstract.** The generalized mean function has been widely used in convex analysis and mathematical programming. This paper studies a further generalization of such a function. A necessary and sufficient condition is obtained for the convexity of a generalized function. Additional sufficient conditions are derived for the purpose of identifying some new classes of generalized mean functions. Since conditions are given in terms of the first and second derivatives of the functions involved, they can be easily checked. We show that some classes of convex functions with certain regularity (such as $S^*$-regularity and self-regularity) can be used as building blocks to construct such generalized functions.

**Key words.** Convexity, mathematical programming, generalized mean function, self-concordant functions, $S^*$-regular functions, self-regular functions.

**AMS subject classifications.** 90C30, 90C25, 52A41, 49J52

## 1. Introduction.

In this paper, we denote the $n$-dimensional Euclidean space by $R^n$, its nonnegative orthant by $R^n_+$, and positive orthant by $R^n_{++}$.

In 1934, Hardy, Littlewood and Pólya ([13]) considered the following function under the name of generalized mean:

$$\Upsilon_w(x) = \phi^{-1}\left(\sum_{i=1}^{n} w_i \phi(x_i)\right) \tag{1.1}$$

where $\phi(\cdot)$ is a real, strictly increasing, convex function defined on a subset of $R$ and $w = (w_1, w_2, \ldots, w_n)^T$ is a given vector in $R^n_+$. Assuming that $\phi > 0$, $\phi' > 0$ and $\phi'' > 0$, they showed an equivalent condition for the convexity of $\Upsilon_w$. When $\phi$ is three times differentiable, Ben-Tal and Teboulle ([2]) established another equivalent condition for $\Upsilon_w$ being convex (see next section for details).

The generalized mean function (1.1) has many applications in optimization. Ben-Tal and Teboulie ([2]) demonstrated an interesting application of (1.1) (in a continuous form) on penalty functions and duality formulation of stochastic nonlinear programming problems. However, the most widely used generalized means are the logorithmic-exponentional and $p$-norm functions:

$$f_w(x) = \log\left(\sum_{i=1}^{n} w_i e^{x_i}\right), \quad p_w(x) = \left(\sum_{i=1}^{n} w_i x_i^p\right)^{1/p} \quad \text{for } x = (x_1, ..., x_n)^T \in R^n.$$

They correspond to the special cases of $\Upsilon_w$ with $\phi(t) = e^t$ and $\phi(t) = t^p$, respectively.

Needless to say that the log-exp function has been widely used in convex analysis and mathematical programming. For example, a geometric program (see Duffin *et*

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*al.*[8] and Boyd and Vandenberghe [6]) can be converted into a convex programming problem by using the log-exp function so that the interior-point algorithms can be developed to solve geometric programs with great efficiency (Kortanek *et al.* [14]). Another example is concerned with the nondifferentiable minimax problem

$$\min_{y\in D}\max_{1\leq i\leq n} g_i(y),$$

where $g_i(\cdot)$, $i=1,...,n$, are real functions defined on a convex set $D$ in $R^m$. Since the recession function of the log-exp function is the "max-function"(see Rockafellar [20]), i.e., $\max_{1\leq i\leq n} x_i = \lim_{\varepsilon\to 0_+}\varepsilon f(\frac{x}{\varepsilon})$ where $f(\cdot)=f_w(\cdot)$ and $w=(1,1,...,1)$, the above nondifferential optimization problem can be approximated by solving the following optimization problem

$$\min_{y\in D}\varepsilon\log\left(\sum_{i=1}^{n} e^{\frac{g_i(y)}{\varepsilon}}\right).$$

The objective function is differentiable and convex, if every $g_i(y)$ is. Other applications of the log-exp function in optimization can be found in Ben-Tal [1], Ben-Tal and Teboulle [3], Zang [25], Bersekas [4], Polyak [19], Fang [9, 10], Li and Fang [15], Peng and Lin [17], Birbil *et al.* [5], Sun and Li [22, 23, 24]), etc.

It is worth mentioning that the conjugate function of the log-exp function happens to be the well-known Shannon's entropy function ([21]) which plays a vital role in so many fields ranging from the image enhancement to economics and from statistical mechanics to nuclear physics (see, Buck and Macaulay [7] and Fang *et al.* [11]).

We consider in this paper a further generalization of (1.1) in the following form:

$$\Gamma_w(x)=\Psi^{-1}\left(\sum_{i=1}^{n} w_i\phi_i(x_i)\right) \tag{1.2}$$

where $\phi_i:\Omega\to R$, $i=1,...,n$, are convex, twice differentiable (but not necessarily being strictly increasing) functions defined on an open convex set $\Omega\subset R$, $\Psi:\Omega\to R$ is convex, twice differentiable and strictly increasing, and $w\in R^n_+$ is a given vector. Clearly, $\Upsilon_w(\cdot)$ is a special case of $\Gamma_w(\cdot)$ with $\phi_1=\phi_2=...=\phi_n=\Psi=\phi$. For convenience, in this paper, we still call $\Gamma_w$ given by (1.2) a generalized mean function, and we call $\phi_i$ the inner function and $\Psi$ the outer function of $\Gamma_w$. To assure the well-definedness of $\Gamma_w$, we naturally require that

$$\sum_{i=1}^{n} Cone[\phi_i(\Omega)]\subseteq\Psi(\Omega)$$

where $Cone[\phi_i(\Omega)]$ denotes the cone generated by the set $\phi_i(\Omega)$.

As in the case of $\Upsilon_w$, we would like to derive certain sufficient and necessary conditions for the function $\Gamma_w$ to be convex. Moreover, we hope to find a systematic way to explicitly construct some classes of convex $\Gamma_w$.

It is interesting to point out that $\Gamma_w$ is by no means a new research subject. In fact, it was essentially studied by W. Fenchel in his lecture notes of "Convex Cones, Sets and Functions" in 1953 [12]. Based on the properties of level sets and characteristic roots of Hessian matrices of functions involved, Fenchel derived some sufficient and necessary conditions for the convexity of the generalized mean function $\Gamma_w$. The conditions he derived, however, are rather complicated, and there is no simple test to decide what kind of functions may admit these complicated properties. Unlike Fenchel's approach, our analysis in this paper depends only on the function value, its first derivative, and second derivative to provide a sufficient and necessary condition for $\Gamma_w$ being convex. The necessary and sufficient condition we derive in this paper can be viewed as a generalization of that in [13] concerning the function (1.1). We can also use related sufficient conditions to explicitly construct concrete examples of convex $\Gamma_w$. Moreover, we show how the self-regular functions [18] (most of such functions are self-concordant functions as defined in [16]) and $S^*$-regular functions (to be defined in this paper) can be used to construct convex generalized mean functions.

The rest of the paper is organized as follows. In Section 2, we investigate the conditions that assure the convexity of the generalized mean function $\Gamma_w$. In Section 3, we identify some classes of functions that satisfy the conditions derived in Section 2, and illustrate how the generalized mean function $\Gamma_w$ can be explicitly constructed. Conclusions are given in the last section.

**2. Necessary and Sufficient Conditions for the Convexity of $\Gamma_w$.** Let us start with a simple lemma (proof omitted) that shows the inverse of an increasing convex function is concave and increasing.

LEMMA 2.1. *Let $\Omega$ be an open convex subset of $R$ and $\Psi : \Omega \to R$ be a real function defined on $\Omega$. Then $\Psi$ is (strictly) convex and strictly increasing if and only if its inverse $\Psi^{-1} : R \to \Omega$ is (strictly) concave and strictly increasing.*

Notice that if $w_i = 0$, for some $i$, then the term $w_i\phi_i(x)$ can be removed from the expression of $\Gamma_w(x)$, and it suffices to consider $\Gamma_w$ defined on $R^{n-1}$. Thus, without loss of generality, we may assume that the vector $w \in R^n_{++}$ throughout the rest of the paper.

To study the convexity of the function $\Gamma_w$, when assuming that $\phi_i$, $i = 1, ..., n$, and $\Psi^{-1}$ are twice differentiable, we need to check its Hessian matrix. Let

$$x_w = \sum_{i=1}^{n} w_i\phi_i(x_i).$$

Since $\frac{\partial x_w}{\partial x_i} = w_i\phi_i'(x_i)$, we have

$$\frac{\partial \Gamma_w}{\partial x_i} = (\Psi^{-1})'(x_w)w_i\phi_i'(x_i).$$

Moreover,

$$\frac{\partial^2 \Gamma_w}{\partial x_i^2} = (\Psi^{-1})''(x_w)(w_i\phi_i'(x_i))^2 + (\Psi^{-1})'(x_w)w_i\phi_i''(x_i),$$

$$\frac{\partial^2 \Gamma_w}{\partial x_i \partial x_j} = (\Psi^{-1})''(x_w)w_iw_j\phi_i'(x_i)\phi_j'(x_j) \quad \text{for} \quad i \neq j.$$

Consequently, the Hessian matrices of $\Gamma_w$ becomes

$$\frac{\partial^2 \Gamma_w}{\partial x^2} = (\Psi^{-1})'(x_w)\begin{bmatrix} w_1\phi_1''(x_1) & 0 & \dots & 0 \\ 0 & w_2\phi_2''(x_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & w_n\phi_n''(x_n) \end{bmatrix}$$

$$+ (\Psi^{-1})''(x_w)\begin{bmatrix} w_1\phi_1'(x_1) \\ w_2\phi_2'(x_2) \\ \dots \\ w_n\phi_n'(x_n) \end{bmatrix}[w_1\phi_1'(x_1), w_2\phi_2'(x_2), \dots, w_n\phi_n'(x_n)]. \tag{2.1}$$

Note that when $\phi_i$, $i = 1, ..., n$, is convex and $\Psi$ is convex and increasing, by Lemma 2.1, we see that the first term on the right-hand side of (2.1) is a positive semidefinite matrix multiplied by a positive coefficient $(\Psi^{-1})'(x_w)$, while the second is a rank one matrix multiplied by a negative coefficient $(\Psi^{-1})''(x_w)$.

Under the conditions of $\phi > 0$, $\phi' > 0$ and $\phi'' > 0$, it is shown in [13] that the function $\Upsilon_w(x)$ is convex if and only if the following condition holds:

$$\sum_{i=1}^{n} w_i \frac{[\phi'(x_i)]^2}{\phi''(x_i)} \leq \frac{[\phi'(y)]^2}{\phi''(y)}, \ y = \Upsilon_w(x).$$

In what follows, we generalize this result for the function $\Gamma_w(x)$. Although the basic idea of our proof is essentialy tied to that of [13], the proof is not straightforward. For completeness, we give a detailed proof for our result.

THEOREM 2.2. *Let $\Omega \subset R$ be open and convex, $\Psi : \Omega \to R$ be convex, twice differentiable and strictly increasing, $\phi_i : \Omega \to R$, $i = 1, ..., n$, be strictly convex and twice differentiable, and $w \in R^n_{++}$ be a given vector. Then the generalized mean function*

$$\Gamma_w(x) = \Psi^{-1}\left(\sum_{i=1}^{n} w_i \phi_i(x_i)\right)$$

*is convex on $\Omega^n := \overbrace{\Omega \times \cdots \times \Omega}^{n}$ if and only if*

$$\Psi''(y)\left(\sum_{i=1}^{n} w_i \frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)}\right) \leq [\Psi'(y)]^2 \quad \text{for} \quad x \in \Omega^n \ \text{and} \ \ y = \Gamma_w(x). \tag{2.2}$$

*Moreover, $\Gamma_w(x)$ is strictly convex if and only if the inequality in (2.2) holds strictly.*

*Proof.* Let $y = \Gamma_w(x) = \Psi^{-1}(x_w)$. Then $x_w = \Psi(y)$ and

$$(\Psi^{-1})'(x_w)\Psi'(y) = 1. \tag{2.3}$$

Differentiating both sides with respect to $y$ and use the above relations, we have

$$\begin{aligned} 0 &= (\Psi^{-1})''(x_w)[\Psi'(y)]^2 + (\Psi^{-1})'(x_w)\Psi''(y) \\ &= (\Psi^{-1})''(x_w)[\Psi'(y)]^2 + \frac{\Psi''(y)}{\Psi'(y)}. \end{aligned}$$

Therefore,

$$(\Psi^{-1})''(x_w) = -\frac{\Psi''(y)}{[\Psi'(y)]^3}. \tag{2.4}$$

Combining (2.3) and (2.4) yields

$$(\Psi^{-1})'(x_w) + (\Psi^{-1})''(x_w)\sum_{i=1}^{n} w_i \frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)} = \frac{[\Psi'(y)]^2 - \left(\sum_{i=1}^{n} w_i \frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)}\right)\Psi''(y)}{[\Psi'(y)]^3}. \tag{2.5}$$

First we prove that $\Gamma_w(x)$ is convex, if (2.2) holds. It suffices to show that the Hessian matrix of $\Gamma_w(x)$ is positive semi-definite.

Notice that for any $d \in R^n$ and $x \in \Omega^n$, the Cauchy-Schwartz inequality implies that

$$\begin{aligned}\left(\sum_{i=1}^n w_i\phi_i'(x_i)d_i\right)^2 &= \left(\sum_{i=1}^n \left[\sqrt{w_i\phi_i''(x_i)}d_i\right]\cdot\sqrt{\frac{w_i}{\phi_i''(x_i)}}\phi_i'(x_i)\right)^2\\ &\le \left(\sum_{i=1}^n w_i\phi_i''(x_i)d_i^2\right)\left(\sum_{i=1}^n w_i\frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)}\right).\end{aligned}$$

By Lemma 2.1, we know $\Psi^{-1}$ is concave and hence $(\Psi^{-1})''(x_w) \le 0$ for all $x_w$. Combining this fact with the above inequality, we see that, for any $d \in R^n$,

$$\begin{aligned}&d^T\frac{\partial^2\Gamma_w}{\partial x^2}d\\ &= (\Psi^{-1})'(x_w)\left(\sum_{i=1}^n w_i\phi_i''(x_i)d_i^2\right) + (\Psi^{-1})''(x_w)\left(\sum_{i=1}^n w_i\phi_i'(x_i)d_i\right)^2\\ &\ge (\Psi^{-1})'(x_w)\left(\sum_{i=1}^n w_i\phi_i''(x_i)d_i^2\right) + (\Psi^{-1})''(x_w)\left(\sum_{i=1}^n w_i\phi_i''(x_i)d_i^2\right)\left(\sum_{i=1}^n w_i\frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)}\right)\\ &= \left(\sum_{i=1}^n w_i\phi_i''(x_i)d_i^2\right)\left[(\Psi^{-1})'(x_w) + (\Psi^{-1})''(x_w)\left(\sum_{i=1}^n \frac{w_i[\phi_i'(x_i)]^2}{\phi_i''(x_i)}\right)\right]\\ &= \left(\sum_{i=1}^n w_i\phi_i''(x_i)d_i^2\right)\frac{[\Psi'(y)]^2 - \left(\sum_{i=1}^n w_i\frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)}\right)\Psi''(y)}{[\Psi'(y)]^3}\\ &\ge 0.\end{aligned}$$

The last equality follows from (2.5) and the last inequality follows from the fact that the first quantity on the right-hand side, i.e., $\sum_{i=1}^n w_i\phi_i''(x_i)d_i^2$, is nonnegative, and the second quantity is also nonnegative due to our assumption. Consequently, we have proven that the Hessian matrix $\frac{\partial^2\Gamma_w}{\partial x^2}$ is positive semi-definite, as desired.

Conversely, we would like to show that inequality (2.2) holds, if $\Gamma_w(x)$ is convex. For any vector $0 \ne d \in R^n$, knowing (2.3), (2.4) and the convexity of $\Gamma_w(x)$, we have

$$\begin{aligned}0 \le d^T\frac{\partial^2\Gamma_w}{\partial x^2}d &= (\Psi^{-1})'(x_w)\left(\sum_{i=1}^n w_i\phi_i''(x_i)d_i^2\right) + (\Psi^{-1})''(x_w)\left(\sum_{i=1}^n w_i\phi_i'(x_i)d_i\right)^2\\ &= \frac{1}{\Psi'(y)}\left(\sum_{i=1}^n w_i\phi_i''(x_i)d_i^2\right) - \frac{\Psi''(y)}{\Psi'(y)^3}\left(\sum_{i=1}^n w_i\phi_i'(x_i)d_i\right)^2\\ &= \left(\sum_{i=1}^n w_i\phi_i''(x_i)d_i^2\right)\left[\frac{1}{\Psi'(y)} - \frac{\Psi''(y)}{\Psi'(y)^3}\frac{\left[\sum_{i=1}^n w_i\phi_i'(x_i)d_i\right]^2}{\sum_{i=1}^n w_i\phi_i''(x_i)d_i^2}\right].\end{aligned} \tag{2.6}$$

Notice that the above inequality holds for any vector $d \in R^n$. In particular, let

$$d_i = \frac{\phi_i'(x_i)}{\phi_i''(x_i)\sum_{k=1}^n w_k\frac{[\phi_k'(x_k)]^2}{\phi_k''(x_k)}},\ i = 1, ..., n.$$

Then, we have

$$\sum_{i=1}^n w_i\phi_i'(x_i)d_i = 1,\ \sum_{i=1}^n w_i\phi_i''(x_i)d_i^2 = \frac{1}{\sum_{i=1}^n w_i\frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)}}.$$

As a result, the inequality (2.6) reduces to

$$0 \leq \left( \frac{1}{\sum_{i=1}^{n} w_i \frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)}} \right) \left[ \frac{1}{\Psi'(y)} - \frac{\Psi''(y)}{\Psi'(y)^3} \left( \sum_{i=1}^{n} w_i \frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)} \right) \right]$$

We see that inequality (2.2) indeed holds. The result about strict convexity can be easily checked out. ∎

Two related sufficiency results of Theorem 2.2 are derived below for convenient usage in constructing convex $\Gamma_w$ (see next section).

THEOREM 2.3. *Let $\Omega$ be an open convex subset of $R$, $\Psi : \Omega \to R$ be strictly increasing, twice differentiable and convex, $\phi_i : \Omega \to R$, $i = 1, ..., n$, be strictly convex and twice differentiable, and $w \in R^n_{++}$ be a given vector. Assume that there exists a scalar $\alpha \in R$ such that*

$$\alpha \Psi(t) \Psi''(t) \leq [\Psi'(t)]^2 \quad \text{for } t \in \Omega. \tag{2.7}$$

*Then the function $\Gamma_w$ is convex on $\Omega^n$ if*

$$\sum_{i=1}^{n} \frac{w_i [\phi_i'(x_i)]^2}{\phi_i''(x_i)} \leq \alpha \Psi(y) \quad \text{for } x \in \Omega^n, \tag{2.8}$$

*where $y = \Gamma_w(x)$.*

*Proof.* Multiplying both sides of (2.8) by $\Psi''(y)$ and applying (2.7), we see that condition (2.2) holds. The result follows from Theorem 2.2 immediately. ∎

THEOREM 2.4. *Let $\Omega$ be an open convex subset of $R$, $\Psi : \Omega \to R$ be strictly increasing, twice differentiable and convex, $\phi_i : \Omega \to R$, $i = 1, ..., n$, be strictly convex and twice differentiable, and $w \in R^n_{++}$ be a given vector. Assume that there exist $0 \neq \alpha_i \in R$, $i = 1, ..., n$, holding the same sign such that*

$$\alpha_i \phi_i(t) \phi_i''(t) \geq [\phi_i'(t)]^2 \quad \text{for } t \in \Omega, \tag{2.9}$$

*and there exists an $\alpha \in R$ such that the inequality (2.7) holds. Then the function $\Gamma_w$ is convex if*

$$\alpha \geq \max_{1 \leq i \leq n} \alpha_i \quad (\text{ when } \alpha_i > 0 \text{ for all } i), \tag{2.10}$$

*or*

$$\alpha \leq \min_{1 \leq i \leq n} \alpha_i \quad (\text{ when } \alpha_i < 0 \text{ for all } i). \tag{2.11}$$

*Proof.* Taking $y = \Gamma_w(x)$, we see two cases.

Case 1: $\alpha_i > 0$ for $i = 1, ..., n$. In this case, (2.9) implies that $\phi_i(t) \geq 0$ for $t \in \Omega$ and (2.10) implies that

$$\sum_{i=1}^{n} w_i \frac{[\phi_i'(t)]^2}{\phi_i''(x_i)} \leq \sum_{i=1}^{n} w_i \alpha_i \phi_i(x_i) \leq \left( \max_{1 \leq i \leq n} \alpha_i \right) \sum_{i=1}^{n} w_i \phi_i(x_i) \leq \alpha \Psi(y).$$

Case 2: $\alpha_i < 0$ for $i = 1, ..., n$. In this case, (2.9) implies that $\phi_i(t) \leq 0$ for $t \in \Omega$ and (2.11) implies that

$$\sum_{i=1}^{n} w_i \frac{[\phi_i'(t)]^2}{\phi_i''(x_i)} \leq \sum_{i=1}^{n} w_i \alpha_i \phi_i(x_i) \leq \left( \min_{1 \leq i \leq n} \alpha_i \right) \sum_{i=1}^{n} w_i \phi_i(x_i) \leq \alpha \Psi(y).$$

Both cases yield (2.8) and the desired result follows from Theorem 2.2. □

A special case of $\phi_1(t) = \phi_2(t) = ... = \Psi(t)$ immediately leads to the next result.

COROLLARY 2.5. *Let $\Omega$ be an open convex set in $R$, $\phi : \Omega \to R$ be a convex, twice differentiable and strictly increasing function, and $w \in R^n_{++}$ be a given vector. If there exists an $\alpha \neq 0$ such that*

$$[\phi'(t)]^2 = \alpha\phi(t)\phi''(t) \quad \textit{for } t \in \Omega, \tag{2.12}$$

*then the function $\Upsilon_w(x) = \phi^{-1}(\sum_{i=1}^n w_i\phi(x_i))$ is convex on $\Omega^n$.*

It is worth making two remarks here:

*Remark* 2.1. The functions satisfying a differential inequality such as (2.7) are related to the so-called self-concordant barrier function as introduced by Nesterov and Nemirovsky [16]. Recall that a $C^3$ function $\xi : (0,\infty) \to R$ is said to be self-concordant if $\xi$ is convex and there exists a constant $\mu_1 > 0$ such that

$$|\xi'''(t)| \leq \mu_1(\xi''(t))^{\frac{3}{2}} \quad \text{for } t \in (0,\infty). \tag{2.13}$$

Moreover, the self-concordant function $\xi$ is called a self-concordant barrier function if there exists a constant $\mu_2 > 0$ such that

$$|\xi'(t)| \leq \mu_2[f''(t)]^{\frac{1}{2}} \quad \text{for } t \in (0,\infty). \tag{2.14}$$

Combining (2.13) and (2.14) yields

$$\xi'(t)\xi'''(t) \leq \mu[\xi''(t)]^2.$$

This indicates that the first-order derivative function of a self-concordant barrier function, i.e., $g(t) := \xi'(t)$, satisfies the inequality (2.7). Our later analysis will show that a self-concordant function $\xi(\cdot)$ itself may also satisfy an inequality like (2.7) or (2.9).

*Remark* 2.2. The functions satisfying a differential inequality such as (2.7) also appear in convexity theory. Given a twice differentiable function $\phi(t) > 0$ on its domain $\Omega$, we consider the convexity of the function $h(t) := \frac{1}{\phi(t)}$ on $\Omega$. Notice that

$$h''(t) = \frac{2[\phi'(t)]^2 - \phi(t)\phi''(t)}{[\phi(t)]^3} \quad \text{for} \quad t \in \Omega.$$

Hence the function $h(t) = \frac{1}{\phi(t)}$ is convex if and only if the inequality $\phi(t)\phi''(t) \leq 2[\phi'(t)]^2$ holds on $\Omega$. Moreover, if $\phi(t)\phi''(t) \leq [\phi'(t)]^2$, the convex function $h(t)$ satisfies a reverse inequality, i.e., $h(t)h''(t) \geq [h'(t)]^2$ on $\Omega$.

From this observation, a related question arises. Given a function $\phi(t) > 0$ on $\Omega$ and a constant $r > 0$, when will the function $h(t) := \frac{1}{\phi(t)^r}$ become convex and satisfy an inequality such as (2.9)? A straightforward analysis leads to the next result.

THEOREM 2.6. *(i) Let $\Omega$ be a convex subset of $R$ and $\phi : \Omega \to (0,\infty)$ be a function. If $\phi(t)\phi''(t) \leq [\phi'(t)]^2$ for $t \in \Omega$, then, for any $r > 0$, the function $h(t) := \frac{1}{\phi(t)^r}$ is convex and $h(t)h''(t) \geq [h'(t)]^2$ for $t \in \Omega$. Conversely, if there exists an $r > 0$ such that $h(t) := \frac{1}{\phi(t)^r}$ is convex and $h(t)h''(t) \geq [h'(t)]^2$ for $t \in \Omega$, then $\phi(t)\phi''(t) \leq [\phi'(t)]^2$ for $t \in \Omega$.*

*(ii) Let $\Omega$ be a convex subset of $R$, $\tau > 0$, and $\phi : \Omega \to (\tau,\infty)$ be a function. If $\phi(t)\phi''(t) \leq [\phi'(t)]^2$ for $t \in \Omega$, then, for any scalar $r > 0$ and $T > 0$, the function $h_T(t) := T + \frac{1}{\phi(t)^r}$ is convex and $\alpha h_T(t)h_T''(t) \geq [h_T'(t)]^2$ for $t \in \Omega$, where $\alpha = \frac{1}{T\tau^r+1}$.*

*Proof.* For case (A), it is sufficient to see that

$$h''(t) = \frac{r^2(\phi'(t))^2 + r[(\phi'(t))^2 - \phi(t)\phi''(t)]}{\phi(t)^{r+2}},$$

and

$$h(t)h''(t) - [h'(t)]^2 = \frac{r[(\phi'(t))^2 - \phi(t)\phi''(t)]}{\phi(t)^{2(r+1)}}.$$

For case (B), it is easy to verify that $h''_T(t) = h''(t)$ and

$$\left(\frac{1}{T\phi(t)^r + 1}\right) h_T(t)h''_T(t) - [h'_T(t)]^2 = \frac{r[(\phi'(t))^2 - \phi(t)\phi''(t)]}{\phi(t)^{2(r+1)}}.$$

Then the desired result follows. $\square$

The above results indicate that if we have a function $\phi$ satisfying the inequality (2.7) with $\alpha = 1$, then we may construct a function $h$ from $\phi$ such that $h$ satisfies the converse differentiable inequality $\alpha h(t)h''(t) \geq [h'(t)]^2$ for some constant $\alpha$. Moreover, if we take a T-translation of the value of the function $h$, then the resulting function satisfies the converse differentiable inequality with an $\alpha$ that can be reduced to be smaller than any threshold given in (0,1) provided a suitable choice of $T > 0$. This fact will be used near the end of Section 3.

Before closing this section, we mention that Ben-Tal and Teboulle [2] also provided a sufficient and necessary condition, under different assumptions, for the convexity of the function $\Upsilon_w(x)$ as defined in (1.1). They showed that $\Upsilon_w(x)$ is convex if and only if $-\phi'/\phi''$ is convex. It is possible to extend their analysis for deriving sufficient conditions for the convexity of $\Gamma_w(x)$ defined by (1.2). For example, the following result can be proved along the line of the proof of "Theorem 2.1" therein.

LEMMA 2.7. *Let $\Psi(t) \in C^3$ and $\phi_i(t) \in C^3$ be strictly increasing and $\rho(t) = -\Psi''(t)/\Psi'(t)$. If $1/\rho(t)$ is convex and $\Psi^{-1}(\phi_i(t))$ is convex, for $i = 1, ..., n$, then $\Gamma_w(x)$ given by (1.2) is convex.*

**3. Constructing convex generalized mean functions $\Gamma_w$.** In this section, we try to identify some classes of functions that satisfy inequality (2.7) and/or inequality (2.9) so that we have building blocks for constructing concrete convex function $\Gamma_w(x)$. First, we give a result that identifies functions satisfying the equation (2.12). Obviously, this class of functions satisfy both inequalities (2.7) and (2.9).

THEOREM 3.1. *Let $\Omega$ be an open set in $R$ and $\phi : \Omega \to R$ be a convex, twice differentiable and strictly increasing function satisfying equation (2.12) with a constant $\alpha \neq 0$. Then,*

*(i) when $\alpha = 1$, $\phi$ is in the form of*

$$\phi(t) = \gamma e^{\frac{t}{\beta}}$$

*for some $\gamma > 0$ and $\beta > 0$.*

*(ii) when $0 < \alpha \neq 1$ with $v^* := \sup_{t\in\Omega} \frac{1-\alpha}{\alpha} t$ being finite, $\phi$ is in the form of*

$$\phi(t) = \gamma \left(\frac{\alpha - 1}{\alpha} t + \beta\right)^{\frac{\alpha}{\alpha-1}}$$

*for some $\gamma > 0$ and $\beta \geq v^*$.*

*(iii) when $\alpha < 0$ with $u^* := \sup_{t\in\Omega} \frac{\alpha-1}{\alpha} t$ being finite, $\phi$ is in the form of*

$$\phi(t) = -\gamma \left(\beta - \frac{\alpha-1}{\alpha} t\right)^{\frac{\alpha}{\alpha-1}}$$

*for some $\gamma > 0$ and $\beta \geq u^*$.*

Note that results (i) and (ii) were pointed out in [2] and [13] and result (iii) can be easily derived. The above result leads to the following consequence related to $\Upsilon_w$.

COROLLARY 3.2. *The following functions can be used to explicitly construct a convex generalized mean function $\Upsilon_w(x) = \phi^{-1}(\sum_{i=1}^n w_i \phi(x_i))$ over $\Omega^n$:*

*(i) $\phi(t) = \gamma e^{\frac{t}{\beta}}$ over $\Omega = R$ with $\gamma > 0$ and $\beta > 0$.*

*(ii) $\phi(t) = \gamma \left(\frac{1}{p} t + \beta\right)^p$ over $\Omega = (\eta, \infty)$ with $p > 1$, $\gamma > 0$ and $\beta \geq -\frac{\eta}{p}$.*

*(iii) $\phi(t) = \frac{\gamma}{(\beta - \frac{1}{p} t)^p}$ over $\Omega = (-\infty, \eta)$ with $p > 0$, $\gamma > 0$ and $\beta \geq -\frac{\eta}{p}$.*

*(iv) $\phi(t) = -\gamma(\beta - \frac{1}{p} t)^p$ over $\Omega = (-\infty, \eta)$ with $0 < p < 1$, $\gamma > 0$ and $\beta \geq \frac{\eta}{p}$.*

Again, results (i) and (ii) were given in [2] and [13] and results (iii) and (iv) can be easily derived. The functions listed in Corollary 3.2 actually form a complete basis in the sense that the function $\phi$ in case (i) satisfies condition (2.12) with $\alpha = 1$; the function $\phi$ in case (ii) satisfies condition (2.12) with $\alpha = \frac{p}{p-1} > 1$; the function $\phi$ in case (iii) satisfies condition (2.12) with $\alpha = \frac{p}{p+1} \in (0, 1)$; and the function $\phi$ in (iv) satisfies condition (2.12) with $\alpha = \frac{p}{p-1} < 0$.

We now try to identify some class of functions that satisfy inequalities (2.7) and/or (2.9). For simplicity, we only consider convex, twice differentiable, strictly increasing functions $\vartheta$ on $\Omega = (0, \infty)$. Let us first define the following four categories of such functions:

$$\begin{aligned}
\mathcal{U}_1 &= \{\vartheta : \text{ There exists } \alpha \in R \text{ such that } \alpha\vartheta(t)\vartheta''(t) \geq [\vartheta'(t)]^2 \text{ for } t \in \Omega\}; \\
\mathcal{U}_2 &= \{\vartheta : \text{ There exists } \alpha \in R \text{ such that } \alpha\vartheta(t)\vartheta''(t) \leq [\vartheta'(t)]^2 \text{ for } t \in \Omega\}; \\
\mathcal{U}_3 &= \{\vartheta : \text{ There exist } \alpha_1 \leq \alpha_2 \text{ such that } \alpha_1\vartheta(t)\vartheta''(t) \leq [\vartheta'(t)]^2 \leq \alpha_2\vartheta(t)\vartheta''(t) \\
&\qquad \text{for } t \in \Omega\}; \\
\mathcal{U}_4 &= \{\vartheta : \text{ There exists } \alpha \in R \text{ such that } \alpha\vartheta(t)\vartheta''(t) = [\vartheta'(t)]^2 \text{ for all } t \in \Omega\}.
\end{aligned}$$

It is evident that

$$\mathcal{U}_4 \subset \mathcal{U}_3 \subset (\mathcal{U}_2 \cap \mathcal{U}_1).$$

As pointed out in Theorem 3.1, the class $\mathcal{U}_4$ can be given explicitly. By allowing $\alpha_1 \neq \alpha_2$, we show that $\mathcal{U}_3$ is much broader than $\mathcal{U}_4$. In fact, many convex functions with certain regularities fall into the category $\mathcal{U}_3$. To start, we introduce a new class of functions with certain regularity properties.

DEFINITION 3.3. *A convex, twice differentiable, strictly increasing function $\delta(t) : (0, \infty) \to R$ is called an $S^*$-regular function if (i) $\delta(t)$ vanishes at $t = 0$ in the sense of*

$$\lim_{t\to 0_+} \delta(0) = \lim_{t\to 0_+} \delta'(0) = \lim_{t\to 0_+} \delta''(0) = 0;$$

*and (ii) there exist positive constants $0 < \beta_1 \leq \beta_2$, $p \geq 1$ and $q \geq 1$ such that*

$$\beta_1[(t+1)^{p-1} - (t+1)^{-1-q}] \leq \delta''(t) \leq \beta_2[(t+1)^{p-1} - (t+1)^{-1-q}], \; t > 0. \tag{3.1}$$

Note that condition (3.1) actually implies the strict convexity of an $S^*$-regular function on $(0, \infty)$. In particular, setting $\beta_1 = \beta_2$, condition (3.1) reduces to an equation

$$\delta''(t) = (t+1)^{p-1} - (t+1)^{-1-q}. \tag{3.2}$$

Taking integration twice and noting that $\lim_{t\to 0_+} \delta(0) = \lim_{t\to 0_+} \delta'(0) = 0$, the unique solution to equation (3.2) is

$$\Delta_{p,q}(t) = \frac{(t+1)^{p+1} - 1}{p(p+1)} - \frac{(t+1)^{1-q} - 1}{q(q-1)} - \frac{p+q}{pq} t \quad \text{for } p \geq 1 \text{ and } \ q > 1. \tag{3.3}$$

In addition, since $\lim_{q\to 1_+} [1 - (t+1)^{1-q}]/(q-1) = \ln(t+1)$, we have

$$\Delta_{p,1}(t) = \frac{(t+1)^{p+1} - 1}{p(p+1)} + \ln(t+1) - \frac{p+1}{p} t \quad \text{for } p \geq 1. \tag{3.4}$$

Taking $p = 1$ in (3.4), we have

$$\Delta_{1,1}(t) = \frac{(t+1)^2 - 1}{2} + \ln(t+1) - 2t = \frac{1}{2} t^2 - t + \ln(t+1). \tag{3.5}$$

Moreover, taking $p = 1$ and $q = 2$ in (3.3) yields

$$\Delta_{1,2}(t) = \frac{1}{2} \left[(t+1)^2 - (t+1)^{-1} - 3t\right]. \tag{3.6}$$

In terms of this particular solution $\Delta_{p,q}(t)$, condition (3.1) can be written as

$$\beta_1 \Delta''_{p,q}(t) \leq \delta''(t) \leq \beta_2 \Delta''_{p,q}(t). \tag{3.7}$$

By integrating and noting that $\lim_{t\to 0_+} \delta'(0) = \lim_{t\to 0_+} \delta(0) = 0$, we further have

$$\beta_1 \Delta'_{p,q}(t) \leq \delta'(t) \leq \beta_2 \Delta'_{p,q}(t) \tag{3.8}$$

and

$$\beta_1 \Delta_{p,q}(t) \leq \delta(t) \leq \beta_2 \Delta_{p,q}(t). \tag{3.9}$$

Therefore, we can see that the class of $S^*$-regular functions is quite broad. Later, by using (3.7)-(3.9), we show that $S^*$-regular functions fall into the category $\mathcal{U}_3$.

It is worth mentioning that for any $p \geq 1, q > 1$ (including the case of $q \to 1_+$) the $S^*$-regular function $\Delta_{p,q}(t)$ is not self-concordant. In fact, the function $\Delta_{p,q}(t)$ does not satisfy the inequality (2.13) since $\delta''(t) \to 0$ and $\delta'''(t) \to p + q$ as $t \to 0_+$.

The $S^*$-regular functions are somewhat analogous to (but different from) the so-called self-regular functions that were defined in [18] to study interior-point algorithms for linear programming problems.

Definition 3.4. *[18] A twice differentiable function $\psi(t) : (0, \infty) \to R$ is self-regular if (i) $\psi(t)$ is strictly convex and vanishes at its global minimal point $t = 1$, i.e., $\psi(1) = \psi'(1) = 0$, and (ii) there exist constants $\nu_2 \geq \nu_1 > 0$ and $p \geq 1, q \geq 1$ such that*

$$\nu_1(t^{p-1} + t^{-1-q}) \leq \psi''(t) \leq \nu_2(t^{p-1} + t^{-1-q}) \quad \textit{for } t \in (0, \infty).$$

Notice that a self-regular function is not necessarily increasing. However, a variable translation leads to a strictly increasing function $g : (0, \infty) \to R$ where $g(t) :=$

$\psi(t+1)$. We call such $g(t)$ a translated self-regular function ($T^*$-regular in short). From Definition 3.4, we see that a $T^*$-regular function $g(t)$ satisfies the following three conditions: (i) $g(t)$ is strictly increasing on $(0,\infty)$, (ii) $\lim_{t\to 0_+} g(t) = \lim_{t\to 0_+} g'(t) = 0$, and (iii) there exists positive constants $\nu_2 \geq \nu_1 > 0$ and $p \geq 1, q \geq 1$ such that

$$\nu_1[(t+1)^{p-1} + (t+1)t^{-1-q}] \leq g''(t) \leq \nu_2[(t+1)^{p-1} + (t+1)^{-1-q}],\ t > 0. \tag{3.10}$$

Since $\lim_{t\to 0_+} g''(t) \neq 0$, $T^*$-regular functions and $S^*$-regular functions belong to different classes, although conditions (3.10) and (3.1) may look alike. In fact, an $S^*$-regular function is not $T^*$-regular, and a $T^*$-regular function is not $S^*$-regular.

As shown in [18], when $\nu_1 = \nu_2$, the self-regular function $\psi(t)$ is given by

$$\psi_{p,q}(t) := \frac{t^{p+1}-1}{p(p+1)} + \frac{t^{1-q}-1}{q(q-1)} + \frac{p-q}{pq}(t-1) \quad \text{for } p \geq 1 \text{ and } q > 1. \tag{3.11}$$

For $T^*$-regular function $G_{p,q}(t) := \psi_{p,q}(t+1)$, condition (3.10) can be written as

$$\nu_1 G''_{p,q}(t) \leq g''(t) \leq \nu_2 G''_{p,q}(t). \tag{3.12}$$

Similar to (3.7) and (3.8), we have

$$\nu_1 G'_{p,q}(t) \leq g'(t) \leq \nu_2 G'_{p,q}(t) \tag{3.13}$$

and

$$\nu_1 G_{p,q}(t) \leq g(t) \leq \nu_2 G_{p,q}(t). \tag{3.14}$$

To compare with the $S^*$-regular functions $\Delta_{p,q}(t)$ (see, (3.3)-(3.6)), we list here a few $T^*$-regular functions:

$$\begin{aligned}
G_{p,q}(t) &= \frac{(t+1)^{p+1}-1}{p(p+1)} + \frac{(t+1)^{1-q}-1}{q(q-1)} + \frac{p-q}{pq}t \quad \text{for } p \geq 1 \text{ and } q > 1.\\
G_{p,1}(t) &= \frac{(t+1)^{p+1}-1}{p(p+1)} - \ln(t+1) + \frac{p-1}{p}t.\\
G_{1,1}(t) &= \frac{(t+1)^2-1}{2} - \ln(t+1) = \frac{1}{2}t^2 + (t - \ln(t+1)).\\
G_{1,2}(t) &= \frac{1}{2}\left[(t+1)^2 + (t+1)^{-1} - t - 2\right].
\end{aligned}$$

As pointed out in [18], the self-regular function (3.11) is self-concordant [16]. Thus, the $T^*$-regular function $G_{p,q}(t)$ can be also viewed as a translated self-concordant function. While $T^*$-regular and $S^*$-regular functions are two different classes functions, the following theorem shows that they both belong to the category of $\mathcal{U}_3$.

Theorem 3.5. *Let $\delta(t) : (0,\infty) \to R$ be either $S^*$-regular or $T^*$-regular on $(0,\infty)$. Then there exist $c_2 \geq c_1 > 0$ such that*

$$c_1 \leq \frac{\delta(t)\delta''(t)}{[\delta'(t)]^2} \leq c_2 \text{ for } t \in (0,\infty), \tag{3.15}$$

*i.e., the function $\delta(t) \in \mathcal{U}_3$.*

*Proof.* We first show that an $S^*$-regular function $\Delta_{p,q}(t)$ satisfies the property (3.15). Actually, we have

$$\frac{\Delta_{p,q}(t)\Delta''_{p,q}(t)}{[\Delta'_{p,q}(t)]^2} = \frac{\left(\frac{(t+1)^{p+1}-1}{p(p+1)} - \frac{(t+1)^{1-q}-1}{q(q-1)} - \frac{p+q}{pq}t\right)[(t+1)^{p-1} - (t+1)^{-1-q}]}{\left(\frac{(t+1)^p}{p} + \frac{(t+1)^{-q}}{q} - \frac{p+q}{pq}\right)^2}.$$

Dividing the numerator and denominator of the right-hand side of the above equation by $(t+1)^{2p} = (t+1)^{p+1}(t+1)^{p-1}$, we have

$$\frac{\Delta_{p,q}(t)\Delta''_{p,q}(t)}{[\Delta'_{p,q}(t)]^2} = \frac{\left(\frac{1-(t+1)^{-(p+1)}}{p(p+1)} + \frac{(t+1)^{-(p+1)}-(t+1)^{-(p+q)}}{q(q-1)} - \frac{(p+q)t}{pq(t+1)^{(p+1)}}\right)\left(1 - \frac{1}{(t+1)^{(p+q)}}\right)}{\left(\frac{1}{p} + \frac{1}{q(t+1)^{(p+q)}} - \frac{p+q}{pq(t+1)^p}\right)^2}.$$

Therefore,

$$\lim_{t\to\infty} \frac{\Delta_{p,q}(t)\Delta''_{p,q}(t)}{[\Delta'_{p,q}(t)]^2} = \frac{p}{p+1}. \tag{3.16}$$

Notice that $\Delta''_{p,q}(t) = (t+1)^{p-1} - (t+1)^{-1-q}$. We have

$$\lim_{t\to 0_+} \Delta'''_{p,q}(t) = \lim_{t\to 0_+} (p-1)(t+1)^{p-2} + (1+q)(t+1)^{-2-q} = p+q.$$

Since $\Delta''_{p,q}(t) \to 0$, $\Delta'_{p,q}(t) \to 0$ and $\Delta_{p,q}(t) \to 0$ as $t \to 0_+$, we have

$$\lim_{t\to 0_+} \frac{(\Delta''_{p,q}(t))^2}{\Delta'_{p,q}(t)} = \lim_{t\to 0_+} \frac{[(\Delta''_{p,q}(t))^2]'}{[\Delta'_{p,q}(t)]'} = \lim_{t\to 0_+} \frac{2\Delta''_{p,q}(t)\Delta'''_{p,q}(t)}{\Delta''_{p,q}(t)} = 2(p+q).$$

Hence

$$\begin{aligned}
\lim_{t\to 0_+} \frac{\Delta_{p,q}(t)}{2\Delta'_{p,q}(t)\Delta''_{p,q}(t)} &= \lim_{t\to 0_+} \frac{[\Delta_{p,q}(t)]'}{[2\Delta'_{p,q}(t)\Delta''_{p,q}(t)]'} \\
&= \lim_{t\to 0_+} \frac{\Delta'_{p,q}(t)}{2[\Delta''_{p,q}(t)]^2 + 2\Delta'_{p,q}(t)\Delta'''_{p,q}(t)} \\
&= \lim_{t\to 0_+} \frac{1}{2\frac{[\Delta''_{p,q}(t)]^2}{\Delta'_{p,q}(t)} + 2\Delta'''_{p,q}(t)} \\
&= \frac{1}{2(2(p+q)+(p+q))} = \frac{1}{6(p+q)}.
\end{aligned}$$

Using the above relations, we further have

$$\begin{aligned}
\lim_{t\to 0_+} \frac{\Delta_{p,q}(t)\Delta''_{p,q}(t)}{[\Delta'_{p,q}(t)]^2} &= \lim_{t\to 0_+} \frac{\left[\Delta_{p,q}(t)\Delta''_{p,q}(t)\right]'}{[(\Delta'_{p,q}(t))^2]'} \\
&= \lim_{t\to 0_+} \frac{\Delta'_{p,q}(t)\Delta''_{p,q}(t) + \Delta_{p,q}(t)\Delta'''_{p,q}(t)}{2\Delta'_{p,q}(t)\Delta''_{p,q}(t)} \\
&= \frac{1}{2} + \lim_{t\to 0_+} \frac{\Delta_{p,q}(t)\Delta'''_{p,q}(t)}{2\Delta'_{p,q}(t)\Delta''_{p,q}(t)} \\
&= \frac{1}{2} + \lim_{t\to 0_+} \frac{\Delta_{p,q}(t)}{2\Delta'_{p,q}(t)\Delta''_{p,q}(t)} \lim_{t\to 0_+} \Delta'''_{p,q}(t) \\
&= \frac{1}{2} + \frac{1}{6(p+q)]}(p+q) \\
&= \frac{2}{3}. \qquad (3.17)
\end{aligned}$$

Notice that $\Delta_{p,q}(t) > 0, \Delta''_{p,q}(t) > 0$ and $\Delta'_{p,q}(t) > 0$ in $(0, \infty)$. From (3.16) and (3.17), we can see by continuity that there exist two constants $\mu_2 \geq \mu_1 > 0$ such that

$$\mu_1 \leq \frac{\Delta_{p,q}(t)\Delta''_{p,q}(t)}{[\Delta'_{p,q}(t)]^2} \leq \mu_2 \quad \text{for } t \in (0, \infty).$$

Together with (3.7) through (3.9), this implies that an $S^*$-regular function $\delta(t)$ satisfies the following inequality:

$$0 < \mu_1\beta_1 \leq \frac{\delta(t)\delta''(t)}{[\delta'(t)]^2} \leq \beta_2\mu_2,$$

Therefore, (3.15) holds with $c_1 := \mu_1\beta_1$ and $c_2 := \mu_2\beta_2$.

Now, for a $T^*$-regular function $G_{p,q}(t)$, we have

$$\frac{G_{p,q}(t)G''_{p,q}(t)}{[G'_{p,q}(t)]^2} = \frac{\left(\frac{(t+1)^{p+1}-1}{p(p+1)} + \frac{(t+1)^{1-q}-1}{q(q-1)} + \frac{p-q}{pq}t\right)\left((t+1)^{p-1} + (t+1)^{-1-q}\right)}{\left(\frac{(t+1)^p}{p} - \frac{(t+1)^{-q}}{q} + \frac{p-q}{pq}\right)^2}$$

Dividing the numerator and denominator of the right-hand side of the above equation by $(t+1)^{2p} = (t+1)^{p+1}(t+1)^{p-1}$, and taking $t \to \infty$, we have

$$\lim_{t\to\infty} \frac{G_{p,q}(t)G''_{p,q}(t)}{[G'_{p,q}(t)]^2} = \frac{p}{p+1}. \tag{3.18}$$

Notice that

$$\lim_{t\to 0_+} G''_{p,q}(t) = \lim_{t\to 0_+} (t+1)^{p-1} + (t+1)^{-1-q} = 2.$$

Since $G_{p,q}(t) \to 0$ and $G'_{p,q}(t) \to 0$ as $t \to 0_+$, thus,

$$\begin{aligned}
\lim_{t\to 0_+} \frac{G_{p,q}(t)G''_{p,q}(t)}{[G'_{p,q}(t)]^2} &= \lim_{t\to 0_+} \frac{G_{p,q}(t)}{[G'_{p,q}(t)]^2} \lim_{t\to 0_+} G''_{p,q}(t) \\
&= \lim_{t\to 0_+} \frac{[G_{p,q}(t)]'}{[(G'_{p,q}(t))^2]'} \lim_{t\to 0_+} G''_{p,q}(t) \\
&= \lim_{t\to 0_+} \frac{G'_{p,q}(t)}{2G'_{p,q}(t)G''_{p,q}(t)} \lim_{t\to 0_+} G''_{p,q}(t) \\
&= \frac{1}{2}. \qquad (3.19)
\end{aligned}$$

Notice that $G_{p,q}(t), G''_{p,q}(t)$, and $G'_{p,q}(t)$ are positive on $(0, \infty)$. From (3.18) and (3.19), we can conclude by continuity that there exist $\lambda_2 \geq \lambda_1 > 0$ such that

$$\lambda_1 \leq \frac{G_{p,q}(t)G''_{p,q}(t)}{[G'_{p,q}(t)]^2} \leq \lambda_2 \quad \text{for } t \in (0, \infty).$$

Together with (3.12) through (3.14), this implies that a $T^*$-regular function satisfies the following inequality:

$$0 < \lambda_1\nu_1 \leq \frac{g(t)g''(t)}{[g'(t)]^2} \leq \lambda_2\nu_2.$$

Hence (3.15) holds with $c_1 := \nu_1\lambda_1$ and $c_2 := \nu_2\lambda_2$. $\square$

A fact that should be pointed out here is that new functions in $\mathcal{U}_1$ or $\mathcal{U}_2$ can be constructed by using the basic operations (addition, multiplication, division and composition) on known functions.

THEOREM 3.6. *(i) If $\phi : (0,\infty) \to (0,\infty)$, $\phi \in \mathcal{U}_1$ with $\alpha = \alpha_1$ and $\varphi : (0,\infty) \to (0,\infty)$, $\varphi \in \mathcal{U}_1$ with $\alpha = \alpha_2$, then $\phi + \varphi \in \mathcal{U}_1$ with $\alpha = 2\max\{\alpha_1, \alpha_2\}$.*

*(ii) If $\phi : (0,\infty) \to (0,\infty)$, $\phi \in \mathcal{U}_1$ with $\alpha_1 \in (0,1]$ and $\varphi : (0,\infty) \to (0,\infty)$, $\varphi \in \mathcal{U}_1$ with $\alpha_2 \in (0,1]$, then the multiplicative function $\phi(t)\cdot\varphi(t) \in \mathcal{U}_1$ with $\alpha = 1$. Similarly, if $\phi \in \mathcal{U}_2$ with $\alpha_1 \geq 1$ and $\varphi \in \mathcal{U}_2$ with $\alpha_2 \geq 1$, then $\phi(t)\cdot\varphi(t) \in \mathcal{U}_2$ with $\alpha = 1$.*

*(iii) If $\phi : (0,\infty) \to (0,\infty)$, $\phi \in \mathcal{U}_2$ with $\alpha_1 \geq 1$ and $\varphi : (0,\infty) \to (0,\infty)$, $\varphi \in \mathcal{U}_1$ with $\alpha_2 \in (0,1]$, then the function $\frac{\phi}{\varphi} \in \mathcal{U}_2$ with $\alpha = 1$. Similarly, if $\phi \in \mathcal{U}_1$ with $\alpha_1 \in (0,1]$ and $\varphi \in \mathcal{U}_2$ with $\alpha_2 \geq 1$, then $\frac{\phi}{\varphi} \in \mathcal{U}_1$ with $\alpha = 1$.*

*(iv) Let $\varphi : (0,\infty) \to \Omega_1 \subset R$ and $\phi : \Omega_1 \to (0,\infty)$ be two convex functions. If $\phi \in \mathcal{U}_1$ with $\alpha > 0$, then the composite function $(\phi \circ \varphi)(t) = \phi(\varphi(t)) \in \mathcal{U}_1$ with the same constant $\alpha$.*

*Proof.* Keep in mind that all functions in $\mathcal{U}_1$ and $\mathcal{U}_2$ are convex, twice differentiable, and strictly increasing. For (i), we note that $\alpha_1, \alpha_2, \phi(t)$ and $\varphi(t)$ are all nonnegative. Thus,

$$\begin{aligned}(\phi'(t) + \varphi'(t))^2 &\leq 2[(\phi'(t))^2 + (\varphi'(t))^2] \\ &\leq 2\alpha_1\phi(t)\phi''(t) + 2\alpha_2\varphi(t)\varphi''(t) \\ &\leq 2\max\{\alpha_1,\alpha_2\}(\phi(t)+\varphi(t))(\phi''(t)+\varphi''(t)).\end{aligned}$$

This indicates that $\phi + \varphi \in \mathcal{U}_1$ with $\alpha = 2\max\{\alpha_1, \alpha_2\}$. The proofs of statements (ii) and (iii) are easy by noting that

$$\begin{aligned}(\phi(t)\varphi(t))[\phi(t)\varphi(t)]'' &= (\phi(t)\phi''(t) - [\phi'(t)]^2)\varphi^2(t) + \phi^2(t)(\varphi(t)\varphi''(t) - [\varphi'(t)]^2) \\ &\quad + [(\phi(t)\varphi(t))']^2\end{aligned}$$

and

$$\frac{\phi(t)}{\varphi(t))}\left(\frac{\phi(t)}{\varphi(t)}\right)'' = \frac{(\phi(t)\phi''(t) - [\phi'(t)]^2)\varphi^2(t) + \phi^2(t)([\varphi'(t)]^2 - \varphi''(t)\varphi(t))}{\varphi^4(t)} + \left[\left(\frac{\phi(t)}{\varphi(t)}\right)'\right]^2.$$

Statement (iv) can also be easily verified. $\square$

To construct examples of the convex function $\Gamma_w$, Theorem 2.4 tells us that it suffices to find functions satisfying the inequalities (2.7) and (2.9) and compare their $\alpha$ values. The next result is to estimate the $\alpha$ values, or equivalently, to estimate the values of $c_1$ and $c_2$ in (3.15). For simplicity, we use the $S^*$-regular and $T^*$-regular functions with $p = 1$ and $q = 1, 2$ to estimate required $c_1$ and $c_2$.

THEOREM 3.7. *(i) The $S^*$-regular functions $\Delta_{1,1}(t)$ and $\Delta_{1,2}(t)$ given by (3.5) and (3.6), respectively, satisfy condition (3.15) with $c_1 = \frac{1}{2}$ and $c_2 = \frac{2}{3}$, that is,*

$$\frac{3}{2}\Delta_{1,1}(t)\Delta''_{1,1}(t) \leq [\Delta'_{1,1}(t)]^2 \leq 2\Delta_{1,1}(t)\Delta''_{1,1}(t) \tag{3.20}$$

*and*

$$\frac{3}{2}\Delta_{1,2}(t)\Delta''_{1,2}(t) \leq [\Delta'_{1,2}(t)]^2 \leq 2\Delta_{1,2}(t)\Delta''_{1,2}(t) \tag{3.21}$$

*for* $t \in (0, \infty)$.

*(ii) The* $T^*$*-regular function* $G_{1,1}(t)$ *satisfies condition (3.15) with* $c_1 = \frac{4}{9}$ *and* $c_2 = \frac{1}{2}$*, while* $G_{1,2}(t)$ *satisfies condition (3.15) with* $c_1 = \frac{12}{29}$ *and* $c_2 = \frac{1}{2}$*, i.e.,*

$$2G_{1,1}(t)G''_{1,1}(t) \le [G'_{1,1}(t)]^2 \le \frac{9}{4}G_{1,1}(t)G''_{1,1}(t) \tag{3.22}$$

*and*

$$2G_{1,2}(t)G''_{1,2}(t) \le [G'_{1,2}(t)]^2 \le \frac{29}{12}G_{1,2}(t)G''_{1,2}(t) \tag{3.23}$$

*for* $t \in (0, \infty)$.

*Proof.* We give a proof to (3.20). (3.21) can be shown similarly. For the function $\Delta_{1,1}(t)$, we have

$$\Delta'_{1,1}(t) = \frac{t^2}{t+1}, \quad \Delta''_{1,1}(t) = \frac{t(t+2)}{(t+1)^2},$$

and hence

$$\zeta(t) := \frac{\Delta_{1,1}(t)\Delta''_{1,1}(t)}{[\Delta'_{1,1}(t)]^2} = \frac{(t+2)\left(\frac{1}{2}t^2 - t + \ln(t+1)\right)}{t^3}.$$

Clearly, we have

$$\zeta'(t) = \frac{(\frac{1}{2}t^2 - t + \ln(t+1))(-2t-6) + \frac{(t+2)t^3}{t+1}}{t^4}.$$

This implies that its stationary (including the maximum or minimum) point $t_*$ on $(0, \infty)$, if exists, satisfies the following equality:

$$\frac{1}{2}t_*^2 - t_* + \ln(t_*+1) = \frac{(t_*+2)t_*^3}{(2t_*+6)(t_*+1)}.$$

The corresponding value of $\zeta(t)$ becomes

$$\zeta(t_*) = \frac{(t_*+2)\left(\frac{1}{2}t_*^2 - t_* + \ln(t_*+1)\right)}{t_*^3} = \frac{(t_*+2)^2}{(2t_*+6)(t_*+1)}.$$

Let

$$\kappa(t) = \frac{(t+2)^2}{(2t+6)(t+1)} \quad \text{for } t \in (0, \infty)$$

It is easy to verify that $\lim_{t\to 0_+} \kappa(t) = \frac{2}{3}$, $\lim_{t\to\infty} \kappa(t) = \frac{1}{2}$, and $\kappa'(t) < 0$ for $t \in (0, \infty)$. Thus $\frac{1}{2} \le \kappa(t) \le \frac{2}{3}$ for $t \in (0, \infty)$. Since $\zeta(t_*) = \kappa(t_*)$, the extremum value of $\zeta(t)$ on $(0, \infty)$ is located in the interval $[\frac{1}{2}, \frac{2}{3}]$. By (3.16) and (3.17), we see that $\zeta(t) \to \frac{2}{3}$ as $t \to 0_+$, $\zeta(t) \to \frac{1}{2}$ as $t \to \infty$. Hence, we have $\frac{1}{2} \le \zeta(t) \le \frac{2}{3}$ for $t \in (0, \infty)$. This validates inequality (3.20).

We now prove (3.22). It is evident that

$$G'_{1,1}(t) = \frac{t(t+2)}{t+1}, \quad G''_{1,1}(t) = \frac{t^2+2t+2}{(t+1)^2},$$

and hence

$$\chi(t) := \frac{G_{1,1}(t)G''_{1,1}(t)}{[G'_{1,1}(t)]^2} = \frac{(\frac{1}{2}t^2 + t - \ln(t+1))(t^2+2t+2)}{(t^2+2t)^2}.$$

It is not difficult to check that

$$\chi'(t) = \frac{-(2t+2)(t^2+2t+4)\left(\frac{1}{2}t^2 + t - \ln(t+1)\right) + \frac{(t^2+2t)^2(t^2+2t+2)}{t+1}}{(2t+t^2)^3}.$$

The stationary point $t_*$ of $\chi(t)$ on $(0,\infty)$, if exists, must satisfy

$$\frac{1}{2}t_*^2 + t_* - \log(t_*+1) = \frac{(2t_*+t_*^2)^2(t_*^2+2t_*+2)}{2(t_*+1)^2(t_*^2+2t_*+4)}.$$

Therefore, if the $\chi(t)$ has an extremum point $t_*$, its extremum value is given by

$$\begin{aligned}
\chi(t_*) &= \frac{\left(\frac{1}{2}t_*^2 + t_* - \log(t_*+1)\right)(t_*^2+2t_*+2)}{(t_*^2+2t_*)^2} \\
&= \frac{(2t_*+t_*^2)^2(t_*^2+2t_*+2)}{2(t_*+1)^2(t_*^2+2t_*+4)} \cdot \frac{(t_*^2+2t_*+2)}{(t_*^2+2t_*)^2} \\
&= \frac{[(t_*+1)^2+1]^2}{2(t_*+1)^2[(t_*+1)^2+3]} \\
&= \omega((t_*+1)^2),
\end{aligned}$$

where $\omega(t) := \frac{(t+1)^2}{2t(t+3)}$ for $t \in (1,\infty)$. It is evident that $\lim_{t\to 1_+}\omega(t) = \frac{1}{2}$ and $\lim_{t\to\infty}\omega(t) = \frac{1}{2}$. We can also check that $\omega'(t) = 0$ has a unique solution over $(1,\infty)$ at $t_* = 3$ with value $\omega(t_*) = \frac{4}{9}$. Therefore, $\frac{4}{9} \le \omega(t) \le \frac{1}{2}$ for $t \in (1,\infty)$. This means that

$$\frac{4}{9} \le \chi(t_*) = \omega((t_*+1)^2) \le \frac{1}{2}.$$

From (3.18) and (3.19), we see that $\lim_{t\to 0_+}\chi(t) = \frac{1}{2} = \lim_{t\to\infty}\chi(t)$. Hence inequality (3.22) holds. The proof of (3.23) that is similar to that of (3.22) is omitted here. ∎

The next result shows that the composition functions of $e^t$ belong to $\mathcal{U}_3$.

Lemma 3.8. *Denote the exponential function $e^t$ by $\exp(t)$ and the composition of $m$ $(m \ge 1)$ exponential functions by*

$$\theta_m(t) := (\overbrace{\exp\circ\exp\circ\cdots\circ\exp}^{m})(t).$$

*Then*

$$\frac{1}{m}\theta_m(t)\theta''_m(t) \le [\theta'_m(t)]^2 \le \theta_m(t)\theta''_m(t) \quad \textit{for } t \in R. \tag{3.24}$$

*Proof.* Let $\alpha_m(t) := [\theta'_m(t)]^2/(\theta_m(t)\theta''_m(t))$ for $t \in R$. Since $\alpha_1(t) \equiv 1$, we can prove the right-hand side inequality of (3.24) using (iv) of Theorem 3.6 and mathematical induction. For the left-hand side inequality, notice that

$$\theta'_m(t) = \theta_m(t)\theta'_{m-1}(t),\ \ \theta''_m(t) = \theta_m(t)(\theta'_{m-1}(t))^2 + \theta_m(t)\theta''_{m-1}(t) \quad \text{for } t \in R.$$

This indicates that

$$\alpha_m(t) = \frac{1}{1 + \frac{1}{\alpha_{m-1}(t)\theta_{m-1}(t)}} > \frac{1}{1 + \frac{1}{\alpha_{m-1}(t)}} \quad \text{for } t \in R.$$

It is easy to check that $\alpha_2(t) \in (\frac{1}{2}, 1)$. The desired result follows by induction. □

We now give the last result concerning how to construct some convex functions $\Gamma_w$.

THEOREM 3.9. *Let $\Omega$ be an open convex subset of $R$.*

*(i) Let $\phi : \Omega \to (0, \infty)$ be a convex, twice differentiable, strictly increasing function on $\Omega$. If $\phi(t)\phi''(t) \le [\phi'(t)]^2$ for $t \in \Omega$, then the generalized mean function*

$$\Gamma_w^{(1)}(x) := \phi^{-1}\left(\sum_{i=1}^n \frac{w_i}{\phi(x_i)^r}\right)$$

*is convex on $\Omega^n$ for any given $w \in R^n_{++}$ and $r > 0$.*

*(ii) Let $\kappa > 0$ be a constant and $\phi : \Omega \to (\kappa, \infty)$ be a convex, twice differentiable, strictly increasing function satisfying the inequality $\phi(t)\phi''(t) \le [\phi'(t)]^2$ for $t \in \Omega$. Then, for any given $w \in R^n_{++}$ and $T > 0, r > 0$, the function*

$$\Gamma_w^{(2)}(x) := \overbrace{\ln \circ \ln \circ \cdots \circ \ln}^{\ell}\left(\sum_{i=1}^n w_i\left(T + \frac{1}{\phi(x_i)^r}\right)\right)$$

*is convex on $\Omega^n$ for any positive integer $\ell \le T\kappa^r + 1$.*

*Proof.* Result (i) comes from part (i) of Theorem 2.6 and Theorem 2.4. Result (ii) follows from Lemma 3.8 and Theorem 2.4, and part (ii) of Theorem 2.6. In fact, it suffices to take the inner function $h_T(t) = T + \frac{1}{\phi(t)^r}$ and outer function $\theta_m(t)$, as defined in Lemma 3.8, whose inverse function is given by $\overbrace{\ln \circ \ln \cdots \circ \ln}^{m}(t)$. □

The above result partially answers the following interesting question: *Given a convex function, how many times of* log*-transformations can be applied while retaining the convexity?*

Using Theorems 2.4, 2.6, 3.7 and 3.9, we have the following examples for convex $\Gamma_w$.

*Example* 3.1.

$$(i)\quad \Delta_{1,j}^{-1}\left[\sum_{i=1}^n \frac{1}{\Delta_{1,j}(x_i)^r}\right],\ G_{1,j}^{-1}\left[\sum_{i=1}^n \frac{1}{G_{1,j}(x_i)^r}\right],\ j = 1, 2.$$

$$(ii)\quad \ln\left(\sum_{i=1}^n \frac{1}{\Delta_{1,j}(x_i)^r}\right),\ \ln\left(\sum_{i=1}^n \frac{1}{G_{1,j}(x_i)^r}\right),\ j = 1, 2.$$

$$(iii)\quad \overbrace{\ln \circ \ln \circ \cdots \circ \ln}^{\ell \le m+1}\left(\sum_{i=1}^n \left(m + e^{-rx_i}\right)\right),\ x \in (0, \infty)^n.$$

$$(iv)\quad \overbrace{\ln \circ \ln \circ \cdots \circ \ln}^{\ell}\left[\sum_{i=1}^n \left(m + \frac{1}{\Delta_{1,1}(x_i)^r}\right)\right],\ x \in (\tau, \infty)^n, \ell \le m\Delta_{1,1}(\tau)^r + 1, \tau > 0.$$

It follows from Corollary 3.2 that the function $x^p$ over $(0, \infty)$ satisfies (2.12) with $\alpha = \frac{p}{p-1}$. Hence, when $1 < p \le 2$, we have $\alpha \ge 2$, and when $1 < p \le \frac{29}{17}$, we have

$\alpha \geq \frac{29}{12} \geq \frac{9}{4}$. By Theorems 3.7, both $\Delta_{1,2}(t)$ and $\Delta_{1,1}(t)$ satisfy condition (2.9) with $\alpha = 2$, and both $G_{1,2}(t)$ and $G_{1,1}(t)$ satisfy condition (2.9) with $\alpha = \frac{29}{12}$. According to Theorem 2.4, we see the functions listed below are examples of convex $\Gamma_w$.

*Example* 3.2. (a) Let $1 < p \leq 2$ and $\delta_i(t) = \Delta_{1,2}(t)$ or $\Delta_{1,1}(t)$, for $t \in (0,\infty)$ and $i = 1, ..., n$. Then $\Gamma_w(x) = (\sum_{i=1}^n w_i \delta_i(x_i))^{\frac{1}{p}}$ is convex on $(0,\infty)^n$.

(b) Let $1 < p \leq \frac{29}{17}$ and $\delta_i(t)$ be $G_{1,2}(t)$, $G_{1,1}(t)$, $\Delta_{1,2}(t)$ or $\Delta_{1,1}(t)$, for $t \in (0,\infty)$ and $i = 1, ..., n$. Then $\Gamma_w(x) = (\sum_{i=1}^n w_i \delta_i(x_i))^{\frac{1}{p}}$ is convex on $(0,\infty)^n$.

Taking the outer function $\Psi(t) = G_{1,1}(t)$ or $G_{1,2}$, and the inner function $\phi(t) = \Delta_{1,1}(t)$ or $\Delta_{1,2}(t)$, Theorems 3.7 shows that conditions required by Theorem 2.4 are satisfied. Also, for these functions, we can take $e^t$ as the inner function. Thus, more examples of convex $\Gamma_w$ can be obtained.

Before closing this section, we briefly illustrate a possible application of involving function $\Gamma_w$ in the regularization method for solving a nonlinear programming problem:

$$\min\{f_0(x) : x \in C\}.$$

For simplicity, we assume that $C$ is a convex set and $f_0$ is a convex function. Let $\mu > 0$ be a positive parameter. Given a strictly convex function $\Gamma_w$, we consider the following problem:

$$\min\{f_0(x) + \mu\Gamma_w(x) : x \in C\}.$$

This problem becomes a strictly convex programming problem with a unique solution, denoted by $x(\mu)$, which comprises of a continuation trajectory $\{x(\mu) : \mu > 0\}$. Under suitable conditions of $f_0, \Phi$ and $\phi$, this trajectory becomes bounded. In this case, by setting $\mu \to 0$, any accumulation point of $x(\mu)$, as $\mu \to 0$, is a solution to the original problem. Thus, a path-following algorithm can be designed to follow this trajectory to achieve the solution of the original problem. The performance of such path following algorithm certainly depends on the choice of the function $\Gamma_w$ with regularity conditions.

**4. Conclusions.** In this paper, we have further extended the theoretical foundation for the generalized mean function. We have established a necessary and sufficient condition for such a generalization to be convex. Some useful and easy-to-test sufficient conditions have also been developed to construct concrete examples of convex $\Gamma_w$. Moreover, a systematic way to explicitly construct convex $\Gamma_w$ has been illustrated. To this end, definitions of $S^*$-regular and $T^*$-regular functions have been introduced. The latter is essentially a transformation of the self-regular function proposed in [18]. It should be noted that most $S^*$-functions are not self-concordant, while the class of $T^*$-regular (self-regular) functions has a large overlap with the class of self-concordant functions [16].

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