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An approximation theory of matrix rank minimization and its application to quadratic equations

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ABSTRACT

Matrix rank minimization problems are gaining plenty of recent attention in both mathematical and engineering fields. This class of problems, arising in various and across-discipline applications, is known to be NP-hard in general. In this paper, we aim at providing an approximation theory for the rank minimization problem, and prove that a rank minimization problem can be approximated to any level of accuracy via continuous optimization (especially, linear and nonlinear semidefinite programming) problems. One of the main results in this paper shows that if the feasible set of the problem has a minimum rank element with the least Frobenius norm, then any accumulation point of solutions to the approximation problem, as the approximation parameter tends to zero, is a minimum rank solution of the original problem. The tractability under certain conditions and convex relaxation of the approximation problem are also discussed. An immediate application of this theory to the system of quadratic equations is presented in this paper. It turns out that the condition for such a system without a nonzero solution can be characterized by a rank minimization problem, and thus the proposed approximation theory can be used to establish some sufficient conditions for the system to possess only zero solution.

1. Introduction

Throughout the paper, let R^n be the n-dimensional Euclidean space, $R^{m\times n}$ be the $m \times n$ real matrix space, and S^n be the set of real symmetric matrices. When $X, Y \in R^{m\times n}$, we use $\langle X, Y\rangle = \mathrm{tr}(X^T Y)$ to denote the inner product of X and Y. $\|X\|_2$ and $\|X\|_F$ denote the spectral norm and Frobenius norm of X, respectively, and $\|X\|_*$ stands for the nuclear norm of X (which is the sum of singular values of X). $\|\cdot\|$ denotes a general norm. $A \succeq 0$ ($\succ 0$) means that $A \in S^n$ is positive semidefinite (positive

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definite). Given an $X \in R^{m\times n}$ with rank r, we use $\sigma(X)$ to denote the vector $(\sigma_1(X), \ldots, \sigma_r(X))$ where $\sigma_1(X) \geqslant \cdots \geqslant \sigma_r(X) > 0$ are the singular values of X.

Let $C \subseteq R^{m\times n}$ be a closed set. Consider the rank minimization problem:

$$\text{Minimize } \{\text{rank}(X) : \quad X \in C\}, \tag{1}$$

which has found many applications in system control [14,4,28,27,20,15,16], matrix completion [6,7,37], machine learning [1,26], image reconstruction and distance geometry [23,35,33,30,11], combinatorial and quadratic optimization [2,38], to name but a few. The recent work on compressive sensing (see e.g. [8,9,13]) also stimulates an extensive investigation of this class of problems. In many applications, C is defined by a linear map $\mathcal{A} : R^{m\times n} \to R^p$. Two typical situations are

$$C = \{X \in R^{m\times n} : \mathcal{A}(X) = b\}, \tag{2}$$

$$C = \{X \in S^n : \mathcal{A}(X) = b, \ X \succeq 0\}. \tag{3}$$

Unless C has a very special structure, the problem (1) is difficult to solve due to the discontinuity and nonconvexity of rank(X). It is NP-hard since it includes the cardinality minimization as a special case [29,30]. The existing algorithms for (1) are largely heuristic-based, such as the alternating projection [19,11], alternating LMIs [32], and nuclear norm minimization (see e.g. [15,16,30,25,34,31]. The idea of the nuclear norm heuristic is to replace the objective of (1) by the nuclear norm $\|X\|_*$, and to solve the following convex optimization problem:

$$\text{Minimize } \{\|X\|_* : X \in C\}. \tag{4}$$

Under some conditions, the solution to the nuclear norm heuristic coincides with the minimum rank solution (see e.g. [15,30,31]). This inspires an extensive and fruitful study on various algorithms for solving the nuclear norm minimization problem [15,30,25,18,34,10,3]. While the nuclear norm $\|X\|_*$ is the convex envelope of rank(X) on the unit ball $\{X : \|X\|_2 \leqslant 1\}$ (see [15,30]), it may have a drastic deviation from the rank of X in many cases since rank(X) is a discontinuous nonconvex function. As a result, the true relationship between (1) and (4) are not known in many situations unless some strong assumptions such as the "restricted isometry property" hold [30].

In this paper, we develop a new approximation theory for rank minimization problems. We first provide a continuous approximation for rank(X), by which rank(X) can be approximated to any prescribed accuracy, and can be even computed exactly by a suitable choice of the approximation parameter. Based on this fact, we prove that (1) can be approximated to any level of accuracy by a continuous optimization problem, typically, a structured linear/nonlinear semidefinite programming (SDP) problem. One of our main results shows that when the feasible set is of the form (3), and if it contains a minimum rank element with the least F-norm (i.e. Frobenius norm), then the rank minimization problem can be approximated to any level of accuracy via an SDP problem, which is computationally tractable. A key feature of the proposed approximation approach is that the inter-relationship between (1) and its approximation counterpart can be clearly displayed in many situations. The approximation theory presented in this paper, aided with modern convex optimization techniques, provides a theoretical basis for (and can directly lead to) both new heuristic and exact algorithms for tackling rank minimization problems.

To demonstrate an application of the proposed approximation theory, let us consider the system

$$x^T A_i x = 0, \ i = 1, \ldots, m, \quad x \in R^n, \tag{5}$$

where $A_i \in S^n$, $i = 1, \ldots, m$. A fundamental question associated with (5) is: when is '$x = 0$' the only solution to (5)? The study of this question (e.g. [17,12,5,36,22]) can be dated back to the late 1930s. For $m = 2$ and $n \geqslant 3$, the answer to the question is well-known: 0 *is the only solution to* $x^T A_1 x = 0$, $x^T A_2 x = 0$ *if and only if* $\mu_1 A_1 + \mu_2 A_2 \succ 0$ *for some* $\mu_1, \mu_2 \in R$. However, this result is not valid for $n = 2$, or for $m \geqslant 3$. In fact, the condition

$$\sum_{i=1}^{m} \mu_i A_i \succ 0 \text{ for some } \mu_1, \ldots, \mu_m \in R \tag{6}$$

implies that 0 is the only solution to (5), but the converse is not true in general. When $n = 2$ and/or $m \geqslant 3$, the sufficient condition (6) may be too strong. Thus finding a mild sufficient condition for the system (5) with only zero solution is posted as an open problem in [21]. We first show that the study of this problem can be transformed equivalently as a rank minimization problem, based on which we use the proposed approximation theory, together with the SDP relaxation and duality theory, to establish some general sufficient conditions for the system with only zero solution.

This paper is organized as follows. In Section 2, an approximation function of $\mathrm{rank}(X)$ (and thus an approximation model for the rank minimization problem) is introduced, and some intrinsic properties of this function are shown. In Section 3, reformulations and modifications of the approximation counterpart of the rank minimization problem are discussed, and their proximity to the original problem is also proved. The application of the approximation theory to the system of quadratic equations has been demonstrated in Section 4. Conclusions are given in the last section.

2. Generic approximation of rank minimization

The objective of this section is to provide an approximation theory that can be applied to general rank minimization problems, without involving a specific structure of the feasible set which is only assumed to be a closed set (and bounded when necessary, but not necessarily convex). In order to get an efficient approximation of the problem (1), it is natural to start with a sensible approximation of $\mathrm{rank}(X)$. Let us consider the function $\phi_\varepsilon : R^{m\times n} \to R$ defined by

$$\phi_\varepsilon(X) = \mathrm{tr}\left(X(X^TX + \varepsilon I)^{-1}X^T\right), \quad \varepsilon > 0. \tag{7}$$

The first result below claims that the rank of a matrix can be approximated (in terms of ϕ_ε) to any prescribed accuracy, as long as the parameter ε is suitably chosen.

Theorem 2.1. *Let $X \in R^{m\times n}$ be a matrix with $\mathrm{rank}(X) = r$, and ϕ_ε be defined by (7). Then for every $\varepsilon > 0$,*

$$\phi_\varepsilon(X) = \sum_{i=1}^{r} \frac{(\sigma_i(X))^2}{(\sigma_i(X))^2 + \varepsilon}, \tag{8}$$

where $\sigma_i(X)$'s are the singular values of X, and the following relation holds:

$$0 \leqslant \mathrm{rank}(X) - \phi_\varepsilon(X) = \sum_{i=1}^{r} \frac{\varepsilon}{(\sigma_i(X))^2 + \varepsilon} \leqslant \varepsilon \sum_{i=1}^{r} \frac{1}{(\sigma_i(X))^2} \quad \text{for all } \varepsilon > 0. \tag{9}$$

Proof. Let $X = U\Sigma V^T$ be the full singular value decomposition, where U, V are orthogonal matrices with dimensions m and n, respectively, and the matrix $\Sigma = \begin{pmatrix} \mathrm{diag}(\sigma(X)) & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix}$ where $0_{p\times q}$ denotes the $p \times q$ zero matrix. Let $\sigma^2(X)$ denote the vector $((\sigma_1(X))^2, \ldots, (\sigma_r(X))^2)$. Note that

$$X^TX + \varepsilon I = V(\Sigma^T\Sigma)V^T + \varepsilon I = V\begin{pmatrix} \mathrm{diag}(\sigma^2(X)) + \varepsilon I_r & 0 \\ 0 & \varepsilon I_{n-r} \end{pmatrix} V^T,$$

where I is partitioned into two small identity matrices I_r and I_{n-r}. Thus, we have

$$\begin{aligned}\phi_\varepsilon(X) &= \mathrm{tr}\left(X(X^TX + \varepsilon I)^{-1}X^T\right) \\ &= \mathrm{tr}\left(U\Sigma \begin{pmatrix} \mathrm{diag}(\sigma^2(X)) + \varepsilon I_r & 0 \\ 0 & \varepsilon I_{n-r} \end{pmatrix}^{-1} \Sigma^T U^T\right)\end{aligned}$$

$$= \mathrm{tr}\left(\begin{pmatrix} \mathrm{diag}(\sigma^2(X)) + \varepsilon I_r & 0 \\ 0 & \varepsilon I_{n-r} \end{pmatrix}^{-1} (\Sigma^T \Sigma)\right)$$

$$= \mathrm{tr}\left(\begin{pmatrix} \mathrm{diag}(\sigma^2(X)) + \varepsilon I_r & 0 \\ 0 & \varepsilon I_{n-r} \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{diag}(\sigma^2(X)) & 0 \\ 0 & 0 \end{pmatrix}\right)$$

$$= \sum_{i=1}^{r} \frac{(\sigma_i(X))^2}{(\sigma_i(X))^2 + \varepsilon}.$$

Clearly, $\phi_\varepsilon(X) \leqslant r = \mathrm{rank}(X)$ for all $\varepsilon > 0$. Note that

$$\mathrm{rank}(X) - \phi_\varepsilon(X) = \sum_{i=1}^{r}\left(1 - \frac{(\sigma_i(X))^2}{\sigma_i(X))^2 + \varepsilon}\right) = \sum_{i=1}^{r} \frac{\varepsilon}{(\sigma_i(X))^2 + \varepsilon} \leqslant \sum_{i=1}^{r} \frac{\varepsilon}{(\sigma_i(X))^2}.$$

Thus the inequality (9) holds. □

From the above result we have $\phi_\varepsilon(X) \leqslant \mathrm{rank}(X)$ and $\lim_{\varepsilon\to 0} \phi_\varepsilon(X) = \mathrm{rank}(X)$. So, we immediately have the following corollary.

Corollary 2.2. *For every matrix $X \in R^{m\times n}$, there exists accordingly a number $\varepsilon^* > 0$ such that* $\mathrm{rank}(X) = \lceil \phi_\varepsilon(X) \rceil$ *for all $\varepsilon \in (0, \varepsilon^*]$.*

This suggests the following scheme which requires only a finite number of iterations to find the exact rank of X: *Step 1*. Choose a small number $\varepsilon > 0$; *Step 2*. Evaluate $\phi_\varepsilon(X)$ at X; *Step 3*. Round up the value of $\phi_\varepsilon(X)$ to the nearest integer; *Step 4*. Set $\varepsilon \leftarrow \beta\varepsilon$ where $\beta \in (0, 1)$ is a given constant, and repeat the steps 2–4 above.

The threshold ε^* in Corollary 2.2 depends on X. This can be seen clearly from the right-hand side of (9). However, the next theorem shows that over the optimal solution set of (1) the approximation is uniformed. Before stating this result, we first show that the optimal solution set of (1) is closed. Note that, in general, the set $\{X \in C : \mathrm{rank}(X) = r\}$ is not closed.

Lemma 2.3. *Let C be a closed set in $R^{m\times n}$. Then the level set $\{X \in C : \mathrm{rank}(X) \leqslant r\}$ is closed for any given number $r \geqslant 0$. In particular, the optimal solution set of (1), i.e., $C^* = \{X \in C : \mathrm{rank}(X) = r^*\}$ is closed, where $r^*(= \min\{\mathrm{rank}(X) : X \in C\})$ is the minimum rank.*

Proof. Suppose that $\{X^k\} \subseteq \{X \in C : \mathrm{rank}(X) \leqslant r\}$ is a sequence convergent to X^0 in the sense that $\|X^k - X^0\| \to 0$ as $k \to \infty$. Let $r^0 = \mathrm{rank}(X^0)$ and $\sigma_1(X^0) \geqslant \cdots \geqslant \sigma_{r^0}(X^0) > 0$ be the nonzero singular values of X^0. Note that the singular value is continuously dependent on the entries of the matrix. It implies that for sufficiently large k, X^k has at least r^0 nonzero singular values. Thus $\mathrm{rank}(X^0) \leqslant \mathrm{rank}(X^k) \leqslant r$ for all sufficiently large k. This together with the closedness of C implies that $X^0 \in \{X \in C : \mathrm{rank}(X) \leqslant r\}$, and thus the level set of $\mathrm{rank}(X)$ is closed. Particularly, it implies that the optimal solution set $\{X \in C : \mathrm{rank}(X) = r^*\} = \{X \in C : \mathrm{rank}(X) \leqslant r^*\}$ is closed. □

We now show that the function $\mathrm{rank}(X)$ can be uniformly approximated by $\phi_\varepsilon(X)$ over the optimal solution set of (1), in the sense that the right-hand side of (9) is independent of the choice of X^*.

Theorem 2.4. *If the optimal solution set, denoted by C^*, of (1) is bounded, then there exists a constant $\delta > 0$ such that for any given $\varepsilon > 0$ the inequality*

$$\phi_\varepsilon(X^*) \leqslant \mathrm{rank}(X^*) \leqslant \phi_\varepsilon(X^*) + \varepsilon\left(\frac{\min\{m, n\}}{\delta^2}\right)$$

holds for all $X^* \in C^*$.

Proof. Let r^* be the minimum rank of (1). Then $r^* = \text{rank}(X^*)$ for all $X^* \in C^*$. Let $\sigma_{r^*}(X^*)$ denote the smallest nonzero singular value of X^*, and denote

$$\sigma_{min} = \min\{\sigma_{r^*}(X^*) : X^* \in C^*\}.$$

We now prove that $\sigma_{min} > 0$. Indeed, if $\sigma_{min} = 0$, then there exists a sequence $\{X_k^*\} \subseteq C^*$ such that $\sigma_{r^*}(X_k^*) \to 0$. Since C^* is bounded, passing to a subsequence if necessary we may assume that $X_k^* \to \widehat{X}$. Thus, $\sigma_{r^*}(\widehat{X}) = 0$, which implies that $\text{rank}(\widehat{X}) < r^*$, contradicting to the closedness of C^* (see Lemma 2.3). Therefore, we have $\sigma_{min} > 0$. Let $\delta > 0$ be a constant satisfying $\delta \leqslant \sigma_{min}$. By (9), we have

$$\text{rank}(X^*) - \phi_\varepsilon(X^*) \leqslant \varepsilon \sum_{i=1}^{r^*} \frac{1}{(\sigma_i(X^*))^2} \leqslant \varepsilon \frac{r^*}{(\sigma_{r^*}(X^*))^2} \leqslant \varepsilon \left(\frac{\min\{m, n\}}{\delta^2}\right),$$

as desired. □

It is easy to see from (7) that $\phi_\varepsilon(X)$ is continuous with respect to (X, ε) over the set $R^{m\times n} \times (0, \infty)$. From Theorem 2.1 and Corollary 2.2, we see that the problem (1) can be approximated by a continuous optimization problem with ϕ_ε. In fact, by replacing $\text{rank}(X)$ by $\phi_\varepsilon(X)$, we obtain the following approximation problem of (1):

$$\begin{aligned} &\text{Minimize } \ \phi_\varepsilon(X) = \text{tr}\left(X(X^TX + \varepsilon I)^{-1}X^T\right) \\ &\quad \text{s.t.} \qquad X \in C \end{aligned} \tag{10}$$

where $\varepsilon > 0$ is a given parameter. From an approximation point of view, some natural questions arise: Does the optimal value (solution) of (10) converges to a minimum rank (solution) of (1) as $\varepsilon \to 0$? How can we solve the problem (10) efficiently, and when this problem is computationally tractable? The remainder of this section and the next section are devoted to answering these questions.

For the convenience of the later analysis, we use notation $\phi_0(X) = \text{rank}(X)$. Before we prove the main result of this section, let us first prove the semicontinuity of the function $\phi_\varepsilon(X)$ at the boundary point $\varepsilon = 0$.

Lemma 2.5. *With respect to* (X, ε), *the function* $\phi_\varepsilon(X)$ *is continuous everywhere in the region* $R^{m\times n} \times (0, \infty)$, *and it is lower semicontinuous at* $(X, 0)$, *i.e.,*

$$\liminf_{(Y,\varepsilon)\to(X,0)} \phi_\varepsilon(Y) \geqslant \phi_0(X) = \text{rank}(X).$$

Proof. The continuity of ϕ_ε in $R^{m\times n} \times (0, \infty)$ is obvious. We only need to prove its lower semicontinuity at $(X, 0)$. Let $\widehat{X}$ be an arbitrary matrix in $R^{m\times n}$ with $\text{rank}(\widehat{X}) = r$. Suppose that $X \to \widehat{X}$. Then it is easy to see that

$$(\sigma_1(X), \ldots, \sigma_r(X)) \to (\sigma_1(\widehat{X}), \ldots, \sigma_r(\widehat{X})) > 0, \tag{11}$$

and $\sigma_i(X) \to 0$ for $i \geqslant r+1$. This implies that $\text{rank}(X) \geqslant \text{rank}(\widehat{X})$ as long as X is sufficiently close to $\widehat{X}$. By (8), we have

$$\phi_\varepsilon(X) - \phi_0(\widehat{X}) = \phi_\varepsilon(X) - \text{rank}(\widehat{X}) = \sum_{i\geqslant r+1} \frac{(\sigma_i(X))^2}{(\sigma_i(X))^2+\varepsilon} + \sum_{i=1}^{r}\left(\frac{(\sigma_i(X))^2}{(\sigma_i(X))^2+\varepsilon} - 1\right). \tag{12}$$

It is not difficult to see that when $(X, \varepsilon) \to (\widehat{X}, 0)$, the right-hand side of (12) does not necessarily tend to zero, when $(\sigma_i(X))^2$ in the first term of the right-hand side of (12) tends to zero no faster than that of ε. For instance, let $(\sigma_1(\widehat{X}), \ldots, \sigma_r(\widehat{X})) = (1, \ldots, 1)$, and consider the sequence $X^k \to \widehat{X}$ where X^k satisfies that $\text{rank}(X^k) = p > r$, $(\sigma_1(X^k), \ldots, \sigma_r(X^k)) = (1, \ldots, 1)$ and $(\sigma_{r+1}(X^k), \ldots, \sigma_p(X^k)) = (1/k, \ldots, 1/k)$. Setting $\varepsilon_k = \frac{1}{k^2}$ and substituting (X^k, ε^k) into (12) yields

$$\phi_{\varepsilon_k}(X^k) - \phi_0(\widehat{X}) = \frac{1}{2}(p-r) - \frac{r}{1+k^2} \to \frac{1}{2}(p-r) > 0 \text{ as } k \to \infty.$$

So, $\phi_\varepsilon(X)$ is not necessarily continuous at $\varepsilon = 0$. However, from (12) we see that

$$\phi_\varepsilon(X) - \phi_0(\widehat{X}) \geqslant \sum_{i=1}^{r} \left(\frac{(\sigma_i(X))^2}{(\sigma_i(X))^2 + \varepsilon} - 1 \right),$$

where $r = \operatorname{rank}(\widehat{X})$. By (11), the right-hand side of the above goes to zero as $X \to \widehat{X}$ and $\varepsilon \to 0$. It follows that

$$\liminf_{(X,\varepsilon)\to(\widehat{X},0)} \phi_\varepsilon(X) \geqslant \liminf_{(X,\varepsilon)\to(\widehat{X},0)} \left(\phi_0(\widehat{X}) + \sum_{i=1}^{r} \left(\frac{(\sigma_i(X))^2}{(\sigma_i(X))^2 + \varepsilon} - 1 \right) \right) = \phi_0(\widehat{X}).$$

The proof is complete. $\square$

It is worth mentioning that for $\varepsilon > 0$ the function

$$\phi_\varepsilon(X) = \operatorname{tr}((X^TX + \varepsilon I)^{-1} X^TX) = \operatorname{tr}(I - (X^TX + \varepsilon I)^{-1}) = n - \operatorname{tr}(X^TX + \varepsilon I)^{-1}$$

is differentiable with respect to X, and it is not difficult to obtain its derivative, for instance, following the matrix calculus rules given in [11]. We now prove the main result of this section, which shows that the rank minimization over a bounded feasible set can be approximated with (10) to any level of accuracy.

Theorem 2.6. *Assume that C is a closed set in $R^{m\times n}$ and the optimal value of (10) is attained. Let r^* be the minimum rank of (1) and for given $\varepsilon > 0$, ϕ^*_ε and $X(\varepsilon)$ be the optimal value and an optimal solution of (10), respectively. Then*

$$\phi^*_\varepsilon \leqslant r^* \text{ for any } \varepsilon > 0. \tag{13}$$

Moveover, when C is bounded, then

$$\lim_{\varepsilon\to 0} \phi^*_\varepsilon = r^*, \tag{14}$$

and any accumulation point of $X(\varepsilon)$, as $\varepsilon \to 0$, is a minimum rank solution of (1).

Proof. Since $X(\varepsilon)$ is an optimal solution to (10), we have

$$\phi^*_\varepsilon = \phi_\varepsilon(X(\varepsilon)) \leqslant \phi_\varepsilon(X) \quad \text{for all } X \in C.$$

Particularly, any optimal solution of (1) satisfies the above inequality. So, $\phi^*_\varepsilon \leqslant \phi_\varepsilon(X^*) \leqslant \operatorname{rank}(X^*) = r^*$ where the second inequality follows from (9), and ε can be any positive number. Thus (13) holds, and

$$\limsup_{\varepsilon\to 0} \phi^*_\varepsilon \leqslant r^*. \tag{15}$$

On the other hand, since $\phi^*_\varepsilon \geqslant 0$, the number $\underline{r} = \liminf_{\varepsilon\to 0} \phi^*_\varepsilon$ is finite. Without loss of generality, assume that the sequence $\{\phi^*_{\varepsilon_k}\}$, where $\varepsilon_k \to 0$ as $k \to \infty$, converges to $\underline{r}$. Note that $X(\varepsilon_k)$ is a minimizer of (10) with $\varepsilon = \varepsilon_k$, i.e., $\phi^*_{\varepsilon_k} = \phi_{\varepsilon_k}(X(\varepsilon_k))$. When C is bounded, the sequence $\{X(\varepsilon_k)\}$ is bounded. Passing to a subsequence if necessary, we assume that $X(\varepsilon_k) \to X_0$ as $k \to \infty$. Clearly, $X_0 \in C$ since C is closed, and hence $\operatorname{rank}(X_0) \geqslant r^*$. Therefore,

$$\underline{r} = \lim_{k\to\infty} \phi^*_{\varepsilon_k} = \lim_{k\to\infty} \phi_{\varepsilon_k}(X_{\varepsilon_k}) \geqslant \liminf_{(X,\varepsilon)\to(X_0,0)} \phi_\varepsilon(X) \geqslant \phi_0(X_0),$$

where the last inequality follows from Lemma 2.5. Thus, $\underline{r} \geqslant \phi_0(X_0) = \operatorname{rank}(X_0) \geqslant r^*$, which together with (15) implies (14).

We now prove that any accumulation point of $X(\varepsilon)$ is a minimum rank solution of (1). Let $\widehat{X}$ (with $\operatorname{rank}(\widehat{X}) = \widehat{r}$) be an arbitrary accumulation point of $X(\varepsilon)$, as $\varepsilon \to 0$. We now prove that $\widehat{X}$ is a minimum rank solution to (1), i.e., $\widehat{r} = r^*$. Consider a convergent sequence $X(\varepsilon_k) \to \widehat{X}$ where $\varepsilon_k \to 0$. Then by (13) and (8), we have

$$r^* \geqslant \phi^*_{\varepsilon_k} = \phi_{\varepsilon_k}(X(\varepsilon_k)) = \sum_{i=1}^{\widehat{r}} \frac{(\sigma_i(X(\varepsilon_k)))^2}{(\sigma_i(X(\varepsilon_k)))^2 + \varepsilon_k} + \sum_{i>\widehat{r}} \frac{(\sigma_i(X(\varepsilon_k)))^2}{(\sigma_i(X(\varepsilon_k)))^2 + \varepsilon_k}$$
$$\geqslant \sum_{i=1}^{\widehat{r}} \frac{(\sigma_i(X(\varepsilon_k)))^2}{(\sigma_i(X(\varepsilon_k)))^2 + \varepsilon_k} \to \sum_{i=1}^{\widehat{r}} \frac{(\sigma_i(\widehat{X}))^2}{(\sigma_i(\widehat{X}))^2 + 0} = \operatorname{rank}(\widehat{X}).$$

Thus any accumulation point of $X(\varepsilon)$ is a minimum rank solution to (1). $\square$

Since r^* is integer, by (13) and (14), we immediately have the following corollary.

Corollary 2.7. *Let r^* and ϕ^*_ε be defined as in Theorem 2.6. If $C \subset R^{m\times n}$ is bounded and closed, then there exists a number $\delta > 0$ such that $r^* = \lceil \phi^*_\varepsilon \rceil$ for all $\varepsilon \in (0, \delta]$.*

The results above provide a theoretical basis for developing new approximation algorithms for rank minimization problems. Such an algorithm can be a heuristic method for general rank minimization, and can be an exact method as indicated by Corollary 2.7. From Theorem 2.6, the set $\{X(\varepsilon)\}$ can be viewed as a trajectory leading to the minimum rank solution set of (1), and thus it is possible to construct a continuation type method (e.g. a path-following method) for rank minimization problems. In the next section, we are going to discuss how and when the approximation problem (10) can be efficiently dealt with from the viewpoint of computation. We prove that under some conditions problem (10) can be either reformulated or relaxed as a tractable optimization problem, typically an SDP problem.

3. Reformulation of the approximation problem (10)

The main result in last section shows that if ε is small enough, the optimal value of the rank minimization problem can be obtained precisely by solving (10) just once, and the solution $X(\varepsilon)$ of (10) is an approximation to the optimal solution of (1). If the problem (10) with a prescribed $\varepsilon > 0$ fails to generate the minimum value of (1), we can reduce the value of ε and solve (10) again. By Corollary 2.7, the minimum rank of (1) can be obtained by solving (10) up to a finite number of times. Thus, roughly speaking, solving a rank minimization problem amounts to solving a continuous optimization problem defined by (10).

In this section, we concentrate on the problem (10) to find out when and how it can be solved efficiently. To this end, we investigate its equivalent formulations together with some useful variants. By doing so, we take into account the structure of C when necessary. Let us start with the reformulation of (10).

Introducing a variable $Y \in S^m$, we first note that (10) can be written as the following nonlinear semidefinite programming problem:

$$\text{Minimize} \left\{ \operatorname{tr}(Y) : \; Y \succeq X(X^TX + \varepsilon I)^{-1}X^T, \; X \in C \right\}. \tag{16}$$

It is easy to see that if (Y^*, X^*) is an optimal solution to (16), then

$$Y^* = X^*((X^*)^TX^* + \varepsilon I)^{-1}(X^*)^T. \tag{17}$$

Thus, we conclude that X^* is an optimal solution to (10) if and only if (Y^*, X^*) is an optimal solution to (16) where Y^* is given by (17). By Schur complement theorem, the problem (16) can be further written as

$$\text{Minimize} \left\{ \operatorname{tr}(Y) : \; \begin{pmatrix} Y & X \\ X^T & X^TX + \varepsilon I \end{pmatrix} \succeq 0, \quad X \in C \right\}, \tag{18}$$

which remains a nonlinear SDP problem. We now introduce the variable $Z = X^TX$, which implies that Z is the optimal solution to the problem $\min_Z\{\operatorname{tr}(Z) : Z \succeq X^TX\}$. By Schur complement theorem again,

$Z \succeq X^T X$ is nothing but $\begin{pmatrix} I & X \\ X^T & Z \end{pmatrix} \succeq 0$. So the problem (10) can be written exactly as a bilevel SDP problem:

$$\begin{aligned} \min_{Y \in S^m, \widetilde{Z} \in S^n, X \in C} \quad & \operatorname{tr}(Y) \\ \text{s.t.} \quad & \begin{pmatrix} Y & X \\ X^T & \widetilde{Z} + \varepsilon I \end{pmatrix} \succeq 0, \\ & \widetilde{Z} = \arg\min_{Z \in S^n} \left\{ \operatorname{tr}(Z) : \begin{pmatrix} I & X \\ X^T & Z \end{pmatrix} \succeq 0 \right\}. \end{aligned} \tag{19}$$

From the discussion above, we conclude that (10) is equivalent to the nonlinear SDP problem (18), and is equivalent to the *linear bilevel SDP problem* (19). As a result, by Theorem 2.6, *the rank minimization over a bounded feasible set is equivalent to the linear bilevel SDP problem of the form (19).* Thus, the level of difficulty for rank minimization can be understood from the perspective of its linear bilevel SDP counterpart. It is worth mentioning that the bilevel programming (in vector form) has been long studied (e.g. [24]), but to our knowledge the bilevel SDP problem remains a new topic so far. The analysis above shows that a bilevel SDP model does arise from rank minimization. However, both (18) and (19) are not convex problems, and hence they are not computationally tractable in general.

This motivates us to consider the next approximation model which can be viewed as a variant of (10). The difficulty of (18) and (19) lies in the hard equality $Z = X^T X$. An immediate idea is to relax it to $Z \succeq X^T X$, yielding the problem:

$$\text{Minimize} \left\{ \operatorname{tr}(Y) : \begin{pmatrix} Y & X \\ X^T & Z + \varepsilon I \end{pmatrix} \succeq 0, \begin{pmatrix} I & X \\ X^T & Z \end{pmatrix} \succeq 0, \quad X \in C \right\}$$

which is a convex problem if C is convex, and an SDP problem if C is defined by (2) or (3). However, for any given $X \in C$ and any number $\beta > 0$, the point $(Y = \beta I, Z = \alpha I)$ is feasible to the above problem when $\alpha > 0$ is sufficiently large. So the optimal value of the above problem is always zero, providing nothing about the minimum rank of the original problem (1). This happens since Z gains too much freedom while $Z = X^T X$ is relaxed to $Z \succeq X^T X$. Thus the value $\operatorname{tr}(Y) = \operatorname{tr}(X(Z + \varepsilon I)^{-1} X^T)$ may significantly deviate from $\phi_\varepsilon(X)(\approx \operatorname{rank}(X))$. To avoid this, some driving force should be imposed on Z so that it is near (or equal) to $X^T X$.

Motivated by this observation, we consider the following problem in which a 'penalty' term is introduced into the objective:

$$\begin{aligned} \text{Minimize} \quad & \operatorname{tr}(Y) + \frac{1}{\gamma}\operatorname{tr}(Z) \\ \text{s.t.} \quad & \begin{pmatrix} Y & X \\ X^T & Z + \varepsilon I \end{pmatrix} \succeq 0, \begin{pmatrix} I & X \\ X^T & Z \end{pmatrix} \succeq 0, \quad X \in C, \end{aligned} \tag{20}$$

where γ is a positive number. The term $\frac{1}{\gamma}\operatorname{tr}(Z)$ acts as a penalty when $Z(\succeq X^T X)$ is deviated away from $X^T X$. Since $\operatorname{tr}(Z) \geqslant \operatorname{tr}(X^T X) = \|X\|_F^2$, this term also drives $\|X\|_F$ to be minimized. Note that when Z is driven near to $X^T X$, it is the first term $\operatorname{tr}(Y)$ of the objective that approximates the rank of X, and returns the approximate value of $\operatorname{rank}(X)$. The advantage of the approximation model (20) is that it is an SDP problem when C is defined by linear constraints (such as (2) or (3)), and hence it is computationally tractable. In what follows, we concentrate on this model and prove that under some conditions the rank minimization can be approximated by (20) to any level of accuracy.

Theorem 3.1. *Let C be a bounded, closed set in $R^{m\times n}$. Suppose that C contains a minimum rank element X^* with the least F-norm, i.e.,* $\mathrm{rank}(X^*) \leqslant \mathrm{rank}(X),\ \|X^*\|_F \leqslant \|X\|_F$ *for all $X \in C$. Let $(Y_{\varepsilon,\gamma}, Z_{\varepsilon,\gamma}, X_{\varepsilon,\gamma})$ denote the optimal solution of the problem (20). Then* $\mathrm{tr}(Y_{\varepsilon,\gamma}) \leqslant \phi_\varepsilon(X^*) \leqslant \mathrm{rank}(X^*)$ *for all $(\varepsilon, \gamma) > 0$ and*

$$\lim_{(\varepsilon,\gamma)\to 0,\, \frac{\gamma}{\varepsilon}\to 0} tr(Y_{\varepsilon,\gamma}) = r^* = \mathrm{rank}(X^*), \quad \lim_{(\varepsilon,\gamma)\to 0,\, \frac{\gamma}{\varepsilon}\to 0} tr(Z_{\varepsilon,\gamma}) = \|X^*\|_F^2,$$

and any accumulation point of the sequence $\{X_{\varepsilon,\gamma}\}$ is a minimum rank solution of (1), as $(\varepsilon, \gamma) \to 0$ and $\frac{\gamma}{\varepsilon} \to 0$.

Proof. Since C is bounded and closed, the sequence $\{X_{\varepsilon,\gamma}\}$ has at least one accumulation point, and any such an accumulation point is in C. Let X^0 be an arbitrary accumulation point of the sequence $\{X_{\varepsilon,\gamma}\}$ as $(\varepsilon, \gamma) \to 0$ and $\frac{\gamma}{\varepsilon} \to 0$. Without loss of generality, we assume that $X_{\varepsilon,\gamma} \to X^0$, as $(\varepsilon, \gamma) \to 0$ and $\frac{\gamma}{\varepsilon} \to 0$. By Schur complement and the structure of the problem (20), it is easy to see that for any given $\varepsilon, \gamma > 0$ the optimal solution $(Y_{\varepsilon,\gamma}, Z_{\varepsilon,\gamma}, X_{\varepsilon,\gamma})$ of (20) satisfies the following relation

$$Y_{\varepsilon,\gamma} = X_{\varepsilon,\gamma}(Z_{\varepsilon,\gamma} + \varepsilon I)^{-1}X_{\varepsilon,\gamma}^T, \quad Z_{\varepsilon,\gamma} \succeq X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma}. \tag{21}$$

Let X^* be an arbitrary minimum rank solution of (1) with the least F-norm. Then the point $(Y_\varepsilon^*, Z^*, X^*)$, where $Y_\varepsilon^* = X^*((X^*)^T X^* + \varepsilon)^{-1}(X^*)^T$ and $Z^* = (X^*)^T X^*$, is feasible to the problem (20). By optimality, we have

$$\mathrm{tr}(Y_{\varepsilon,\gamma}) + \frac{1}{\gamma}\mathrm{tr}(Z_{\varepsilon,\gamma}) \leqslant \mathrm{tr}(Y_\varepsilon^*) + \frac{1}{\gamma}\mathrm{tr}(Z^*) = \phi_\varepsilon(X^*) + \frac{1}{\gamma}\|X^*\|_F^2. \tag{22}$$

It follows from (21) that

$$\mathrm{tr}(Z_{\varepsilon,\gamma}) \geqslant \mathrm{tr}(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma}) = \|X_{\varepsilon,\gamma}\|_F^2 \geqslant \|X^*\|_F^2 \text{ for all } \varepsilon, \gamma > 0. \tag{23}$$

Combining (22) and (23) yields

$$\mathrm{tr}(Y_{\varepsilon,\gamma}) \leqslant \phi_\varepsilon(X^*), \tag{24}$$

$$0 \leqslant \mathrm{tr}(Z_{\varepsilon,\gamma}) - \|X^*\|_F^2 \leqslant \gamma(\phi_\varepsilon(X^*) - \mathrm{tr}(Y_{\varepsilon,\gamma})) \leqslant \gamma\phi_\varepsilon(X^*) \leqslant \gamma \min\{m, n\}, \tag{25}$$

for all $(\varepsilon, \gamma) > 0$. The last inequality of (25) follows from $\phi_\varepsilon(X^*) \leqslant \mathrm{rank}(X^*) \leqslant \min\{m, n\}$. Let $X_{\varepsilon,\gamma} = X^0 + \Delta_{\varepsilon,\gamma}$ where $\Delta_{\varepsilon,\gamma} \to 0$ since $X_{\varepsilon,\gamma} \to X^0$. Then

$$X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} = (X^0)^T X^0 + (\Delta_{\varepsilon,\gamma}^T X^0 + (X^0)^T \Delta_{\varepsilon,\gamma} + \Delta_{\varepsilon,\gamma}^T \Delta_{\varepsilon,\gamma}) = (X^0)^T X^0 + G(\Delta_{\varepsilon,\gamma}),$$

where $G(\Delta_{\varepsilon,\gamma}) = \Delta_{\varepsilon,\gamma}^T X^0 + (X^0)^T \Delta_{\varepsilon,\gamma} + \Delta_{\varepsilon,\gamma}^T \Delta_{\varepsilon,\gamma}$. Thus by (21) we have

$$Z_{\varepsilon,\gamma} \succeq X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} = (X^0)^T X^0 + G(\Delta_{\varepsilon,\gamma}). \tag{26}$$

Note that $\mathrm{tr}(G(\Delta_{\varepsilon,\gamma})) \to 0$ as $\Delta_{\varepsilon,\gamma} \to 0$. By (25) and (26), we have

$$\|X^*\|_F^2 = \lim_{(\varepsilon,\gamma)\to 0,\gamma/\varepsilon\to 0} \mathrm{tr}(Z_{\varepsilon,\gamma}) \geqslant \lim_{(\varepsilon,\gamma)\to 0,\gamma/\varepsilon\to 0} \mathrm{tr}((X^0)^T X^0 + G(\Delta_{\varepsilon,\gamma})) = \|X^0\|_F^2.$$

Thus, X^0 is a least F-norm element in C. On the other hand, from (26), we see that $\widehat{\Delta}_{\varepsilon,\gamma} := Z_{\varepsilon,\gamma} - X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} \succeq 0$. Thus, by (26) and (25) again, we have

$$\|\widehat{\Delta}_{\varepsilon,\gamma}\|_2 \leqslant \mathrm{tr}(\widehat{\Delta}_{\varepsilon,\gamma}) = \mathrm{tr}(Z_{\varepsilon,\gamma}) - \|X_{\varepsilon,\gamma}\|_F^2 \leqslant \mathrm{tr}(Z_{\varepsilon,\gamma}) - \|X^*\|_F^2 \leqslant \gamma \min\{m, n\}.$$

The first inequality above follows from the fact $\widehat{\Delta}_{\varepsilon,\gamma} \succeq 0$, and the second follows from $\|X_{\varepsilon,\gamma}\|_F \geqslant \|X^*\|_F$. Therefore,

$$\|\widehat{\Delta}_{\varepsilon,\gamma}\|_2/\varepsilon \to 0, \text{ as } (\varepsilon, \gamma) \to 0 \text{ and } \gamma/\varepsilon \to 0. \tag{27}$$

When $M \in R^{n\times n}$ and $\|M\|_2 < 1$, it is well-known that $(I+M)^{-1} = I - M + M^2 - M^3 + \cdots = I + \sum_{i=1}^{\infty}(-1)^i M^i$. Thus, for any $U, V \in R^{n\times n}$ where U is nonsingular, if $\|VU^{-1}\|_2 < 1$ we have

$$(U+V)^{-1} = U^{-1}(I+VU^{-1})^{-1} = U^{-1} + U^{-1}\left(\sum_{i=1}^{\infty}(-1)^i (VU^{-1})^i\right). \tag{28}$$

As $(\varepsilon, \gamma) \to 0$ and $\gamma/\varepsilon \to 0$, it follows from (27) that

$$\|\widehat{\Delta}_{\varepsilon,\gamma}(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\|_2 \leqslant \|\widehat{\Delta}_{\varepsilon,\gamma}\|_2 \|(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\|_2 \leqslant \|\widehat{\Delta}_{\varepsilon,\gamma}\|_2/\varepsilon \to 0.$$

Thus, substituting $U = X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I$ and $V = \widehat{\Delta}_{\varepsilon,\gamma}$ into (28) yields

$$\begin{aligned}
&(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \widehat{\Delta}_{\varepsilon,\gamma} + \varepsilon I)^{-1} - (X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1} \\
&\quad = (X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\left(\sum_{i=1}^{\infty}(-1)^i\left(\widehat{\Delta}_{\varepsilon,\gamma}(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\right)^i\right).
\end{aligned}$$

Note that $\|(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\|_2 \leqslant 1/\varepsilon$ and $\left\|\left(I - \varepsilon(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\right)\right\|_2 \leqslant 1$. When $\left\|\widehat{\Delta}_{\varepsilon,\gamma}\right\|_2/\varepsilon < 1$, we have

$$\begin{aligned}
&\left\|X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma}\left((X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \widehat{\Delta}_{\varepsilon,\gamma} + \varepsilon I)^{-1} - (X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\right)\right\|_2 \\
&\quad = \left\|\left[(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I) - \varepsilon I\right]\left((X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \widehat{\Delta}_{\varepsilon,\gamma} + \varepsilon I)^{-1} - (X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\right)\right\|_2 \\
&\quad = \left\|\left(I - \varepsilon(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\right)\left(\sum_{i=1}^{\infty}(-1)^i\left(\widehat{\Delta}_{\varepsilon,\gamma}(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\right)^i\right)\right\|_2 \\
&\quad \leqslant \left\|\left(I - \varepsilon(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\right)\right\|_2 \left\|\sum_{i=1}^{\infty}(-1)^i\left(\widehat{\Delta}_{\varepsilon,\gamma}(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\right)^i\right\|_2 \\
&\quad \leqslant \left\|\sum_{i=1}^{\infty}(-1)^i\left(\widehat{\Delta}_{\varepsilon,\gamma}(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\right)^i\right\|_2 \\
&\quad \leqslant \sum_{i=1}^{\infty}\left\|\widehat{\Delta}_{\varepsilon,\gamma}\right\|_2^i \left\|(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}\right\|_2^i \leqslant \sum_{i=1}^{\infty}\left(\|\widehat{\Delta}_{\varepsilon,\gamma}\|_2/\varepsilon\right)^i = \frac{\left(\|\widehat{\Delta}_{\varepsilon,\gamma}\|_2/\varepsilon\right)}{1 - \left(\|\widehat{\Delta}_{\varepsilon,\gamma}\|_2/\varepsilon\right)}.
\end{aligned}$$

Thus, by (27), we have

$$\operatorname{tr}\left(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma}\left[\left(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \widehat{\Delta}_{\varepsilon,\gamma} + \varepsilon I\right)^{-1} - \left(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I\right)^{-1}\right]\right) \to 0 \tag{29}$$

as $(\varepsilon, \gamma) \to 0$ and $\gamma/\varepsilon \to 0$. By (24), (9) and (21), we have

$$\begin{aligned}
\operatorname{rank}(X^*) &\geqslant \phi_\varepsilon(X^*) \geqslant \operatorname{tr}(Y_{\varepsilon,\gamma}) = \operatorname{tr}\left(X_{\varepsilon,\gamma}\left(Z_{\varepsilon,\gamma} + \varepsilon I\right)^{-1} X_{\varepsilon,\gamma}^T\right) \\
&= \operatorname{tr}\left(X_{\varepsilon,\gamma}\left(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \widehat{\Delta}_{\varepsilon,\gamma} + \varepsilon I\right)^{-1} X_{\varepsilon,\gamma}^T\right) \\
&= \operatorname{tr}\left(X_{\varepsilon,\gamma}\left[\left(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \widehat{\Delta}_{\varepsilon,\gamma} + \varepsilon I\right)^{-1} - \left(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I\right)^{-1}\right] X_{\varepsilon,\gamma}^T\right) \\
&\quad + \operatorname{tr}\left(X_{\varepsilon,\gamma}\left(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I\right)^{-1} X_{\varepsilon,\gamma}^T\right) \\
&= \operatorname{tr}\left(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma}\left[\left(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \widehat{\Delta}_{\varepsilon,\gamma} + \varepsilon I\right)^{-1} - \left(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I\right)^{-1}\right]\right) + \phi_\varepsilon(X_{\varepsilon,\gamma}).
\end{aligned}$$

which together with (29) and Lemma 2.5 implies that

$$\begin{aligned}\operatorname{rank}(X^*) &\geqslant \limsup_{\varepsilon,\gamma\to 0,\,\frac{\gamma}{\varepsilon}\to 0} \operatorname{tr}(Y_{\varepsilon,\gamma}) \geqslant \liminf_{\varepsilon,\gamma\to 0,\,\frac{\gamma}{\varepsilon}\to 0} \operatorname{tr}(Y_{\varepsilon,\gamma}) \\ &= \liminf_{\varepsilon,\gamma\to 0,\,\frac{\gamma}{\varepsilon}\to 0} \{\operatorname{tr}(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma}[(X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \widehat{\Delta}_{\varepsilon,\gamma} + \varepsilon I)^{-1} - (X_{\varepsilon,\gamma}^T X_{\varepsilon,\gamma} + \varepsilon I)^{-1}]) \\ &\quad + \phi_\varepsilon(X_{\varepsilon,\gamma})\} \\ &= \liminf_{\varepsilon,\gamma\to 0,\,\frac{\gamma}{\varepsilon}\to 0} \phi_\varepsilon(X_{\varepsilon,\gamma}) \geqslant \Phi_0(X^0) = \operatorname{rank}(X^0).\end{aligned}$$

Since X^* is a minimum rank solution, all inequalities above must be equalities, and thus X^0 is a minimum rank solution, and $\lim_{\varepsilon,\gamma\to 0,\,\frac{\gamma}{\varepsilon}\to 0} \operatorname{tr}(Y_{\varepsilon,\gamma}) = \operatorname{rank}(X^*)$. $\square$

By Theorem 3.1, we may simply set $\gamma = \gamma(\varepsilon)$ as a function of ε, for instance, $\gamma = \varepsilon^p$ where $p > 1$ is a constant. Then (20) becomes the problem below:

$$\begin{aligned}&\text{Minimize } \operatorname{tr}(Y) + \frac{1}{\gamma(\varepsilon)}\operatorname{tr}(Z) \\ &\text{s.t.} \quad \begin{pmatrix} Y & X \\ X^T & Z + \varepsilon I \end{pmatrix} \succeq 0, \begin{pmatrix} I & X \\ X^T & Z \end{pmatrix} \succeq 0, \quad X \in C,\end{aligned} \tag{30}$$

which includes only one parameter. An immediate corollary from Theorem 3.1 is given as follows, which shows that the minimum rank of (1) can be obtained exactly by solving (30) with a suitable chosen parameter ε.

Corollary 3.2. *Let $C \subset R^{m\times n}$ be a bounded and closed set, containing an element X^* with the minimum rank $r^* = \operatorname{rank}(X^*)$ and the least F-norm. Let $\gamma : (0, \infty) \to (0, \infty)$ be a function satisfying $\gamma(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$. If $(Y_\varepsilon, Z_\varepsilon, X_\varepsilon)$ is the optimal solution of (30), then $tr(Y_\varepsilon) \leqslant r^*$ for all ε, and*

$$\lim_{\varepsilon\to 0} \operatorname{tr}(Y_\varepsilon) = r^*, \quad \lim_{\varepsilon\to 0} \operatorname{tr}(Z_\varepsilon) = \|X^*\|_F^2,$$

and any accumulation point of the sequence $\{X_\varepsilon\}$ is a minimum rank solution of (1). Moreover, there exists a threshold $\delta > 0$ such that $r^* = \lceil \operatorname{tr}(Y_\varepsilon) \rceil$ for every $\varepsilon \in (0, \delta]$.

From the above results, we see that a rank minimization problem can be tractable under some conditions. We summarize this result as follows.

Corollary 3.3. *When C is defined by linear constraints (such as (2) and (3)), and if C contains a minimum rank element with the least F-norm, the rank minimization problem (1) is equivalent to the SDP problem (20) by a suitable choice of the parameter (η, ε).*

Note that the first term of the objective of (20) is to estimate $\operatorname{rank}(X)$ and the second term is to measure the least F-norm. So from Theorem 3.1 we may roughly say that under some conditions minimizing $\operatorname{rank}(X)$ over C is equivalent to minimizing $\operatorname{rank}(X) + \beta\|X\|_F^2$ over C for some β. This is true, as shown by the next result below.

Theorem 3.4. *Let the feasible set be of the form $C = \mathcal{F} \cap \{X : \gamma_1 \leqslant \|X\|_F \leqslant \gamma_2\}$, where $0 < \gamma_1 \leqslant \gamma_2$ are constants and $\mathcal{F} \subseteq R^{m\times n}$ is a closed set.*

(i) The following two problems are equivalent in the sense that they yield the same minimum rank solution:

$$\text{Minimize } \{\operatorname{rank}(X) : X \in C = \mathcal{F} \cap \{X : \gamma_1 \leqslant \|X\|_F \leqslant \gamma_2\}\}, \tag{31}$$

$$\text{Minimize}\left\{\text{rank}(X)+\frac{1}{\eta}\|X\|_F^2 : X\in C=\mathcal{F}\cap\{X:\gamma_1\leqslant\|X\|_F\leqslant\gamma_2\}\right\}, \tag{32}$$

where $\eta>\gamma_2-\gamma_1$ is a given number

(ii) If $\mathcal{F}$ is a cone, then the set C contains a minimum rank matrix X^* with the least F-norm.

Proof. (i) Assume that X^* is a minimizer of (31) with the minimum rank r^*, and assume that $\widetilde{X}$ is an arbitrary minimizer of the problem (32). We show that $\text{rank}(\widetilde{X})=r^*$. In fact, if this is not true, then $\text{rank}(\widetilde{X})\geqslant r^*+1$, and thus

$$\begin{aligned}&\text{rank}(\widetilde{X})+(1/\eta)\|\widetilde{X}\|_F^2\\ &\quad\geqslant \text{rank}(X^*)+1+(1/\eta)\|\widetilde{X}\|_F^2\\ &\quad= \text{rank}(X^*)+(1/\eta)\|X^*\|_F^2+1+(1/\eta)\left(\|\widetilde{X}\|_F^2-\|X^*\|_F^2\right)\\ &\quad> \text{rank}(X^*)+(1/\eta)\|X^*\|_F^2,\end{aligned} \tag{33}$$

where the last inequality above follows from the fact $X^*,\widetilde{X}\in\{X:\gamma_1\leqslant\|X\|_F\leqslant\gamma_2\}$ which implies that $1+(1/\eta)\left(\|\widetilde{X}\|_F^2-\|X^*\|_F^2\right)\geqslant 1+(1/\eta)(\gamma_1-\gamma_2)>0$ by the choice of η. Thus, (33) contradicts to the fact of $\widetilde{X}$ being a minimizer of (32).

(ii) Suppose that $\mathcal{F}$ is cone. Consider the F-norm minimization problem:

$$\text{Minimize } \{\|X\|_F^2 : X\in C=\mathcal{F}\cap\{X:\gamma_1\leqslant\|X\|_F\leqslant\gamma_2\}\}.$$

Since the feasible set of the problem is closed and bounded, the least F-norm solution, denoted by $\overline{X}$, exists. Let X^* be a minimum rank element in C. Then $\|X^*\|_F\geqslant\|\overline{X}\|_F\geqslant\gamma_1>0$. Thus, there is a positive number $1\geqslant\alpha>0$ such that $\alpha\|X^*\|_F=\|\overline{X}\|_F$. Note that $\alpha X^*\in\mathcal{F}$ (since $\mathcal{F}$ is a cone), and that $\text{rank}(\alpha X^*)=\text{rank}(X^*)$. Thus, αX^* is a minimum rank matrix with the least F-norm in C. □

Before we close this section, let us make some further comments on the situation where C is the intersection of a cone and a bounded set defined by matrix norm, as discussed in Theorem 3.4. This situation does arise in the study of quadratic (in)equality systems and quadratic optimization. First of all, it is worth pointing out the following fact. Its proof is evident and omitted.

Theorem 3.5. *Let $\mathcal{F}$ be a cone in $R^{m\times n}$, and let $0<\gamma_1\leqslant\gamma_2$ be two positive numbers. Then the minimum rank r^* of the rank minimization problem*

$$r^*=\min\{\text{rank}(X):\ X\in C=\mathcal{F}\cap\{X:\gamma_1\leqslant\|X\|\leqslant\gamma_2\}\} \tag{34}$$

is independent of the choice of γ_1,γ_2 and the norm $\|\cdot\|$.

In another word, no matter what matrix norms and the positive numbers γ_1,γ_2 are used, the problem of the form (34) yields the same minimum rank. So, in theory, all these rank minimization problems are equivalent. From a computation point of view, however, the choice of the norm $\|\cdot\|$ does matter. For instance, when $\mathcal{F}$ is a subset of the positive semidefinite cone, there are some benefits of using the nuclear norm $\|X\|_*$ in (34). Since $\|X\|_*=\text{tr}(X)$ in positive semidefinite cone, the constraint $\gamma_1\leqslant\|X\|_*\leqslant\gamma_2$ in this case coincides with the linear constraint $\gamma_1\leqslant\text{tr}(X)\leqslant\gamma_2$. As a result, the approximation counterpart, defined by (20), of the problem (34) is an SDP problem for this case, and hence it can be solved efficiently. However, when the nuclear norm is used in (34), the problem (34) may not satisfy the condition of Theorem 3.1.

When C is defined by a cone, from Theorem 3.4(ii) the problem (34) satisfies the condition of Theorem 3.1. However, when the F-norm is used, the problem (20) is not convex in general. To handle this nonconvexity, we may consider the relaxation of (34). For instance, when $\mathcal{F}$ in (34) is a cone contained in the positive semidefinite cone, we define

$$\begin{cases} \delta_1 = \min\{\mathrm{tr}(X) : \gamma_1 \leqslant \|X\|_F \leqslant \gamma_2,\ X \succeq 0\}, \\ \delta_2 = \max\{\mathrm{tr}(X) : \gamma_1 \leqslant \|X\|_F \leqslant \gamma_2,\ X \succeq 0\} \end{cases} \tag{35}$$

where $\gamma_1 > 0$. Clearly, δ_1 and δ_2 exist and are positive. Thus the problem (34) is relaxed to

$$l^* = \min\{\mathrm{rank}(X) : \quad X \in C = \mathcal{F} \cap \{X : \delta_1 \leqslant \mathrm{tr}(X) \leqslant \delta_2\}\}.$$

When $\mathcal{F}$ is defined by linear constraints, the approximation counterpart (20) of this relaxation problem is an SDP problem. Denote the optimal solution of this SDP problem by $(Y_{\varepsilon,\eta}, Z_{\varepsilon,\eta}, X_{\varepsilon,\eta})$. Then by Theorem 3.1 it provides a lower bound for the minimum rank of the above relaxation problem, and hence a lower bound for the minimum rank of the original problem (34), i.e., $tr(Y_{\varepsilon,\eta}) \leqslant l^* \leqslant r^*$.

4. Application to the system of quadratic equations

Given a finite number of matrices $A_i \in S^n$, $i = 1, \ldots, m$, we consider the development of sufficient conditions for the following assertion:

$$x^T A_i x = 0, \quad i = 1, \ldots, m \Longrightarrow x = 0, \tag{36}$$

i.e., 0 is the only solution to (5). At the first glance, it seems that (5) and (36) have nothing to do with a rank minimization problem. In this section, however, we show that (4.1) can be equivalently formulated as a rank minimization problem, based on which we may derive some sufficient conditions for (36) by applying the approximation theory developed in previous sections. Note that system (5) can be written as $\langle A_i, xx^T\rangle = 0,\ i = 1, \ldots, m$. Since $X = xx^T$ is either 0 (when $x = 0$) or a positive semidefinite rank-one matrix (when $x \neq 0$), it is natural to consider the linear system:

$$\langle A_i, X\rangle = 0, \quad i = 1, \ldots, m,\ X \succeq 0, \tag{37}$$

which is a homogeneous system. The set $\{X : \langle A_i, X\rangle = 0,\ i = 1, \ldots, m,\ X \succeq 0\}$ is a convex cone. It is evident that *the system (5) has a nonzero solution if and only if the system (37) has a rank-one solution. In another word,* 0 *is the only solution to (5) if and only if (37) has no rank-one solution.* There are only two cases for the system (37) with no rank-one solution: either $X = 0$ is the only matrix satisfying (37) or the minimum rank of the nonzero matrices satisfying (37) is greater than or equal to 2. As a result, let us consider the following rank minimization problem:

$$r^* = \min\left\{\mathrm{rank}(X) : \langle A_i, X\rangle = 0,\ i = 1, \ldots, m,\ \delta_1 \leqslant \|X\| \leqslant \delta_2,\ X \succeq 0\right\}, \tag{38}$$

where $0 < \delta_1 \leqslant \delta_2$ are two given positive constants. Clearly, $X = 0$ is the only matrix satisfying (37) if and only if the problem (38) is infeasible, in which case we set $r^* = \infty$. It is also easy to see that system (37) has a solution $X \neq 0$ if and only if the problem (38) is feasible, in which case r^* is finite and $1 \leqslant r^* \leqslant n$. Thus for the problem (38), we have either $r^* = \infty$ or $1 \leqslant r^* \leqslant n$.

From the above discussion, we immediately have the following result.

Lemma 4.1. 0 *is the only solution to system (5) if and only if* $r^* \geqslant 2$ *where* r^* *is the minimum rank of (38).*

Thus developing a sufficient condition for (36) can be achieved by identifying the condition under which the minimum rank of (38) is greater than or equal to 2. We follow this idea to establish some sufficient conditions for (36). By Theorem 3.5, the optimal value r^* of (38) is independent of the choice of δ_1, δ_2 and $\|\cdot\|$. Thus Lemma 4.1 holds for any given $0 < \delta_1 \leqslant \delta_2$ and any prescribed matrix norm in (38). So we have a freedom to choose δ_1, δ_2 and the matrix norm in (38) without affecting the value of r^* in (38). Thus, by setting $\delta_1 = \delta_2 = 1$ for simplicity and using the F-norm in (38), we have the problem

$$r^* = \min\left\{\mathrm{rank}(X) : \langle A_i, X\rangle = 0,\ i = 1, \ldots, m,\ \|X\|_F = 1,\ X \succeq 0\right\}. \tag{39}$$

By Theorem 3.4(ii), the feasible set of this problem contains a minimum rank solution with the least F-norm (which is equal to 1 for this case). From Theorem 3.1 and its corollary, the rank minimization (39) can be approximated by the following continuous optimization problem (as $(\eta, \varepsilon) \to 0$ and $\eta/\varepsilon \to 0$):

$$\begin{aligned} &\text{Minimize } \operatorname{tr}(Y) + (1/\eta)\operatorname{tr}(Z) \\ &\text{s.t.} \quad \begin{pmatrix} Y & X \\ X & Z+\varepsilon I \end{pmatrix} \succeq 0, \begin{pmatrix} I & X \\ X & Z \end{pmatrix} \succeq 0, \\ &\qquad \langle A_i, X\rangle = 0, \quad i = 1, \ldots, m, \ \|X\|_F = 1, \ X \succeq 0. \end{aligned} \tag{40}$$

(All results later in this section can be stated without involving the parameter η by setting, for instance, $\eta = \varepsilon^2$ for the simplicity). By Corollary 3.2, the first term of the objective in the above problem provides a lower bound for the minimum rank of (39). However, the constraint $\|X\|_F = 1$ makes the problem (40) difficult to be solved directly. So let us consider a relaxation of this constraint. Similar to (35), we define two constants:

$$\delta_1 = \min\{\operatorname{tr}(X) : \|X\|_F = 1, X \succeq 0\}, \quad \delta_2 = \max\{\operatorname{tr}(X) : \|X\|_F = 1, X \succeq 0\}. \tag{41}$$

It is easy to verify that $\delta_1 = 1$ and $\delta_2 = \sqrt{n}$. In fact, in terms of eigenvalues of X, the above two extreme problems are nothing but minimizing and maximizing, respectively, the function $\sum_{i=1}^n \lambda_i$ subject to $\sum_{i=1}^n \lambda_i^2 = 1, \lambda_i \geqslant 0, i = 1, \ldots, n$. The optimal values of these two problems are 1 and $\sqrt{n}$, respectively. Therefore, we conclude that

$$\{X : \ \|X\|_F = 1, \ X \succeq 0\} \subseteq \{X : \ 1 \leqslant \operatorname{tr}(X) \leqslant \sqrt{n}, \ X \succeq 0\}.$$

Thus, the following SDP problem is a relaxation of (40):

$$\begin{aligned} &\text{Minimize } \operatorname{tr}(Y) + (1/\eta)\operatorname{tr}(Z) \\ &\text{s.t.} \quad \begin{pmatrix} Y & X \\ X & Z+\varepsilon I \end{pmatrix} \succeq 0, \begin{pmatrix} I & X \\ X & Z \end{pmatrix} \succeq 0, \\ &\qquad \langle A_i, X\rangle = 0, \ i = 1, \ldots, m, \ 1 \leqslant \operatorname{tr}(X) \leqslant \sqrt{n}, \ X \succeq 0. \end{aligned} \tag{42}$$

The optimal value of (42) is a lower bound for that of (40). It is not difficult to verify that the dual problem of (42) is given by

$$\begin{aligned} &\text{Maximize} \quad \operatorname{tr}(\Phi) - \varepsilon \operatorname{tr}(Q) + t_1 + \sqrt{n} t_2 \\ &\text{s.t.} \qquad t_1 \geqslant 0, \ t_2 \leqslant 0, \\ &\begin{pmatrix} V + V^T + \sum_{i=1}^m y_i A_i + (t_1+t_2)I & U_1 & U_2 & U_3 & U_4 \\ U_1^T & \Phi & \Theta - V & U_5 & U_6 \\ U_2^T & \Theta^T - V^T & Q - \frac{1}{\eta} I & U_7 & U_8 \\ U_3^T & U_5^T & U_7^T & -I & -\Theta \\ U_4^T & U_6^T & U_8^T & -\Theta^T & -Q \end{pmatrix} \preceq 0. \end{aligned} \tag{43}$$

All blocks in the above matrix are $n \times n$ submatrices. Also, note that (43) is always feasible and satisfies the Slater's condition, for instance, ($\Theta = V = 0$, $\Phi = -I$, $Q = \frac{1}{2\eta} I$, $t_1 = 1$, $t_2 = -2$, $y_i = 0$ for all $i = 1, \ldots, m$, and $U_i = 0$ for all $i = 1, \ldots, 8$) is a strictly feasible point. So there is no duality gap between (42) and (43). We have the following result.

Theorem 4.2. *If there exist $(\eta,\varepsilon) > 0$ and $t_1, t_2, \mu_i, i = 1, \ldots, m$ and matrices $\Phi, Q \in S^{n\times n}$, $V, \Theta \in R^{n\times n}$ and $M_i \in R^{n\times n}$, $i = 1, \ldots, 8$ such that the following conditions hold*

$$\left\lceil \mathrm{tr}(\Phi) - \varepsilon \mathrm{tr}(Q) + t_1 + \sqrt{n}t_2 - \frac{1}{\eta} \right\rceil \geqslant 2, \quad t_1 \geqslant 0,\ t_2 \leqslant 0, \tag{44}$$

$$\begin{pmatrix} \sum_{i=1}^m \mu_i A_i - (t_1+t_2)I - (V+V^T) & M_1 & M_2 & M_3 & M_4 \\ M_1^T & -\Phi & V-\Theta & M_5 & M_6 \\ M_2^T & V^T - \Theta^T & \frac{1}{\eta}I - Q & M_7 & M_8 \\ M_3^T & M_5^T & M_7^T & I & \Theta \\ M_4^T & M_6^T & M_8^T & \Theta^T & Q \end{pmatrix} \succeq 0, \tag{45}$$

then 0 *is the only solution to the quadratic equation (5).*

Proof. Let X^* be the minimum rank solution of (39) with the least norm $\|X^*\|_F = 1$. Let $(Y_{\eta,\varepsilon}, X_{\eta,\varepsilon}, Z_{\eta,\varepsilon})$ be the optimal solution to (40), by Theorem 3.1, we have $r^* \geqslant \lceil Y_{\eta,\varepsilon} \rceil$ for every $(\eta,\varepsilon) > 0$, where r^* is the minimum rank of (39). Since (42) is a relaxation of (40), the optimal value of (42), denoted by $v^*(\eta,\varepsilon)$, provides a lower bound for that of (40), i.e.,

$$\mathrm{tr}(Y_{\eta,\varepsilon}) + (1/\eta)\mathrm{tr}(Z_{\eta,\varepsilon}) \geqslant v^*(\eta,\varepsilon), \tag{46}$$

which holds for any given $(\eta,\varepsilon) > 0$. Note that (43) is the dual problem of (42). If the conditions (44) and (45) hold, then for this (η,ε), the point $(t_1, t_2, y_i = -\mu, i = 1, \ldots, m, \Phi, V, \Theta, U_i = -M_j, j = 1, \ldots, 8)$ is feasible to the dual problem (43). Thus, by duality theory we have

$$v^*(\eta,\varepsilon) \geqslant \mathrm{tr}(\Phi) - \varepsilon\mathrm{tr}(Q) + t_1 + \sqrt{n}t_2. \tag{47}$$

Notice that (Y^*, Z^*, X^*), where $Y^* = X^*((X^*)^T X^* + \varepsilon I)^{-1}(X^*)^T$ and $Z^* = (X^*)^T X^*$, is a feasible point of (40). Thus

$$\mathrm{tr}(Y_{\eta,\varepsilon}) + (1/\eta)\mathrm{tr}(Z_{\eta,\varepsilon}) \leqslant \mathrm{tr}(Y^*) + (1/\eta)\mathrm{tr}(Z^*) = \phi_\varepsilon(X^*) + (1/\eta), \tag{48}$$

where the last equality follows from that $\mathrm{tr}(Y^*) = \phi_\varepsilon(X^*)$ and $\mathrm{tr}(Z^*) = \|X^*\|_F^2 = 1$. Combining (46), (47) and (48) yields

$$\phi_\varepsilon(X^*) + (1/\eta) \geqslant \mathrm{tr}(\Phi) - \varepsilon\mathrm{tr}(Q) + t_1 + \sqrt{n}t_2.$$

This together with (9) implies that $\mathrm{rank}(X^*) \geqslant \mathrm{tr}(\Phi) - \varepsilon\mathrm{tr}(Q) + t_1 + \sqrt{n}t_2 - (1/\eta)$. Thus, under the conditions (44) and (45), we see that

$$r^* = \mathrm{rank}(X^*) \geqslant \left\lceil \mathrm{tr}(\Phi) - \varepsilon\mathrm{tr}(Q) + t_1 + \sqrt{n}t_2 - (1/\eta) \right\rceil \geqslant 2.$$

By Lemma 4.1, we conclude that (36) holds, i.e., 0 is the only solution to (5). □

From the above result, a number of sufficient conditions stronger than (44)–(45) can be obtained. For example, we have the following corollary.

Corollary 4.3. *Let $A_i \in S^n, i = 1, \ldots, m$ be a given set of matrices. If there exist $(\eta,\varepsilon) > 0$, $t_1, t_2, \mu_1, \ldots, \mu_m \in R$, $Q, \Phi \in S^{n\times n}$ and $V, \Theta \in R^{n\times n}$ such that*

$$\left\lceil tr(\Phi) - \varepsilon tr(Q) + t_1 + \sqrt{n}t_2 - 1/\eta \right\rceil \geqslant 2, \quad t_1 \geqslant 0,\ t_2 \leqslant 0, \tag{49}$$

$$\begin{pmatrix} -\Phi & V-\Theta \\ V^T - \Theta^T & \frac{1}{\eta}I - Q \end{pmatrix} \succeq 0, \quad \begin{pmatrix} I & \Theta \\ \Theta^T & Q \end{pmatrix} \succeq 0, \tag{50}$$

$$\sum_{i=1}^m \mu_i A_i - (t_1+t_2)I - (Y+Y^T) \succeq 0, \tag{51}$$

then 0 *is the only solution to the system (5).*

We now point out that (6) implies (49)–(51). Let $\eta > 0$ be a given number. If $\sum_{i=1}^{m} t_i A_i \succ 0$ for some t_i, $i = 1, \ldots, m$, then we choose $\mu_i = \alpha t_i$ where α can be any large positive number such that $\sum_{i=1}^{m} \mu_i A_i \succ t_1 I$ where $t_1 = 2 + \frac{1}{\eta}$. Then conditions (49)–(51) hold with $V = \Phi = Q = \Theta = 0$ and $t_2 = 0$. Thus, the known condition (6) indeed implies (49)–(50). For $m = 2$ and $n \geqslant 3$, since the condition (36) is equivalent to $\mu_1 A_1 + \mu_2 A_2 \succ 0$, the sufficient conditions in Theorem 4.2 and Corollary 4.3 are also necessary conditions for (36).

Remark 4.4. To get more simple sufficient conditions for (36), we may continue to reduce the freedom of the variables in (49)–(51). For instance, (50) can be replaced by a stronger version like $\Phi \preceq 0$, $\frac{1}{\varepsilon^2} I \succeq Q$, $Q \succeq Y^T Y$ without involving the matrix Θ. It is also worth stressing that checking the new sufficient conditions developed in this section can be achieved by solving an SDP problem. For instance, if the optimal value of the SDP problem (43) is greater than $\frac{1}{\eta} + 1$, then the conditions (44)–(45) hold. Similarly, if the optimal value of the SDP problem (43) with $M_i = 0, i = 1, \ldots, 8$ is greater than $\frac{1}{\eta} + 1$, then the conditions (49)–(51) hold.

5. Conclusions

Since rank(X) is a discontinuous function with an integer value, this makes the rank minimization problem hard to be solved directly. In this paper, we have presented a generic approximation approach for rank minimization problems through the approximation function $\phi_\varepsilon(X)$. In particular, we have shown that when the feasible set is bounded the rank minimization problem can be approximated to any level of accuracy by a nonlinear SDP problem or a linear bilevel SDP problem with a special structure. To obtain a tractable approximation of the rank minimization with linear constraints, the approximation model (20) is introduced, and is proved to be efficient for locating the minimum rank solution of the problem if the feasible set contains a minimum rank element with the least F-norm. In this case, the rank minimization problem is equivalent to an SDP problem. This theory was applied to a system of quadratic equations which can be formulated as a rank minimization. Based on its approximation counterpart, we have developed some sufficient conditions for such a system with zero being its unique solution.

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