A GLOBALLY AND LOCALLY SUPERLINEARLY CONVERGENT NON-INTERIOR-POINT ALGORITHM FOR P_0 LCPS *

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Abstract. Based on the concept of the regularized central path, a new non-interior-point path-following algorithm is proposed for solving the P_0 linear complementarity problem (P_0 LCP). The condition ensuring the global convergence of the algorithm for P_0 LCPs is weaker than most previously used ones in the literature. This condition can be satisfied even when the strict feasibility condition, which has often been assumed in most existing non-interior-point algorithms, fails to hold. When the algorithm is applied to P_* and monotone LCPs, the global convergence of this method requires no assumption other than the solvability of the problem. The local superlinear convergence of the algorithm can be achieved under a nondegeneracy assumption. The effectiveness of the algorithm is demonstrated by our numerical experiments.

Key words. linear complementarity problem, non-interior-point algorithm, Tikhonov regularization, P_0 matrix, regularized central path

AMS subject classifications. 90C30, 90C33, 90C51, 65K05

1. Introduction.

We consider a new path-following algorithm for the linear complementarity problem (LCP):

$$x \geq 0, \ Mx + d \geq 0, \ x^T(Mx + d) = 0,$$

where M is an n by n matrix and d a vector in R^n. This problem is said to be a P_0 LCP if M is a P_0 matrix, and a P_* LCP if M is a P_* matrix. We recall that M is said to be a P_0 matrix (see [13]) if

$$\max_{1 \leq i \leq n} x_i(Mx)_i \geq 0 \ \text{ for any } \ 0 \neq x \in R^n.$$

M is said to be a P_* matrix (see [26]) if there exists a nonnegative constant $\tau \geq 0$ such that

$$(1 + \tau) \sum_{i \in I_+} x_i(Mx)_i + \sum_{i \in I_-} x_i(Mx)_i \geq 0 \ \text{ for any } \ 0 \neq x \in R^n,$$

where $I_+ = \{i : x_i(Mx)_i > 0\}$ and $I_- = \{i : x_i(Mx)_i < 0\}$.

We first give a synopsis of non-interior-point methods and related results for complementarity problems. The first non-interior-point path-following method for LCPs was proposed by Chen and Harker [6]. This method was improved by Kanzow [24] who also studied other closely related methods, later called the Chen-Harker-Kanzow-Smale (CHKS) smoothing function method (see [20]). The CHKS function $\phi : R^3 \to R$ is defined by

$$\phi(t_1, t_2, \mu) = t_1 + t_2 - \sqrt{(t_1 - t_2)^2 + 4\mu}.$$

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Based on such a function, Hotta and Yoshise [20] studied the structural properties of a non-interior-point trajectory and proposed a globally convergent path-following algorithm for a class of P_0 LCPs. However, no rate of convergence was reported in these papers. The first global linear convergence result for the LCP with a P_0 and R_0 matrix was obtained by Burke and Xu [3], who also proposed in [4] a non-interior-point predictor-corrector algorithm for monotone LCPs which was both globally linearly and locally quadratically convergent under certain assumption. Further development of non-interior-point methods can be found in [35, 5, 40, 33, 8, 7, 21]. It is worth mentioning that Chen and Xiu [8] and Chen and Chen [7] proposed a class of non-interior-point methods using the Chen-Mangasarian smoothing function family [9] that includes the CHKS smoothing function as a special case.

Since most existing non-interior-point path-following algorithms are based on the CHKS function, these methods actually follow the central path to locate a solution of the LCP. The central path is the set of solutions of the following system as the parameter $\mu > 0$ varies:

$$x > 0,\ Mx + d > 0,\ X(Mx + d) = \mu e,$$

where $X = \mathrm{diag}(x)$ and $e = (1, ..., 1)^T$. For P_0 LCPs, it is shown (see [42, 43]) that most assumptions used for non-interior-point algorithms, for instance, the Condition 1.5 in [25], Condition 1.2 in Hotta and Yoshise [20], and the $P_0 + R_0$ assumption in Burke and Xu [3] and Chen and Chen [7], imply that the solution set of the problem is bounded. As showed by Ravindran and Gowda in [34] the P_0 complementarity problem with a bounded solution set must have a strictly feasible point, i.e., there exists an x^0 such that $Mx^0 + d > 0$. (This implies that a P_0 LCP with no strictly feasible point either has no solution or has an unbounded solution set.) We conclude that the above-mentioned conditions all imply that the problem has a strictly feasible point. Thus, for a solvable P_0 LCP without strictly feasible point (in this case, the central path does not exists), it is unknown whether most existing non-interior-point algorithms are globally convergent or not. An interesting problem is how to improve these algorithms so that they are able to handle those P_0 problems with unbounded solution sets or without strictly feasible points.

Recently, Zhao and Li [42] proposed a new continuation trajectory for complementarity problems, which is defined as follows:

$$x > 0,\ Mx + d + \theta^p x > 0,\ x_i[(Mx + d)_i + \theta^p x_i] = \theta^q a_i,\ i = 1, ..., n,$$

where θ is a parameter in (0,1], $p \in (0, \infty)$ and $q \in [1, \infty)$ are two fixed scalars, and $a = (a_1, ..., a_n)^T \in R^n_{++}$ is a fixed vector, for example, $a = e$. For a P_0 matrix M, it turns out (see [42]) that the above system has a unique solution for each given parameter θ, and this solution, denoted by $x(\theta)$, is continuously differentiable on (0,1). Thus, the set $\{x(\theta) : \theta \in (0, 1]\}$ forms a smooth path approaching to the solution set of the P_0 LCP as θ tends to zero. Notice that for a given θ, the term $Mx + d + \theta^p x$ is the Tikhonov regularization of $Mx + d$, which has been used to study complementarity problems by several authors such as Isac [22], Venkateswaran [36], Facchinei and Kanzow [14], Ravindran and Gowda [34], and Zhao and Li [42]. We may refer the above smooth path to *regularized central path.* A good feature of the regularized central path is that its existence and boundedness can be guaranteed under a very weak assumption. In particular, the boundedness of the solution set and the strict feasibility condition are not needed for the existence of this path. Combining the CHKS function and Tikhonov regularization method, Zhao and Li [43] extended the results in [42] to non-interior-point methods, and studied the existence as well as the limiting behavior of a new non-interior-point smooth path.

The theoretical results established in [43] motivate us to construct a new non-interior-point path-following algorithm for P_0 LCPs. The purpose of this paper is to provide such a practical numerical algorithm. It is worth stressing the differences between the proposed method in this paper and previous algorithms in the literature. i) The proposed algorithm follows the regularized central path instead of the central path. ii) The condition ensuring the global convergence of the algorithm for P_0 LCPs is strictly weaker than those ones used in most existing non-interior-point methods. The local superlinear convergence of the algorithm can be achieved under a standard nondegeneracy assumption that was used in many works such as [38, 39, 33]. In particular, we also study the important special case of P_* LCPs, and derive some stronger results than that of the P_0 case.

The paper is organized as follows. In section 2, we introduce some basic results and describe the algorithm. In section 3, we prove the global convergence of the algorithm for a class of P_0 LCPs. The local convergence analysis of the algorithm is given in section 4. The special case of P_* LCPs is discussed in section 5, and some numerical results are reported in section 6.

Notation: R^n denotes the n-dimensional Euclidean space. R^n_+ and R^n_{++} denote the nonnegative orthant and positive orthant, respectively. A vector $x \geq 0$ ($x > 0$) means $x \in R^n_+$ ($x \in R^n_{++}$). All the vectors, unless otherwise stated, are column vectors. T denotes the transpose of a vector. For any vector x, the capital X denotes the diagonal matrix $\mathrm{diag}(x)$, and for any index set $I \subseteq \{1, ..., n\}$, x_I denotes the sub-vector made up of the components x_i for $i \in I$. The symbol e denotes the vector in R^n with all of its components equal to one. For given vectors u, w, v in R^n, the triplet (u, w, v) (the pair (x, y)) denotes the column vector $(u^T, w^T, v^T)^T$ ($(x^T, y^T)^T$). For any $u \in R^n$, the symbol u^p denotes the pth power of the vector u, i.e., the vector $(u_1^p, ..., u_n^p)^T$ where $p > 0$ is a positive scalar, and U^p denotes the diagonal matrix $\mathrm{diag}(u^p)$. For any vector $x \leq y$, we denote by $[x, y]$ the rectangular box $[x_1, y_1] \times ... \times [x_n, y_n]$.

2. A non-interior-point path-following algorithm. Let p and q be two given positive scalars. Define the map $\mathcal{H} : R^n_+ \times R^{2n} \to R^{3n}$ as follows:

$$\mathcal{H}(u, x, y) = \begin{pmatrix} u \\ x + y - \sqrt{(x-y)^2 + 4u^q} \\ y - (Mx + d + U^p x) \end{pmatrix}, \ (u, x, y) \in R^n_+ \times R^{2n}, \tag{2.1}$$

where $U^p = \mathrm{diag}(u^p)$ and all the algebraic operations are performed componentwise. The above homotopy map first appeared in [43]. Clearly, if $\mathcal{H}(u, x, y) = 0$ then (x, y) is a solution to the LCP; conversely, if (x, y) is a solution to the LCP, then $(0, x, y)$ is a solution to the equation $\mathcal{H}(u, x, y) = 0$. Thus, an LCP can be solved via locating a solution of the nonlinear equation $\mathcal{H}(u, x, y) = 0$. From the discussion in [43], we can conclude that it is a judicious choice to use the above version of the homotopy formulation in order to deal with the LCP with an unbounded solution set.

Before embarking on stating the algorithm, we first introduce some results established in [43]. Let $(a, b, c) \in R^n_{++} \times R^{2n}$ be given. Consider the following system with the parameter θ :

$$\mathcal{H}(u, x, y) = \theta(a, b, c), \tag{2.2}$$

where $\theta \in (0, 1]$. For P_0 LCPs, it is shown in [43] that for each given $\theta \in (0, 1]$ the above system has a unique solution denoted by $(u(\theta), x(\theta), y(\theta))$, which is also continuously differentiable with respect to θ. Therefore, the following set

$$\{(u(\theta), x(\theta), y(\theta)) : \mathcal{H}(u, x, y) = \theta(a, b, c),\ \theta \in (0, 1]\} \tag{2.3}$$

forms a smooth path. Also, in this paper, we refer this path to as the *regularized central path.* The existence of such a smooth path for P_0 LCPs needs no assumption (see Theorem 2.1 below). An additional condition is assumed to guarantee the boundedness of this path. We now introduce such a condition proposed in [43].

For given $(a, b, c) \in R^n_{++} \times R^{2n}$ and $\theta \in (0, 1]$, we define a mapping $\mathcal{F}_{(a,b,c,\theta)} : R^{2n} \to R^{2n}$ as follows:

$$\mathcal{F}_{(a,b,c,\theta)}(x, y) = \begin{pmatrix} Xy \\ y - M(x + \frac{1}{2}\theta b) - d - \theta^p A^p x - \theta c \end{pmatrix},$$

where $X = \mathrm{diag}(x)$ and $A^p = \mathrm{diag}(a^p)$.

CONDITION 2.1. *For any given $(a, b, c) \in R^n_{++} \times R^{2n}$ and scalar $\hat{t}$, there exists a scalar $\theta^* \in (0, 1]$ such that*

$$\bigcup_{\theta \in (0,\theta^*]} \mathcal{F}^{-1}_{(a,b,c,\theta)}(D_\theta)$$

is bounded, where

$$\mathcal{F}^{-1}_{(a,b,c,\theta)}(D_\theta) := \{(x, y) \in R^{2n}_{++} : \mathcal{F}_{(a,b,c,\theta)}(x, y) \in D_\theta\}$$

and $D_\theta := [0, \theta a^q] \times [-\theta \hat{t} e, \theta \hat{t} e] \subseteq R^n_+ \times R^n$.

The following result states that Condition 2.1 is weaker than most known assumptions used for non-interior-point methods. An example was given in [43] to show that Condition 2.1 can be satisfied even when the strict feasibility condition fails to hold.

PROPOSITION 2.1. [43] *Let $f = Mx + d$ where M is a P_0-matrix. If one of the following conditions holds, then Condition 2.1 is satisfied.*

a) Condition 1.5 of Kojima et al [25].

b) Condition 2.2 of Hotta and Yoshise [20].

c) Assumption 2.2 of Chen and Chen [7].

d) f is a P_0 and R_0 function [3, 7].

e) f is a P_*-function (i.e., M is a P_* matrix) and there is a strictly feasible point [26].

f) f is a uniform P-function, i.e., M is a P-matrix [27].

g) The solution set of the LCP is nonempty and bounded.

The converse, however, is not true, i.e., Condition 2.1 cannot imply any one of the above conditions.

Restricted to LCPs, the main result established in [43] are summarized as follows.

THEOREM 2.1. [43] *Let M be a P_0-matrix.*

(i) For each $\theta \in (0, 1]$, the system (2.2) has a unique solution denoted by $(u(\theta)$, $x(\theta)$, $y(\theta))$, which is also continuously differentiable in θ.

(ii) If Condition 2.1 is satisfied, then the regularized central path (2.3) is bounded. Hence, there exists a subsequence $(u(\theta^k), x(\theta^k), y(\theta^k))$ that converges, as $\theta^k \to 0$, to $(0, x^*, y^*)$ where x^* is a solution to the LCP.

For P_* LCPs, the only condition for the result (ii) above is the solvability of the problem.

THEOREM 2.2. [43] *Let M be P_*-matrix. Assume that the solution set of the LCP is nonempty.*

(i) If $p \le 1$ and $q \in [1, \infty)$, then the regularized central path (2.3) is bounded.

(ii) If $p > 1$, $q \in [1, \infty)$ and $c \in R^n_{++}$, then the regularized central path (2.3) is bounded.

The boundedness of the path (2.3) implies that the problem has a solution. Combining this fact and the above result, we may conclude that the solvability of a P_* LCP, roughly speaking, is a necessary and sufficient condition for the boundedness of the regularized central path. For monotone LCPs, we have a much stronger result than the above, i.e., the entire path (2.3) is convergent as $\theta \to 0$. The property of the limiting point of this path, as $\theta \to 0$, depends on the choice of the scalars p and q.

THEOREM 2.3. [43] *Let M be a positive semi-definite matrix. Assume that the solution set of the LCP is nonempty.*

(i) If $p \le 1$ and $q \in [1, \infty)$, then the regularized central path (2.3) converges, as $\theta \to 0$, to the unique least N-norm solution of the LCP, where $N = A^{p/2}$.

(ii) If $p > 1$, $q \in [1, \infty)$ and $c \in R^n_{++}$, then the regularized central path (2.3) converges, as $\theta \to 0$, to a maximally complementary solution of the LCP.

We now introduce the algorithm. We choose the following neighborhood around the regularized central path $\{(u(\theta), x(\theta), y(\theta)) : \theta \in (0, 1]\}$:

$$\mathcal{N}(\beta) = \{(u, x, y) : \|u - \theta a\| = 0,\ \|\mathcal{H}(u, x, y) - \theta(a, b, c)\| \le \beta\theta,\ \theta \in (0, 1]\}.$$

Denote

$$G_\theta(x, y) = \begin{pmatrix} x + y - \sqrt{(x - y)^2 + 4(\theta a)^q} \\ y - (Mx + d + \theta^p A^p x) \end{pmatrix}. \tag{2.4}$$

Then, the above neighborhood reduces to

$$\mathcal{N}(\beta) = \{(x, y) : \|G_\theta(x, y) - \theta(b, c)\| \le \beta\theta,\ \theta \in (0, 1]\},$$

where G_θ is given by (2.4). For a given $\theta \in (0, 1]$, we denote

$$\mathcal{N}(\beta, \theta) = \{(x, y) : \|G_\theta(x, y) - \theta(b, c)\| \le \beta\theta\}.$$

Throughout the paper, $\nabla G_\theta(x, y)$ denotes the Jacobian of $G_\theta(x, y)$ with respect to (x, y). Let $\varepsilon > 0$ be a given tolerance. We now describe the algorithm as follows.

ALGORITHM 2.1: Let $p \in (0, \infty)$, $q \in [1, \infty)$, $\sigma \in (0, 1)$ and $\alpha \in (0, 1)$ be given.

Step 1. Select $(a, b, c) \in R^n_{++} \times R^{2n}$, $(x^0, y^0) \in R^{2n}$, $\theta^0 \in (0, 1)$, and $\beta > 0$ such that $(x^0, y^0) \in \mathcal{N}(\beta, \theta^0)$.

Step 2 (Approximate Newton Step). If $\|G_0(x^k, y^k)\| \le \varepsilon$, stop; otherwise, let $(d\hat{x}^k, d\hat{y}^k)$ solve the equation

$$G_0(x^k, y^k) + \nabla G_{\theta^k}(x^k, y^k)(dx, dy) = 0. \tag{2.5}$$

Let

$$(\hat{x}^{k+1}, \hat{y}^{k+1}) = (x^k, y^k) + (d\hat{x}^k, d\hat{y}^k).$$

If $\|G_0(\hat{x}^{k+1}, \hat{y}^{k+1})\| \le \varepsilon$, stop; otherwise, if

$$(\hat{x}^{k+1}, \hat{y}^{k+1}) \in \mathcal{N}(\beta, (\theta^k)^2),$$

then set

$$\theta^{k+1} = (\theta^k)^2,\ (x^{k+1}, y^{k+1}) = (\hat{x}^{k+1}, \hat{y}^{k+1}).$$

Set $k := k + 1$, and repeat step 2. Otherwise, go to step 3.

Step 3 (Centering Step). If $G_{\theta^k}(x^k, y^k) = \theta^k(b, c)$, set $(x^{k+1}, y^{k+1}) = (x^k, y^k)$, and go to step 4. Otherwise, let (dx^k, dy^k) be the solution to the equation

$$G_{\theta^k}(x^k, y^k) - \theta^k(b, c) + \nabla G_{\theta^k}(x^k, y^k)(dx, dy) = 0. \tag{2.6}$$

Let λ_k be the maximum among the values of $1, \alpha, \alpha^2, ...$ such that

$$\|G_{\theta^k}(x^k + \lambda_k dx^k, y^k + \lambda_k dy^k) - \theta^k(b, c)\| \leq (1 - \sigma\lambda_k)\|G_{\theta^k}(x^k, y^k) - \theta^k(b, c)\|.$$

Set

$$(x^{k+1}, y^{k+1}) = (x^k, y^k) + \lambda_k(dx^k, dy^k).$$

Step 4 (Reduce θ^k). Let γ_k be the maximum among the values $1, \alpha, \alpha^2, ...$ such that

$$(x^{k+1}, y^{k+1}) \in \mathcal{N}(\beta, (1 - \gamma_k)\theta^k),$$

i.e.,

$$\|G_{(1-\gamma_k)\theta^k}(x^{k+1}, y^{k+1}) - (1 - \gamma_k)\theta^k(b, c)\| \leq \beta(1 - \gamma_k)\theta^k.$$

Set $\theta^{k+1} = (1 - \gamma_k)\theta^k$. Set $k := k + 1$ and go to step 2.

Remark 2.1. (i) To start the algorithm, we need an initial point within the neighborhood of the regularized central path. Such an initial point can be found at no cost for the above algorithm. For example, let (a, b, c) be an arbitrary triplet in $R^n_{++} \times R^{2n}$, (x^0, y^0) be an arbitrary vector in R^{2n}, and θ^0 be an arbitrary scalar in (0,1). Then, choose β such that

$$\beta \geq \frac{\|G_{\theta^0}(x^0, y^0) - \theta^0(b, c)\|}{\theta^0}.$$

Clearly, this initial point satisfies $(x^0, y^0) \in \mathcal{N}(\beta, \theta^0)$.

(ii) The step 3 of the algorithm is a centering step in the sense that it forces the iterate close to the regularized central path such that the iterate is always confined in the neighborhood of the path. In the next section, we show that step 3 together with step 4 guarantees the global convergence of the algorithm. Step 2 is an approximate Newton step which was shown to have good local convergence properties (see, for example, [10, 11]). This step is used to accelerate the iteration such that a local rapid convergence can be achieved. Similar strategies were used in several works such as [38, 39, 28, 7, 8]. We also note that linear systems (2.5) and (2.6) have the same coefficient matrix, and thus only one matrix factorization is needed at each iteration.

We now show that the algorithm is well-defined.

PROPOSITION 2.2. *Algorithm 2.1 is well-defined and satisfies the following properties: (i) θ^k is monotonically decreasing, and (ii) $\|G_{\theta^k}(x^k, y^k) - \theta^k(b, c)\| \leq \beta\theta^k$ for all $k \geq 0$, i.e., $(x^k, y^k) \in \mathcal{N}(\beta, \theta^k)$ for all $k \geq 0$.*

Proof. We verify that each step of the algorithm is well-defined. As we pointed out in Remark 2.1, step 1 of the algorithm is well-defined. Consider the following $2n \times 2n$ matrix

$$\nabla G_{\theta^k}(x^k, y^k) = \begin{pmatrix} I - (X^k - Y^k)D^k & I + (X^k - Y^k)D^k \\ -(M + (\theta^k)^p A^p) & I \end{pmatrix}, \tag{2.7}$$

where $X^k = \operatorname{diag}(x^k)$, $Y^k = \operatorname{diag}(y^k)$, and $D^k = \operatorname{diag}(d^k)$ with $d^k = (d_1^k, ..., d_n^k)^T$ where

$$d_i^k = \frac{1}{\sqrt{(x_i^k - y_i^k)^2 + 4(\theta^k)^q a_i^q}}, \; i = 1,\ 2,\ ...,\ n.$$

Since $a \in R^n_{++}$, for each given $\theta^k \in (0,1)$ it is easy to see that $I - (X^k - Y^k)D^k$ and $I + (X^k - Y^k)D^k$ are positive diagonal matrices for every $(x^k, y^k) \in R^{2n}$. Thus, by Lemma 5.4 in Kojima et al. [25], the matrix $\nabla G_{\theta^k}(x^k, y^k)$ is nonsingular when M is a P$_0$-matrix. Thus, step 2 is well-defined.

Since (dx^k, dy^k) is a descent direction for the following function at (x^k, y^k),

$$\tilde{f}(x, y) = \frac{1}{2}\|G_{\theta^k}(x, y) - \theta^k(b, c)\|_2^2,$$

the line search in step 3 is well-defined, and thus the whole step 3 is well-defined.

We finally prove that the step 4 is well-defined. For any scalar $\mu_1 > \mu_2 \geq 0$, we have

$$\begin{aligned}
&\|G_{\mu_1}(x, y) - G_{\mu_2}(x, y)\| \\
&= \left\| \begin{pmatrix} x + y - \sqrt{(x-y)^2 + 4\mu_1^q a^q} \\ y - (Mx + d + \mu_1^p A^p x) \end{pmatrix} - \begin{pmatrix} x + y - \sqrt{(x-y)^2 + 4\mu_2^q a^q} \\ y - (Mx + d + \mu_2^p A^p x) \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} \sqrt{(x-y)^2 + 4\mu_1^q a^q} - \sqrt{(x-y)^2 + 4\mu_2^q a^q} \\ (\mu_1^p - \mu_2^p) A^p x \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} \frac{4(\mu_1^q - \mu_2^q)a^q}{\sqrt{(x-y)^2 + 4\mu_1^q a^q} + \sqrt{(x-y)^2 + 4\mu_2^q a^q}} \\ (\mu_1^p - \mu_2^p) A^p x \end{pmatrix} \right\| \\
&\leq \left\| \begin{pmatrix} \frac{4(\mu_1^q - \mu_2^q)a^q}{\sqrt{4\mu_1^q a^q}} \\ (\mu_1^p - \mu_2^p) A^p x \end{pmatrix} \right\|
\end{aligned}$$

$$\leq \max\{\mu_1^{q/2}(1 - (\mu_2/\mu_1)^q), \mu_1^p(1 - (\mu_2/\mu_1)^p)\}\|(2a^{q/2}, A^p x)\|. \tag{2.8}$$

In particular, setting $(x, y) = (x^k, y^k)$, $\mu_1 = \theta^k > 0$, and $\mu_2 = (1 - \gamma)\theta^k$ with $\gamma \in (0, 1)$, we then have

$$\begin{aligned}
&\|G_{\theta^k}(x^k, y^k) - G_{(1-\gamma)\theta^k}(x^k, y^k)\| \\
&\leq \max\{(\theta^k)^{q/2}(1 - (1 - \gamma)^q),\ (\theta^k)^p(1 - (1 - \gamma)^p)\}\|(2a^{q/2}, A^p x^k)\|.
\end{aligned} \tag{2.9}$$

There are two cases to be considered.

Case (i): $G_{\theta^k}(x^k, y^k) = \theta^k(b, c)$ in step 3. Then, $(x^{k+1}, y^{k+1}) = (x^k, y^k)$. By (2.9), for all sufficiently small γ we have

$$\begin{aligned}
&\|G_{(1-\gamma)\theta^k}(x^{k+1}, y^{k+1}) - (1 - \gamma)\theta^k(b, c)\| \\
&= \|G_{(1-\gamma)\theta^k}(x^k, y^k) - G_{\theta^k}(x^k, y^k) + \theta^k(b, c) - (1 - \gamma)\theta^k(b, c)\| \\
&\leq \|G_{(1-\gamma)\theta^k}(x^k, y^k) - G_{\theta^k}(x^k, y^k)\| + \gamma\theta^k\|(b, c)\| \\
&\leq \max\{(\theta^k)^{q/2}(1 - (1 - \gamma)^q), (\theta^k)^p(1 - (1 - \gamma)^p)\}\|(2a^{q/2}, A^p x^k)\| + \gamma\theta^k\|(b, c)\| \\
&\leq \beta(1 - \gamma)\theta^k.
\end{aligned}$$

Case (ii): $G_{\theta^k}(x^k, y^k) \neq \theta^k(b, c)$ in step 3. For this case, according to step 3 we have

$$\begin{aligned}
\|G_{\theta^k}(x^{k+1}, y^{k+1}) - \theta^k(b, c)\| &\leq (1 - \sigma\lambda_k)\|G_{\theta^k}(x^k, y^k) - \theta^k(b, c)\| \\
&\leq (1 - \sigma\lambda_k)\beta\theta^k.
\end{aligned}$$

The second inequality follows from the fact that $\|G_{\theta^k}(x^k, y^k) - \theta^k(b,c)\| \le \beta\theta^k$, which is evident from the construction of the algorithm. Notice that $1 - \sigma\lambda_k < 1$. By (2.9) and the above inequality, for all sufficiently small γ we have

$$
\begin{aligned}
&\|G_{(1-\gamma)\theta^k}(x^{k+1}, y^{k+1}) - (1-\gamma)\theta^k(b,c)\| \\
&\le \|G_{(1-\gamma)\theta^k}(x^{k+1}, y^{k+1}) - G_{\theta_k}(x^{k+1}, y^{k+1})\| + \|G_{\theta^k}(x^{k+1}, y^{k+1}) - \theta^k(b,c)\| \\
&+\gamma\theta^k\|(b,c)\| \\
&\le \max\{(\theta^k)^{q/2}(1-(1-\gamma)^q), (\theta^k)^p(1-(1-\gamma)^p)\}\|(2a^{q/2}, A^p x^{k+1})\| \\
&+(1-\sigma\lambda_k)\beta\theta^k + \gamma\theta^k\|(b,c)\| \\
&\le (1-\gamma)\beta\theta^k.
\end{aligned}
$$

Thus, the step 4 is well-defined.

We now show that all the iterates are in the neighborhood defined by the algorithm. By the construction of the algorithm, it is evident that either $\theta^{k+1} = (\theta^k)^2$ or $\theta^{k+1} = (1-\gamma_k)\theta^k$. So, θ^k is monotonically decreasing. When $k = 0$, it follows from step 1 that $(x^0, y^0) \in \mathcal{N}(\beta, \theta^0)$. Assume that this property holds for k, i.e., $(x^k, y^k) \in \mathcal{N}(\beta, \theta^k)$. We show that it holds for $k+1$. Indeed, if step 2 is accepted, then the criterion $(x^{k+1}, y^{k+1}) \in \mathcal{N}(\beta, \theta^{k+1})$ is satisfied, where $\theta^{k+1} = (\theta^k)^2$. If step 2 is rejected, then (x^{k+1}, y^{k+1}) is created by step 3 together with step 4. It follows from step 4 that $(x^{k+1}, y^{k+1}) \in \mathcal{N}(\beta, \theta^{k+1})$ where $\theta^{k+1} = (1-\gamma_k)\theta^k$. Thus, for all $k \ge 0$, we have that $(x^k, y^k) \in \mathcal{N}(\beta, \theta^k)$, i.e., $\|G_{\theta^k}(x^k, y^k) - \theta^k(b,c)\| \le \beta\theta^k$. □

3. Global convergence for $\mathbf{P_0}$ LCPs. We now show that the proposed algorithm is globally convergent for P_0 LCPs provided that Condition 2.1 is satisfied. By Proposition 2.2, for every $k \ge 1$, the iterate (x^k, y^k) satisfies the following:

$$\|G_{\theta^k}(x^k, y^k) - \theta^k(b,c)\| \le \beta\theta_k, \ \theta^k = (1-\gamma_{k-1})\theta^{k-1} \ \text{ or } \ \theta^k = (\theta^{k-1})^2. \tag{3.1}$$

Let $(b^k, c^k) \in R^{2n}$ be two auxiliary vectors determined by

$$(b^k, c^k) = \frac{G_{\theta^k}(x^k, y^k) - \theta^k(b,c)}{\theta^k} \ \text{ for all } \ k. \tag{3.2}$$

Then, $\{(b^k, c^k)\}$ is uniformly bounded. In fact, by (3.1), we have that $\|(b^k, c^k)\| \le \beta$, and hence

$$-\beta e \le b^k \le \beta e, \ -\beta e \le c^k \le \beta e. \tag{3.3}$$

By the definition of (2.4), we can write (3.2) as

$$
\begin{aligned}
&x^k + y^k - \sqrt{(x^k - y^k)^2 + 4(\theta^k)^q a^q} = \theta^k(b + b^k), \\
&y^k - (Mx^k + d + (\theta^k)^p A^p x^k) = \theta^k(c + c^k).
\end{aligned}
$$

By the property of CHKS-function (see Lemma 1 in [20]), the system above is equivalent to

$$x^k - \frac{1}{2}\theta^k(b + b^k) > 0, \ y^k - \frac{1}{2}\theta^k(b + b^k) > 0, \tag{3.4}$$

$$\left[X^k - \frac{1}{2}\theta^k(B + B^k)\right]\left(y^k - \frac{1}{2}\theta^k(b + b^k)\right) = (\theta^k)^q a^q, \tag{3.5}$$

$$y^k = Mx^k + d + (\theta^k)^p A^p x^k + \theta^k(c + c^k), \tag{3.6}$$

where X^k, B and B^k are diagonal matrices corresponding to x^k, b and b^k, respectively.

Remark 3.1. The fact that all iterates generated by Algorithm 2.1 satisfy the system (3.4)-(3.6) plays a key role in the analysis throughout the paper. By continuity, from (3.6) it follows that $\{y^k\}$ is bounded if $\{x^k\}$ is. Thus, if the sequence (x^k, y^k) is unbounded, then $\{x^k\}$ must be unbounded.

The following result is a minor revision of Lemma 1 in [34].

LEMMA 3.1. [42, 44] *Let M be a P_0 matrix. Let $\{z^k\}$ be an arbitrary sequence with $\|z^k\| \to \infty$ and $z^k \geq \bar{z}$ for all k, where $\bar{z} \in R^n$ is a fixed vector. Then there exist a subsequence of $\{z^k\}$, denoted by $\{z^{k_j}\}$, and a fixed index i_0 such that $z_{i_0}^{k_j} \to \infty$ and $(Mz^{k_j} + d)_{i_0}$ is bounded from below.*

The next result shows that the iterative sequence $\{(x^k, y^k)\}$ generated by Algorithm 2.1 is bounded under Condition 2.1.

THEOREM 3.1. *Let M be a P_0 matrix. If Condition 2.1 is satisfied, then the iterative sequence $\{(x^k, y^k)\}$ generated by Algorithm 2.1 is bounded.*

Proof. We prove this result by contradiction. Assume that $\{(x^k, y^k)\}$ is unbounded. Then $\{x^k\}$ is unbounded (see Remark 3.1). Without loss of generality, we may assume that $\|x^k\| \to \infty$. Notice that $\theta^k < 1$ and $\|b^k\| \leq \beta$. It follows from (3.4) that

$$x^k \geq \frac{1}{2}\theta^k(b + b^k) \geq -\frac{1}{2}(\|b\| + \beta)e \quad \text{for all} \quad k.$$

Thus, by Lemma 3.1, there exist a subsequence of $\{x^k\}$, denoted also by $\{x^k\}$, and an index m such that $x_m^k \to \infty$ and $(Mx^k + d)_m$ is bounded from below. By (3.5), for each i we have

$$\left(x_i^k - \frac{1}{2}\theta^k(b_i + b_i^k)\right)\left(y_i^k - \frac{1}{2}\theta^k(b_i + b_i^k)\right) = (\theta^k)^q a_i^q,$$

and thus,

$$y_m^k - \frac{1}{2}\theta^k(b_m + b_m^k) = \frac{(\theta^k)^q a_m^q}{x_m^k - \theta^k(b_m + b_m^k)/2}.$$

By using (3.6), the above equation can be further written as

$$\begin{aligned}&(Mx^k + d)_m + \theta^k(c_m + c_m^k) - \frac{1}{2}\theta^k(b_m + b_m^k) - \frac{(\theta^k)^q a_m^q}{x_m^k - \theta^k(b_m + b_m^k)/2}\\ &= -(\theta^k)^p a_m^p x_m^k.\end{aligned}$$

Since b^k and c^k are bounded, $x_m^k \to \infty$, and $(Mx^k + d)_m$ is bounded from below, we conclude that the left-hand side of the above equation is bounded from below. This implies that $\theta^k \to 0$ (since otherwise the right-hand side tends to $-\infty$).

In what follows, we denote by

$$\bar{x}^k = x^k - \frac{1}{2}\theta^k(b + b^k), \ \bar{y}^k = y^k - \frac{1}{2}\theta^k(b + b^k). \tag{3.7}$$

¿From (3.4) and (3.5), we see that $(\bar{x}^k, \bar{y}^k) > 0$ for all k, and

$$\bar{X}^k \bar{y}^k = (\theta^k)^q a^q. \tag{3.8}$$

Since $\|b^k\| \leq \beta$, it follows that

$$\left\| \frac{M(x^k - \bar{x}^k - \theta^k b/2)}{\theta^k} \right\| \leq \frac{1}{2}\beta\|M\|. \tag{3.9}$$

By using (3.6) and (3.7), we have

$$\begin{aligned}
&\bar{y}^k - M(\bar{x}^k + \frac{1}{2}\theta^k b) - d - (\theta^k)^p A^p \bar{x}^k - \theta^k c \\
&\quad = Mx^k + d + (\theta^k)^p A^p x^k + \theta^k (c + c^k) - \frac{1}{2}\theta^k (b + b^k) \\
&\quad\quad - M(\bar{x}^k + \frac{1}{2}\theta^k b) - d - (\theta^k)^p A^p \bar{x}^k - \theta^k c \\
&\quad = \theta^k \left(\frac{M(x^k - \bar{x}^k - \theta^k b/2)}{\theta^k} - \frac{1}{2}(b + b^k) + c^k + \frac{1}{2}(\theta^k)^p A^p (b + b^k) \right).
\end{aligned}$$

By (3.9) and the boundedness of θ^k, b^k and c^k, there exists a scalar $\hat{t} > 0$ such that

$$-\hat{t}e \leq \frac{M(x^k - \bar{x}^k - \theta^k b/2)}{\theta^k} - \frac{1}{2}(b + b^k) - c^k + \frac{1}{2}(\theta^k)^p A^p (b + b^k) \leq \hat{t}e$$

for all k. Therefore,

$$\bar{y}^k - M(\bar{x}^k + \frac{1}{2}\theta^k b) - d - (\theta^k)^p A^p \bar{x}^k - \theta^k c \;\in\; \theta^k[-\hat{t}e, \hat{t}e]$$

for all k. Notice that $q \in [1, \infty)$. Combination of (3.8) and the above leads to

$$\begin{aligned}
\mathcal{F}_{(a,b,c,\theta^k)}(\bar{x}^k, \bar{y}^k) \;&=\; \begin{pmatrix} \bar{X}^k \bar{y}^k \\ \bar{y}^k - M(\bar{x}^k + \frac{1}{2}\theta^k b) - d - (\theta^k)^p A^p \bar{x}^k - \theta^k c \end{pmatrix} \\
&\in\; \theta^k[0, a^q] \times \theta^k[-\hat{t}e, \hat{t}e] \\
&=: D_{\theta^k}
\end{aligned}$$

for all k. Thus,

$$(\bar{x}^k, \bar{y}^k) \in \mathcal{F}^{-1}_{(a,b,c,\theta^k)}(D_{\theta^k}) \;\; \text{for all} \;\; k.$$

By Condition 2.1, there exists a θ^* such that

$$\bigcup_{\theta \in (0,\theta^*]} \mathcal{F}^{-1}_{(a,b,c,\theta)}(D_\theta)$$

is bounded. Since $\theta^k \to 0$, there exists some k_0 such that for all $k \geq k_0$ we have $\theta^k \leq \theta^*$. Thus,

$$\{(\bar{x}^k, \bar{y}^k)\}_{k \geq k_0} \subseteq \bigcup_{\theta^k \leq \theta^*} \mathcal{F}^{-1}_{(a,b,c,\theta^k)}(D_{\theta^k}) \subseteq \bigcup_{\theta \in (0,\theta^*]} \mathcal{F}^{-1}_{(a,b,c,\theta)}(D_\theta).$$

The right-hand side of the above is bounded. This contradicts the left-hand side which (by assumption) is an unbounded sequence. □

We are ready to prove the global convergence of Algorithm 2.1 for P_0 LCPs.

THEOREM 3.2. *Let M be a P_0-matrix. Assume that Condition 2.1 is satisfied. If (x^k, y^k, θ^k) is generated by Algorithm 2.1, then $\{(x^k, y^k)\}$ has at least one accumulation point, and*

$$\lim_{k\to\infty} \theta^k \to 0, \quad \lim_{k\to\infty} \|G_{\theta^k}(x^k, y^k)\| \to 0. \tag{3.10}$$

Thus, every accumulation point of (x^k, y^k) is a solution to the LCP.

Proof. By Theorem 3.1, the iterative sequence $\{(x^k, y^k)\}$ generated by the algorithm is bounded, and hence it has at least one accumulation point. By Proposition 2.2, we have

$$\begin{aligned}\|G_{\theta^k}(x^k, y^k)\| &\le \|G_{\theta^k}(x^k, y^k) - \theta^k(b, c)\| + \theta^k\|(b, c)\| \\ &\le \theta^k[\beta + \|(b, c)\|].\end{aligned}$$

Thus, to show the second limiting property in (3.10) it is sufficient to show that $\theta^k \to 0$. By the construction of the algorithm, we have either $\theta^{k+1} = (1-\gamma_k)\theta^k$ or $\theta^{k+1} = (\theta^k)^2$. Thus θ^k is monotonically decreasing, and thus there exists a scalar $1 > \bar\theta \ge 0$ such that $\theta^k \to \bar\theta$. If $\bar\theta = 0$, the desired result follows.

Assume the contrary that $\bar\theta > 0$. We now derive a contradiction. Since $\bar\theta > 0$, the algorithm eventually phases out the approximate Newton step, and takes only step 3 and step 4. In fact, if step 2 is accepted infinite many times, then there exists a subsequence $\{k_j\}$ such that $\theta^{k_j+1} = (\theta^{k_j})^2$ which implies that $\bar\theta = \bar\theta^2$. This is impossible since $0 < \bar\theta < 1$. Thus, there exists a k_0 such that for all $k \ge k_0$, the iterates $\{(x^k, y^k)\}_{k\ge k_0}$ are generated only by step 3, and hence $\theta^{k+1} = (1-\gamma_k)\theta^k$ for all $k \ge k_0$. Since $\theta^k \to \bar\theta > 0$, it follows that $\gamma_k \to 0$. Thus, for all sufficiently large k, we have $(x^{k+1}, y^{k+1}) \notin \mathcal{N}(\beta, (1-\frac{1}{\alpha}\gamma_k)\theta^k$, that is

$$\left\| G_{(1-\frac{1}{\alpha}\gamma_k)\theta^k}(x^{k+1}, y^{k+1}) - \left(1 - \frac{1}{\alpha}\gamma_k\right)\theta^k(b, c) \right\| > \beta\left(1 - \frac{1}{\alpha}\gamma_k\right)\theta^k.$$

Since the iterate (x^{k+1}, y^{k+1}) is bounded, taking a subsequence if necessary we may assume that this sequence converges to some $(\hat x, \hat y)$. Notice that $\gamma_k \to 0$. Taking the limit in the above inequality, we have

$$\|G_{\bar\theta}(\hat x, \hat y) - \bar\theta(b, c)\| \ge \beta\bar\theta > 0.$$

Since $\bar\theta > 0$, the matrix $\nabla G_{\bar\theta}(\hat x, \hat y)$ is nonsingular. Let $(d\hat x, d\hat y)$ be the solution to

$$G_{\bar\theta}(\hat x, \hat y) - \bar\theta(b, c) + \nabla G_{\bar\theta}(\hat x, \hat y)(dx, dy) = 0.$$

Then, $(d\hat x, d\hat y)$ is a strictly descent direction for $\|G_{\bar\theta}(x, y) - \bar\theta(b, c)\|$ at $(\hat x, \hat y)$. As a result, the line search steplengths, $\hat\lambda$ (in step 3) and $\hat\gamma$ (in step 4), are both positive constants. Since G and ∇G are continuous in the neighborhood of $(\hat x, \hat y)$, it follows that $(dx^k, dy^k, \lambda_k, \gamma_k) \to (d\hat x, d\hat y, \hat\lambda, \hat\gamma)$, and therefore λ_k, γ_k must be uniformly bounded from below by some positive constant for all sufficiently large k. This contradicts the fact $\gamma_k \to 0$. Therefore, $\theta^k \to 0$ must hold. Assume that $(\hat x, \hat y)$ is an arbitrary accumulation point of (x^k, y^k), then by (3.10),

$$0 = \lim_{k\to\infty} \|G_{\theta^k}(x^k, y^k)\| = \|G_0(\hat x, \hat y)\|,$$

which implies that $(\hat x, \hat y)$ is a solution to the LCP. $\square$

Remark 3.2. We have pointed out that the global convergence of most existing non-interior-point methods for P_0 LCPs actually requires the boundedness assumption of the solution set, in which case the P_0 problem must have a strictly feasible point. In order to relax this requirement, Chen and Ye [12] designed a big-M smoothing method for P_0 LCPs. They proved that if the P_0 LCP has a solution and if certain condition such as "$\bar{x}_{n+2} - \bar{y}_{n+2} \neq -2\bar{\varepsilon}$" is satisfied at the accumulation point of their iterative sequence, then their algorithm is globally convergent. We note that Condition 2.1 in this paper is quite different from the Chen and Ye's. However, it is not clear what relation is between the two conditions.

While the global convergence for P_0 LCPs is proved under Condition 2.1, it should be pointed out that this condition is not necessary for the global convergence of P_* problems. We can prove that Algorithm 2.1 is globally convergent provided that the P_* LCP has a solution. Since this result cannot follow from Theorem 3.2, and since its proof is not straightforward, we postpone the discussion for this special case till the local convergence analysis for P_0 LCPs is complete.

4. Local behavior of the algorithm. Under a nondegeneracy assumption, we show in this section the local superlinear convergence of the algorithm when $p = 2 \leq q$. Let (x^*, y^*) be an accumulation point of the iterative sequence (x^k, y^k) generated by Algorithm 2.1. We make use of the following assumption that can be found also in [38, 39, 33].

CONDITION 4.1. *Assume that (x^*, y^*) is strictly complementary, i.e., $x^* + y^* > 0$, where $y^* = Mx^* + d$, and the matrix M_{II} is nonsingular, where $I = \{i : x_i^* > 0\}$.*

While this condition for local convergence has been used by several authors, it is stronger than some existing non-interior-point algorithms. Let M be a P_0 matrix. Under the above condition, it is easy to verify the nonsingularity of the matrix:

$$\nabla G_0(x^*, y^*) = \begin{pmatrix} I - W & I + W \\ -M & I \end{pmatrix}, \tag{4.1}$$

where $W = \mathrm{diag}(w)$ is a diagonal matrix with $w_i = 1$ if $x_i^* > 0$, and $w_i = -1$ otherwise. If Condition 4.1 is satisfied, it follows easily from Proposition 2.5 of Qi [32] that the solution (x^*, y^*) is a locally isolated solution. On the other hand, it is well-known that that a P_0 complementarity problem has a unique solution when it has a locally isolated solution (Jones and Gowda [23] and Gowda and Sznajder [17]). Thus, Condition 4.1 implies the uniqueness of the solution for a P_0 LCP, and hence it implies Condition 2.1. By Theorem 3.2, we conclude that under Conditions 4.1 the entire sequence (x^k, y^k), generated by Algorithm 2.1, converges to the unique solution of the P_0 LCP, i.e., $(x^k, y^k) \to (x^*, y^*)$. By continuity of ∇G_θ and nonsingularity of $\nabla G_0(x^*, y^*)$, there exists a local neighborhood of (x^*, y^*), denoted by $N(x^*, y^*)$, such that for all $(x, y) \in N(x^*, y^*)$ and all sufficiently small θ the matrix $\nabla G_\theta(x, y)$ is nonsingular, and there exists a constant C and $\hat{\theta} \in (0, 1)$ such that

$$\|\nabla G_\theta(x, y)^{-1}\| \leq C \text{ for all } (x, y) \in N(x^*, y^*) \text{ and } \theta \in (0, \hat{\theta}].$$

The following result is very useful for our local convergence analysis.

LEMMA 4.1. *Let M be a P_0 matrix. Under Condition 4.1, there exists a neighborhood $N(x^*, y^*)$ of (x^*, y^*) such that for all $(x^k, y^k) \in N(x^*, y^*)$ we have*

(i) $\|\nabla G_{\theta^k}(x^k, y^k) - \nabla G_0(x^k, y^k)\| \leq \kappa \max\{(\theta^k)^q, (\theta^k)^p\}$, *where κ is a constant.*

(ii) $G_0(x^k, y^k) - G_0(x^*, y^*) - \nabla G_0(x^k, y^k)[(x^k, y^k) - (x^*, y^*)] = 0$.

Proof. Let

$$I = \{i : x_i^* > 0\},\ J = \{j : y_j^* > 0\}.$$

Then, by strict complementarity, $I \cap J$ is empty and $I \cup J = \{1, 2, ..., n\}$. Denote

$$\eta = \frac{1}{2}\min\{\|x_I^*\|_\infty,\ \|y_J^*\|_\infty\}.$$

We show first the following inequality:

$$\|W - (X^k - Y^k)D^k\| \le \frac{2}{\eta^2}(\theta^k)^q\|a^q\|_\infty \text{ for all } (x^k, y^k) \in N(x^*, y^*), \tag{4.2}$$

where W is given as in (4.1) and D^k is defined as in (2.7). As we have pointed out, under Conditions 4.1 the sequence $\{(x^k, y^k)\}$ converges to (x^*, y^*). For all $(x^k, y^k) \in N(x^*, y^*)$, without loss of generality, we may assume that

$$x_i^k - y_i^k \ge \eta > 0 \ \text{ for } i \in I;\ \ -(x_i^k - y_i^k) \ge \eta > 0 \ \text{ for } i \in J.$$

Hence, when k is sufficiently large, for each $i \in I$ we have

$$\begin{aligned}
&|W_i - (x_i^k - y_i^k)d_i^k| \\
&= |1 - (x_i^k - y_i^k)d_i^k| \\
&= \frac{|\sqrt{(x_i^k - y_i^k)^2 + 4(\theta^k)^q a_i^q} - (x_i^k - y_i^k)|}{\sqrt{(x_i^k - y_i^k)^2 + 4(\theta^k)^q a_i^q}} \\
&= \frac{4(\theta^k)^q a_i^q}{\sqrt{(x_i^k - y_i^k)^2 + 4(\theta^k)^q a_i^q}\left(\sqrt{(x_i^k - y_i^k)^2 + 4(\theta^k)^q a_i^q} + x_i^k - y_i^k\right)} \\
&\le \frac{4(\theta^k)^q a_i^q}{\sqrt{\eta^2 + 4(\theta^k)^q a_i^q}\left(\sqrt{\eta^2 + 4(\theta^k)^q a_i^q} + \eta\right)} \\
&\le \frac{2}{\eta^2}(\theta^k)^q a_i^q. \qquad (4.3)
\end{aligned}$$

Similarly, for $j \in J$ we have

$$|W_j - (x_j^k - y_j^k)d_j^k| \le \frac{2}{\eta^2}(\theta^k)^q a_j^q,$$

which together with (4.3) yields the desired inequality (4.2). On the other hand, by strict complementarity, for every sufficiently large k it is evident that $(X^k - Y^k)\overline{D}^k = W$ where $\overline{D}^k = \text{diag}(\bar{d}^k)$ where $(\bar{d}^k)_i = 1/\sqrt{(x_i^k - y_i^k)^2}(i = 1, ..., n)$. Thus, for every sufficiently large k we have

$$\begin{aligned}
&\nabla G_0(x^k, y^k) - \nabla G_0(x^*, y^*) \\
&= \begin{pmatrix} I - (X^k - Y^k)\overline{D}^k & I + (X^k - Y^k)\overline{D}^k \\ -M & I \end{pmatrix} - \begin{pmatrix} I - W & I + W \\ -M & I \end{pmatrix} \\
&= \begin{pmatrix} W - (X^k - Y^k)\overline{D}^k & (X^k - Y^k)\overline{D}^k - W \\ 0 & 0 \end{pmatrix} \\
&= 0. \qquad (4.4)
\end{aligned}$$

By using (2.7), (4.2) and (4.4), for every sufficiently large k we have

$$\begin{aligned}\|\nabla G_{\theta^k}(x^k,y^k)-\nabla G_0(x^k,y^k)\| &= \|\nabla G_{\theta^k}(x^k,y^k)-\nabla G_0(x^*,y^*)\| \\ &= \left\|\begin{pmatrix} W-(X^k-Y^k)D^k & (X^k-Y^k)D^k-W \\ -(\theta^k)^p A^p & O\end{pmatrix}\right\| \\ &\le 2\|W-(X^k-Y^k)D^k\|+\|(\theta^k)^pA^p\| \\ &\le \frac{4}{\eta^2}(\theta^k)^q\|a^q\|_\infty+(\theta^k)^p\|a^p\|_\infty \\ &\le \kappa\max\{(\theta^k)^q,(\theta^k)^p\},\end{aligned}$$

where $\kappa=(4\|a^q\|_\infty)/\eta^2+\|a^p\|_\infty$ is a constant independent of k. Result (i) is proved.

We now prove the result (ii). By the strict complementarity and the definition of W, it is easy to see that for every sufficiently large k the following holds:

$$\begin{aligned}&x^k+y^k-\sqrt{(x^k-y^k)^2}=(I-W)(x^k-x^*)+(I+W)(y^k-y^*),\\ &y^k-Mx^k-d=-M(x^k-x^*)+y^k-y^*.\end{aligned}$$

Therefore, by using (4.4) and the above two equations, we have

$$\begin{aligned}&G_0(x^k,y^k)-G_0(x^*,y^*)-\nabla G_0(x^k,y^k)((x^k,y^k)-(x^*,y^*))\\ &=\begin{pmatrix} x^k+y^k-\sqrt{(x^k-y^k)^2} \\ y^k-Mx^k-d\end{pmatrix}-\begin{pmatrix} I-W & I+W \\ -M & I\end{pmatrix}\begin{pmatrix} x^k-x^* \\ y^k-y^*\end{pmatrix}\\ &=0,\end{aligned}$$

as desired. □

In the next result, we show that under Condition 4.1 the algorithm is at least locally superlinear. The key of the proof is to show that the algorithm eventually rejects the centering step and finally switches to step 2 when the iterate approaches the solution set.

THEOREM 4.1. *Let M be a P_0-matrix. Let $p=2\le q$ and $\beta>2\|a^{q/2}\|+\|(b,c)\|$. Assume that Condition 4.1 is satisfied. Then there exists a k_0 such that $\theta^{k+1}=(\theta^k)^2$ for all $k\ge k_0$, and*

$$\lim_{k\to\infty}\frac{\|x^{k+1}-x^*\|}{\|x^k-x^*\|}=0,$$

which implies that the algorithm is locally superlinearly convergent.

Proof. Let $N(x^*,y^*)$ be a neighborhood of (x^*,y^*) defined as in Lemma 4.1. We first show that for all $(x^k,y^k)\in N(x^*,y^*)$, there exists a constant $\delta>0$ such that

$$\|(\hat{x}^{k+1},\hat{y}^{k+1})-(x^*,y^*)\|\le\delta\max\{(\theta^k)^q,(\theta^k)^p\}\|(x^k,y^k)-(x^*,y^*)\|.$$

As we have pointed out, Condition 4.1 implies that $(x^k,y^k)\to(x^*,y^*)$, and there exist constants C and $\hat{\theta}$ such that

$$\|\nabla G_{\theta^k}(x^k,y^k)^{-1}\|\le C$$

for all $(x^k,y^k)\in N(x^*,y^*)$ and $\theta^k\in(0,\hat{\theta}]$. Therefore, for all sufficiently large k, by Lemma 4.1 we have

$$\|(\hat{x}^{k+1},\hat{y}^{k+1})-(x^*,y^*)\|$$

$$
\begin{aligned}
&= \|(x^k, y^k) - (x^*, y^*) - \nabla G_{\theta^k}(x^k, y^k)^{-1} G_0(x^k, y^k)\| \\
&= \|\nabla G_{\theta^k}(x^k, y^k)^{-1}\{[\nabla G_{\theta^k}(x^k, y^k) + \nabla G_0(x^k, y^k)]((x^k, y^k) - (x^*, y^*)) \\
&\quad - G_0(x^k, y^k) + G_0(x^*, y^*) - \nabla G_0(x^k, y^k)((x^k, y^k) - (x^*, y^*))\}\| \\
&\le \|\nabla G_{\theta^k}(x^k, y^k)^{-1}[\nabla G_{\theta^k}(x^k, y^k) - \nabla G_0(x^k, y^k)]((x^k, y^k) - (x^*, y^*))\| \\
&\quad + \|\nabla G_{\theta^k}(x^k, y^k)^{-1}[G_0(x^k, y^k) - G_0(x^*, y^*) - \nabla G_0(x^k, y^k)((x^k, y^k) - (x^*, y^*))]\| \\
&\le C\|\nabla G_{\theta^k}(x^k, y^k) - \nabla G_0(x^k, y^k)\|\|(x^k, y^k) - (x^*, y^*)\| \\
&\le C\kappa \max\{(\theta^k)^q, (\theta^k)^p\}\|(x^k, y^k) - (x^*, y^*)\|.
\end{aligned}
$$

Set $\delta = C\kappa$. The desired inequality follows. The above inequality implies that the sequence $(\hat{x}^{k+1}, \hat{y}^{k+1})$ also converges to (x^*, y^*). Notice that $\theta^k \to 0$ (by Theorem 3.2). To show the local superlinear convergence of Algorithm 2.1, the above inequality implies that it is sufficient to show that the algorithm eventually takes the approximate Newton step alone.

Since (x^*, y^*) is a strictly complementary solution, $G_0(x, y)$ is continuously differentiable in the neighborhood of (x^*, y^*), and thus it must be Lipschitzian in the neighborhood of (x^*, y^*). Hence, there exists a constant $L > 0$ such that for all sufficiently large k

$$
\begin{aligned}
\|G_0(\hat{x}^{k+1}, \hat{y}^{k+1}) - G_0(x^*, y^*)\| &\le L\|(\hat{x}^{k+1}, \hat{y}^{k+1}) - (x^*, y^*)\| \\
&\le L\delta \max\{(\theta^k)^q, (\theta^k)^p\}\|(x^k, y^k) - (x^*, y^*)\| \\
&= \tau_k \max\{(\theta^k)^q, (\theta^k)^p\}.
\end{aligned}
$$

where $\tau_k = L\delta\|(x^k, y^k) - (x^*, y^*)\| \to 0$ as $k \to \infty$. That is,

$$
\|G_0(\hat{x}^{k+1}, \hat{y}^{k+1})\| \le \tau_k \max\{(\theta^k)^q, (\theta^k)^p\} \tag{4.5}
$$

for all sufficiently large k. Setting $(\mu_1, \mu_2) = (\mu, 0)$ in (2.8), where $\mu \in (0, 1)$, we see from the first inequality in (2.8) that

$$
\|G_\mu(x, y) - G_0(x, y)\| \le \mu\|(2\mu^{q/2-1}a^{q/2}, \mu^{p-1}A^p x)\| \text{ for all } (x, y) \in R^{2n}. \tag{4.6}
$$

Thus, by using (4.5) and (4.6), for all sufficiently large k we have

$$
\begin{aligned}
&\|G_{(\theta^k)^2}(\hat{x}^{k+1}, \hat{y}^{k+1}) - (\theta^k)^2(b, c)\| \\
&\le \|G_{(\theta^k)^2}(\hat{x}^{k+1}, \hat{y}^{k+1}) - G_0(\hat{x}^{k+1}, \hat{y}^{k+1})\| + \|G_0(\hat{x}^{k+1}, \hat{y}^{k+1})\| \\
&\quad + (\theta^k)^2\|(b, c)\| \\
&\le (\theta^k)^2\|(2[(\theta^k)^2]^{q/2-1}a^{q/2}, [(\theta^k)^2]^{p-1}A^p\hat{x}^{k+1}\|) + \tau_k \max\{(\theta^k)^q, (\theta^k)^p\} \\
&\quad + (\theta^k)^2\|(b, c)\| \\
&\le (\theta^k)^2(2\|a^{q/2}\| + \|(b, c)\| + [(\theta^k)^2]^{p-1}\|A^p\hat{x}^{k+1}\|) + \tau_k \max\{(\theta^k)^q, (\theta^k)^p\} \\
&= \beta(\theta^k)^2\Big[\frac{2\|a^{q/2}\| + \|(b, c)\|}{\beta} + \frac{[(\theta^k)^2]^{p-1}\|A^p\hat{x}^{k+1}\|}{\beta} \\
&\quad + \frac{\tau_k \max\{(\theta^k)^{q-2}, (\theta^k)^{p-2}\}}{\beta}\Big] \\
&\le \beta(\theta^k)^2. \qquad (4.7)
\end{aligned}
$$

The third inequality follows from that $q \ge 2$ and $[(\theta^k)^2]^{q/2-1} \le 1$. The last inequality follows from the fact that $p = 2 \le q$, $\beta > 2\|a^{q/2}\| + \|(b, c)\|$, $\tau_k \to 0$ and

$$
\lim_{k \to 0} \frac{[(\theta^k)^2]^{p-1}\|A^p\hat{x}^{k+1}\|}{\beta} = 0.
$$

Thus, from (4.7), the approximate Newton step is accepted at kth step provided that k is a large number. Therefore, the next iterate $(x^{k+1}, y^{k+1}) = (\hat{x}^{k+1}, \hat{y}^{k+1})$. Repeating the above proof, we can see that at the $(k+1)$th step $(x^{k+2}, y^{k+2}) = (\hat{x}^{k+2}, \hat{y}^{k+2})$, i.e., the approximate Newton step is still accepted at $(k+1)$th step. By induction, we conclude that the algorithm eventually takes only the approximate Newton step. Hence, for some k_0, we have $\theta^{k+1} = (\theta^k)^2$ for all $k \geq k_0$, and $\lim_{k\to 0} \|x^{k+1} - x^*\|/\|x^k - x^*\| = 0$. □

The proof above shows that if an iterate (x^k, y^k) lies in a sufficiently small neighborhood of (x^*, y^*), then the next iterate still falls in this neighborhood, and much closer to the solution (x^*, y^*) than (x^k, y^k). Since the centering step is gradually phased out and only approximate Newton steps are executed at the end of iteration, the superlinear convergence of the algorithm can be achieved.

5. Special cases. In this section, we show some much deeper global convergence results than Theorem 3.2 when the algorithm is applied to P_* LCPs. For the special case, the only assumption to assure the global convergence is the nonemptyness of the solution set. In other words, this algorithm is able to solve any P_* LCP provided that a solution exists. For a given LCP, we denote by

$$I = \{i : x_i^* > 0 \text{ for some solution } x^*\}, \tag{5.1}$$

$$J = \{j : (Mx^* + d)_j > 0 \text{ for some solution } x^*\}, \tag{5.2}$$

$$K = \{k : x_k^* = (Mx^* + d)_k = 0 \text{ for all solution } x^*\}. \tag{5.3}$$

The above partition of the set $\{1, 2, ..., n\}$ is unique for a given P_* LCP. Consider the affine set:

$$\overline{S} = \{(x, y) \in R^{2n} : \ x_{J\cup K} = 0, \ y_{I\cup K} = 0, \ y = Mx + d\}.$$

In fact, $\overline{S}$ is the affine hull of the solution set of the LCP, i.e., the smallest affine set containing the solution set. For any $(\tilde{x}, \tilde{y}) \in \overline{S}$, it is easy to see that $\tilde{x}_i \tilde{y}_i = 0$ for all $i = 1, ..., n$. We now prove a very useful result.

LEMMA 5.1. *Let $(\tilde{x}, \tilde{y})$ be an arbitrary vector in $\overline{S}$. Let M be a P_*-matrix. Let $\{(x^k, y^k, \theta^k)\}$ be generated by Algorithm 2.1 and $(\bar{x}^k, \bar{y}^k)$ be defined by (3.7). Then*

$$\begin{aligned} \tilde{x}^T \bar{y}^k + \tilde{y}^T \bar{x}^k \leq (\theta^k)^q (1 + \tau n) e^T a^q - \tau n \left(\min_{1\leq i\leq n} \rho_i^k \right) - (\bar{x}^k - \tilde{x})^T [(\theta^k)^p A^p \bar{x}^k \\ + \theta^k (c + c^k) + \frac{1}{2}\theta^k (M - I + (\theta^k)^p A^p)(b + b^k)], \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} \rho_i^k = {} & \bar{x}_i^k \tilde{y}_i + \tilde{x}_i \bar{y}_i^k + (\bar{x}_i^k - \tilde{x}_i)\{(\theta^k)^p a_i^p \bar{x}_i^k + \theta^k (c_i + c_i^k) \\ & + \frac{1}{2}\theta^k [(M - I + (\theta^k)^p A^p)(b + b^k)]_i\}. \end{aligned}$$

Proof. Since $(\bar{x}^k, \bar{y}^k) > 0$ and $\tilde{x}_i \tilde{y}_i = 0$ for all $i = 1, ..., n$, by (3.8) we have

$$\begin{aligned} (\bar{x}_i^k - \tilde{x}_i)(\bar{y}_i^k - \tilde{y}_i) &= \bar{x}_i^k \bar{y}_i^k - \bar{x}_i^k \tilde{y}_i - \tilde{x}_i \bar{y}_i^k + \tilde{x}_i \tilde{y}_i \\ &= (\theta^k)^q a_i^q - \bar{x}_i^k \tilde{y}_i - \tilde{x}_i \bar{y}_i^k. \end{aligned}$$

It is easy to verify that

$$\bar{y}^k = M\bar{x}^k + d + (\theta^k)^p A^p \bar{x}^k + \theta^k (c + c^k) + \frac{1}{2}\theta^k (M - I + (\theta^k)^p A^p)(b + b^k).$$

Thus, we have

$$
\begin{aligned}
(\bar{x}_i^k - \tilde{x}_i)[M(\bar{x}^k - \tilde{x})]_i &= (\bar{x}_i^k - \tilde{x}_i)[(M\bar{x}^k + d)_i - \tilde{y}_i] \\
&= (\bar{x}_i^k - \tilde{x}_i)\{\bar{y}_i^k - (\theta^k)^p a_i^p \bar{x}_i^k - \theta^k (c_i + c_i^k) \\
&\quad - \frac{1}{2}\theta^k [(M - I + (\theta^k)^p A^p)(b + b^k)]_i - \tilde{y}_i\} \\
&= (\bar{x}_i^k - \tilde{x}_i)(\bar{y}_i^k - \tilde{y}_i) - (\bar{x}_i^k - \tilde{x}_i)\{(\theta^k)^p a_i^p \bar{x}_i^k \\
&\quad + \theta^k (c_i + c_i^k) + \frac{1}{2}\theta^k [(M - I + (\theta^k)^p A^p)(b + b^k)]_i\} \\
&\le (\theta^k)^q a_i^q - \bar{x}_i^k \tilde{y}_i - \tilde{x}_i \bar{y}_i^k - (\bar{x}_i^k - \tilde{x}_i)\{(\theta^k)^p a_i^p \bar{x}_i^k \\
&\quad + \theta^k (c_i + c_i^k) + \frac{1}{2}\theta^k [(M - I + (\theta^k)^p A^p)(b + b^k)]_i\} \\
&\le (\theta^k)^q e^T a^q - \min_{1 \le i \le n} \rho_i^k, \qquad (5.5)
\end{aligned}
$$

where,

$$
\begin{aligned}
\rho_i^k = {} & \bar{x}_i^k \tilde{y}_i + \tilde{x}_i \bar{y}_i^k + (\bar{x}_i^k - \tilde{x}_i)\{(\theta^k)^p a_i^p \bar{x}_i^k + \theta^k (c_i + c_i^k) \\
& + \frac{1}{2}\theta^k [(M - I + (\theta^k)^p A^p)(b + b^k)]_i\}.
\end{aligned}
$$

Therefore, by (3.8), (5.5) and the definition of the P_* matrix, we have

$$
\begin{aligned}
\tilde{x}^T \bar{y}^k + \tilde{y}^T \bar{x}^k &= -(\bar{x}^k - \tilde{x})^T (\bar{y}^k - \tilde{y}) + (\bar{x}^k)^T \bar{y}^k \\
&= -(\bar{x}^k - \tilde{x})^T [M\bar{x}^k + d + (\theta^k)^p A^p \bar{x}^k + \theta^k (c + c^k) \\
&\quad + \frac{1}{2}\theta^k (M - I + (\theta^k)^p A^p)(b + b^k) - \tilde{y}] + (\theta^k)^q e^T a^q \\
&= -(\bar{x}^k - \tilde{x})^T (M\bar{x}^k + d - \tilde{y}) - (\bar{x}^k - \tilde{x})^T [(\theta^k)^p A^p \bar{x}^k + \theta^k (c + c^k) \\
&\quad + \frac{1}{2}\theta^k (M - I + (\theta^k)^p A^p)(b + b^k)] + (\theta^k)^q e^T a^q \\
&= -(\bar{x}^k - \tilde{x})^T M(\bar{x}^k - \tilde{x}) - (\bar{x}^k - \tilde{x})^T [(\theta^k)^p A^p \bar{x}^k + \theta^k (c + c^k) \\
&\quad + \frac{1}{2}\theta^k (M - I + (\theta^k)^p A^p)(b + b^k)] + (\theta^k)^q e^T a^q \\
&\le \tau \sum_{i \in I_+} (\bar{x}_i^k - \tilde{x}_i)[M(\bar{x}^k - \tilde{x})]_i - (\bar{x}^k - \tilde{x})^T [(\theta^k)^p A^p \bar{x}^k \\
&\quad + \theta^k (c + c^k) + \frac{1}{2}\theta^k (M - I + (\theta^k)^p A^p)(b + b^k)] + (\theta^k)^q e^T a^q \\
&\le \tau n \left((\theta^k)^q e^T a^q - \min_{1 \le i \le n} \rho_i^k \right) - (\bar{x}^k - \tilde{x})^T [(\theta^k)^p A^p \bar{x}^k \\
&\quad + \theta^k (c + c^k) + \frac{1}{2}\theta^k (M - I + (\theta^k)^p A^p)(b + b^k)] + (\theta^k)^q e^T a^q \\
&= (\theta^k)^q (1 + \tau n) e^T a^q - \tau n \left(\min_{1 \le i \le n} \rho_i^k \right) - (\bar{x}^k - \tilde{x})^T [(\theta^k)^p A^p \bar{x}^k \\
&\quad + \theta^k (c + c^k) + \frac{1}{2}\theta^k (M - I + (\theta^k)^p A^p)(b + b^k)].
\end{aligned}
$$

The proof is complete. ◻

The following result shows that under a suitable choice of parameters our algorithm can locate a solution of the P_* LCP as long as a solution exists.

THEOREM 5.1 *Let M be a P_*-matrix. Assume that the solution set of the P_* LCP is nonempty. If one of the following holds,*

(i) $p \le 1$,

(ii) $p > 1, c > \frac{1}{2}\|(M-I)b\|e$, *and* $0 < \beta < \min_{1\le i\le n} \frac{c_i-(1/2)\|(M-I)b\|}{1+\|M-I\|/2}$,

then the sequence $\{(x^k, y^k, \theta^k)\}$, *generated by Algorithm 2.1, is bounded, and*

$$\lim_{k\to\infty} \theta^k \to 0, \ \lim_{k\to\infty} \|G_{\theta^k}(x^k, y^k)\| = 0.$$

Therefore, any accumulation point of (x^k, y^k) *is a solution to the LCP.*

Proof. We focus on the proof of the boundedness of $\{(x^k, y^k)\}$. Let (x^*, y^*) be an arbitrary solution to the LCP. Set $(\tilde{x}, \tilde{y}) = (x^*, y^*)$ in Lemma 5.1. Since for this case $\bar{y}_i^k x_i^* + \bar{x}_i^k y_i^* \ge 0$, we have that

$$\rho_i^k \ge \eta_i^k := (\bar{x}_i^k - x_i^*)\left\{(\theta^k)^p a_i^p \bar{x}_i^k + \theta^k(c_i + c_i^k) + \frac{1}{2}\theta^k[(M - I + (\theta^k)^p A^p)(b + b^k)]_i\right\}.$$

This, together with (5.4), implies that

$$\begin{aligned}(x^*)^T\bar{y}^k + (y^*)^T\bar{x}^k &\le (\theta^k)^q(1+\tau n)e^Ta^q - \tau n\left(\min_{1\le i\le n}\eta_i^k\right) - (\bar{x}^k - x^*)^T\{(\theta^k)^pA^p\bar{x}^k \\ &\quad +\theta^k(c + c^k) + \frac{1}{2}\theta^k(M - I + (\theta^k)^pA^p)(b + b^k)\}. \qquad (5.6)\end{aligned}$$

Dividing both sides of the above by $(\theta^k)^p$ and noting that the left-hand side is nonnegative, we have

$$\begin{aligned}&(\bar{x}^k - x^*)^TA^p\bar{x}^k + \tau n\left(\min_{1\le i\le n}\frac{\eta_i^k}{(\theta^k)^p}\right) \\ &+(\theta^k)^{1-p}(\bar{x}^k - x^*)^T\left[c + c^k + \frac{1}{2}(M - I + (\theta^k)^pA^p)(b + b^k)\right] \\ &\le (\theta^k)^{q-p}(1+\tau n)e^Ta^q. \qquad (5.7)\end{aligned}$$

If $p \le 1$, the right-hand side of the above inequality is bounded since $q \ge 1$ and $\theta_k < 1$. This implies that the sequence $\{\bar{x}^k\}$ is bounded (otherwise the left-hand side is unbounded from above), and thus $\{x^k\}$ is bounded. So is $\{y^k\}$ by (3.6). The boundedness of $\{(x^k, y^k)\}$ under (i) is proved.

We now prove the boundedness of (x^k, y^k) in the case (ii). Consider two subcases.

Subcase 1: $\theta_k \not\to 0$. In this case, there exists a constant $\hat{\theta} > 0$ such that $1 > \theta^k \ge \hat{\theta}$. It is easy to see from (5.7) that the sequence $\{\bar{x}^k\}$ is bounded, and thus (x^k, y^k) is bounded.

Subcase 2: $\theta_k \to 0$. In this case, by the choice of p, β and c, it is easy to see that

$$\begin{aligned}c_i + c_i^k + \frac{1}{2}[(M-I)(b+b^k)]_i &> c_i - \beta - \frac{1}{2}\|(M-I)(b+b^k)\| \\ &\ge c_i - \beta - \frac{1}{2}\|(M-I)b\| - \frac{1}{2}\|(M-I)b^k\| \\ &\ge c_i - \frac{1}{2}\|(M-I)b\| - \beta\left(1 + \frac{1}{2}\|M-I\|\right) \\ &> 0. \qquad (5.8)\end{aligned}$$

Since $\theta_k \to 0$, for all sufficiently large k it follows that

$$c_i + c_i^k + \frac{1}{2}[(M - I + (\theta_k)^p A^p)(b + b^k)]_i > 0.$$

Thus, for all sufficiently large k we have

$$\begin{aligned} \frac{\eta_i^k}{\theta^k} &\geq -x_i^* \left\{ (\theta^k)^{p-1} a_i^p \bar{x}_i^k + c_i + c_i^k + \frac{1}{2}[(M - I + (\theta^k)^p A^p)(b + b^k)]_i \right\} \\ &\geq -(\theta^k)^{p-1} (x^*)^T A^p \bar{x}^k \\ &\quad - \max_{1 \leq i \leq n} x_i^* \left\{ c_i + c_i^k + \frac{1}{2}[(M - I + (\theta^k)^p A^p)(b + b^k)]_i \right\}. \end{aligned} \tag{5.9}$$

Since the left-hand side of (5.6) is nonnegative, dividing both sides of (5.6) by θ^k and using (5.9), we have

$$\begin{aligned} 0 &\leq (\theta^k)^{q-1}(1 + \tau n) e^T a^q - \tau n \left(\min_{1 \leq i \leq n} \frac{\eta_i^k}{\theta^k} \right) + (\theta^k)^{p-1} (x^*)^T A^p \bar{x}^k \\ &\quad -(\bar{x}^k - x^*)^T (c + c^k) - \frac{1}{2}(\bar{x}^k - x^*)^T (M - I + (\theta^k)^p A^p)(b + b^k) \\ &\leq (\theta^k)^{q-1}(1 + \tau n) e^T a^q + \tau n \max_{1 \leq i \leq n} x_i^* \left\{ c_i + c_i^k + \frac{1}{2}[(M - I + (\theta^k)^p A^p)(b + b^k)]_i \right\} \\ &\quad +(\theta^k)^{p-1}(1 + \tau n)(x^*)^T A^p \bar{x}^k - (\bar{x}^k - x^*)^T (c + c^k) \\ &\quad -\frac{1}{2}(\bar{x}^k - x^*)^T (M - I + (\theta^k)^p A^p)(b + b^k). \end{aligned}$$

It follows that

$$\begin{aligned} &(\bar{x}^k)^T \left[c + c^k - (\theta^k)^{p-1}(1 + \tau n) A^p x^* + \frac{1}{2}(M - I + (\theta^k)^p A^p)(b + b^k) \right] \\ &\leq (\theta^k)^{q-1}(1 + \tau n) e^T a^q + \tau n \max_{1 \leq i \leq n} x_i^* \Big\{ c_i + c_i^k + \frac{1}{2}[(M - I \\ &\quad +(\theta^k)^p A^p)(b + b^k)]_i \Big\} + (x^*)^T \left[c + c^k + \frac{1}{2}(M - I + (\theta^k)^p A^p)(b + b^k) \right]. \end{aligned} \tag{5.10}$$

Since $p > 1$ and $\theta^k \to 0$, by a proof similar to (5.8), for all sufficiently large k we have

$$\begin{aligned} &c + c^k - (\theta^k)^{p-1}(1 + \tau n) A^p x^* + \frac{1}{2}(M - I + (\theta^k)^p A^p)(b + b^k) \\ &\geq \frac{1}{2} \left\{ c - \frac{1}{2}\|(M - I)b\| e - \beta \left(1 + \frac{1}{2}\|M - I\| \right) e \right\} \\ &> 0. \end{aligned}$$

Since the right-hand side of (5.10) is bounded and $\bar{x}^k > 0$, from the above inequality and (5.10) it follows that $\{\bar{x}^k\}$ is bounded, and hence (x^k, y^k) is bounded.

Based on the boundedness of $\{(x^k, y^k)\}$, repeating the proof of Theorem 3.2 we can prove that $\theta^k \to 0$. □

Remark 5.1. It is worth mentioning the difference between (i) and (ii) of the above theorem. In the case (i), there is no restriction on the parameter $\beta > 0$. Thus, β can be assigned a large number so that the neighborhood is wide enough to ensure a large steplength at each iteration. For the case (ii), however, the parameter β is required to be relatively small. To satisfy this requirement, the initial point of Algorithm 2.1 can be also obtained easily. For example, set $x^0 = 0, a \in R^n_{++}, \theta^0 \in (0,1)$, and choose $y^0 \in R^n_{++}$ to be large enough such that $c > \frac{1}{2}\|(M-I)b\|e$ where

$$b = \frac{x^0 + y^0 - \sqrt{(x^0-y^0)^2 + 4(\theta^0)^q a^q}}{\theta^0} = \frac{4(\theta^0)^q a^q}{y^0 + \sqrt{(y^0)^2 + 4(\theta^0)^q a^q}},$$

$$c = \frac{y^0 - (f(x^0) + (\theta^0)^p A^p x^0)}{\theta^0} = \frac{y^0 - f(0)}{\theta^0}.$$

The above choice implies that $\|G_{\theta^0}(x^0, y^0)\| = 0$. Thus, $(x^0, y^0) \in \mathcal{N}(\beta, \theta^0)$ for any $\beta > 0$. In particular, β can be taken such that

$$0 < \beta < \min_{1 \le i \le n} \frac{c_i - (1/2)\|(M-I)b\|}{1 + \|M-I\|/2}.$$

In the rest of this section, we characterize the accumulation point of the sequence $\{(x^k, y^k)\}$. We first recall some concepts. Let S denote the solution set of the LCP. An element x^* of S is said to be the *N-norm least solution,* where N is a positive, definite, symmetric matrix, if $\|N^{1/2}x^*\| \le \|N^{1/2}u\|$ for all $u \in S$. In particular, if $N = I$, the solution x^* is called the *least 2-norm solution* of S. An element x^* of S is said to be the *least element* of S if $x^* \le u$ for all $u \in S$ (see, for example, [30, 13]). The solution x^* is called a maximally complementary solution if $x_i^* > 0$ for all $i \in I$, $(Mx^* + d)_i > 0$ for all $i \in J$ and $x_i^* = (Mx^* + d)_i = 0$ for all $i \in K$. Clearly, a strictly complementary solution is a maximally complementary solution with $K = \emptyset$.

THEOREM 5.2. *Let M be a P_* matrix. Assume that the solution set of the LCP is nonempty.*

(i) If $p < 1$, then every accumulation point $(\hat{x}, \hat{y})$ of the sequence (x^k, y^k) satisfies the following property: For any solution x^*, there exists a corresponding index i_0 such that

$$(\hat{x})^T A^p (\hat{x} - x^*) + \tau n a_{i_0}^p \hat{x}_{i_0} (\hat{x}_{i_0} - x^*_{i_0}) \le 0. \tag{5.11}$$

Moreover, if the least element solution exists, then the entire sequence (x^k, y^k) is convergent, and its accumulation point coincides with the least element solution.

(ii) If $p > 1, c > \frac{1}{2}\|(M-I)b\|e$, $0 < \beta < \lim_{1 \le i \le n} \frac{c_i - (1/2)\|(M-I)b\|}{1 + \|M-I\|/2}$, and $q = 1$, then each accumulation point is a maximally complementary solution of the LCP.

Proof. For $p < 1$, by the result (i) of Theorem 5.1, $\{x^k\}$ is bounded and $\theta^k \to 0$. Let $(\hat{x}, \hat{y})$ be an arbitrary accumulation point of $\{(x^k, y^k)\}$. Taking the limit in (5.7) where x^* is an arbitrary solution of the LCP, we see that there exists an index i_0 such that

$$(\hat{x})^T A^p (\hat{x} - x^*) + \tau n a_{i_0}^p \hat{x}_{i_0} (\hat{x}_{i_0} - x^*_{i_0}) \le 0.$$

Moreover, if the least element solution exists, setting x^* to be the least element, we conclude from the above inequality that $\hat{x}$ is equal to the least element. Since such an element is unique, the sequence $\{x^k\}$ is convergent.

We now consider the case (ii). By result (ii) of Theorem 5.1, the sequence (x^k, y^k) is bounded, $\theta^k \to 0$, and each accumulation point of (x^k, y^k) is a solution to the LCP.

Let (x^*, y^*) be a maximally complementary solution and I, J, K be defined by (5.1)-(5.3). Then we have

$$\begin{aligned}(x^*)^T\bar{y}^k + (y^*)^T\bar{x}^k &= (x_I^*)^T\bar{y}_I^k + (y_J^*)^T\bar{x}_J^k \\ &= (x_I^*)^T(\bar{X}_I^k)^{-1}\bar{X}_I^k\bar{y}_I^k + (y_J^*)^T(\bar{Y}_J^k)^{-1}\bar{Y}_J^k\bar{x}_J^k \\ &= (\theta^k)^q\left[(x_I^*)^T(\bar{X}_I^k)^{-1}a_I^q + (y_J^*)^T(\bar{Y}_J^k)^{-1}a_J^q\right].\end{aligned}$$

By (5.6) and the above inequality, we have

$$\begin{aligned}&(x_I^*)^T(\bar{X}_I^k)^{-1}a_I^q + (y_J^*)^T(\bar{Y}_J^k)^{-1}a_J^q \\ &\le (1+\tau n)e^Ta^q - \tau n\left(\min_{1\le i\le n}\frac{\eta_i^k}{(\theta^k)^q}\right) - (\bar{x}^k - x^*)^T[(\theta^k)^{p-q}A^p\bar{x}^k \\ &\quad +(\theta^k)^{1-q}(c+c^k) + \frac{1}{2}(\theta^k)^{1-q}(M - I + (\theta^k)^pA^p)(b+b^k)].\end{aligned}$$

Let $(\hat{x}, \hat{y})$ be an arbitrary accumulation point of the iterates. Since $\theta^k \to 0$ and $p > 1 = q$, we can see that $\eta_i^k/(\theta^k)^q$ is bounded. The right-hand side of the above inequality is bounded. Since $(x_I^*, y_J^*) > 0$, we conclude that $\bar{x}_I^k \to \hat{x}_I > 0$; otherwise, if $\hat{x}_i = 0$ for some $i \in I$, then $x_i^*/\bar{x}_i \to \infty$, and hence the left-hand side tends to infinity, contradicting the boundedness of the right-hand side. By a similar way, we have that $\hat{y}_I > 0$. Thus, $(\hat{x}, \hat{y})$ is a maximally complementary solution. □

Since every positive semi-definite matrix is a P_*-matrix with $\tau = 0$, the result (i) above can be further improved for monotone LCPs. In fact, from Theorem 2.3, the following result is natural since the algorithm follows the regularized central path approximately.

THEOREM 5.3. *Let M be a positive semi-definite matrix. Assume that the solution set of the LCP is nonempty. For $p < 1$, the entire sequence (x^k, y^k), generated by Algorithm 2.1, converges to $(\hat{x}, \hat{y})$ where $\hat{x}$ is the least N-norm solution with $N = A^{p/2}$. In particular, if $a = e$ is taken, the sequence converges to the (unique) least 2-norm solution.*

Proof. For the case of $p < 1$, setting $\tau = 0$ in (5.11) we have

$$(\hat{x})^TA^p(\hat{x} - x^*) \le 0,$$

which implies that $\|A^{p/2}\hat{x}\| \le \|A^{p/2}x^*\|$. Since x^* is an arbitrary solution, it follows that the solution $\hat{x}$ is the least N-norm solution where $N = A^{p/2}$. It is also easy to see from the above inequality that the solution $\hat{x}$ is unique, and thus the entire sequence is convergent. □

Remark 5.2. For P_* LCPs, the boundedness assumption of the solution set (or the strict feasibility condition) is not required for the global convergence of our algorithm. Further, all results in this section can be easily extended to nonlinear P_* complementarity problems. We notice that Ye's homogeneous model [41] for monotone LCPs, which was later generalized to nonlinear monotone complementarity problems by Andersen and Ye [2], also does not require the boundedness of the solution set (or the strict feasibility) of the original problem. However, it is unknown whether the Ye's algorithm can be generalized to the nonlinear P_* problems.

6. Numerical examples. Algorithm 2.1 were tested on some LCPs, nonlinear complementarity problems (NCPs), and nonlinear programming problems (NLPs) which can be written as complementarity problems by KKT optimality conditions. For all test examples, common parameters and initial points were used in our algorithm.

From the analysis of section 4 and our experiments, the value of parameters p and q should be relatively large for the sake of rapid convergence. The constant σ should be taken relatively small such that a possible large steplength λ_k can be taken. The vector $(a, b, c) \in R^n_{++} \times R^{2n}$ and the initial point $(x^0, y^0) \in R^{2n}$ can be chosen freely. In general cases, the value of β should be taken relatively large to ensure that the neighborhood is wide enough to permit a large iterative steplength. Thus, the parameters used in our code were set as $p = 2,\ q = 3,\ \sigma = 0.001$ and $\alpha = 0.9$. The vectors $a,\ b,\ c$ were set as $a = b = c = e$. The initial values of (x^0, y^0, θ^0) were set as $\theta^0 = 0.9$ and $x^0 = y^0 = e$. The parameter β was given by

$$\beta = \frac{\|G_{\theta^0}(x^0, y^0) - \theta^0(b, c)\|}{\theta^0} + 100.$$

Since $G_0(x^*, y^*) = 0$ if and only if (x^*, y^*) is a solution to the complementarity problem, we use $\|G_0(x^k, y^k)\| < \varepsilon$ as the stopping criterion, where $\varepsilon > 0$ is a given tolerance. In our experiments, $\varepsilon = 10^{-14}$ was taken for all numerical examples. All results were undertaken on a DEC Alpha V4.0 workstation by Fortran 90, and all the arithmetic operations were performed in double precision for precaution of round-off errors. We recorded the following aspects to examine the effectiveness of the algorithm: The dimension of problems, the total number of iterations, the total number of functions called, the CPU time used, the final value of θ^k, and the residual, i.e, the final value of $\|G_0(x^k, y^k)\|$. All CPU times reported here include time for input and output. We now introduce test examples and give out the numerical results for them.

Linear complementarity problems:

LCP1. This is the Watson's first problem [37].

LCP2. This is the Watson's second problem [37].

LCP3. The matrix M_1 is a P_*-matrix given in (6.1), and $d = -e$. The solution set is unbounded. There is no strictly feasible point for this LCP. The central path does not exist for this problem. However, Algorithm 2.1 deals with this problem very efficiently.

LCP4. This is a P_0 LCP given by Chen and Ye [12]. The matrix M_2 is given in (6.1), and $d = (0, 0, 1)$. The solution set is unbounded.

LCP5. This is a P_0 LCP with the matrix M_3 given in (6.1), and $d = (0, 0, 1)$. This problem has no strictly feasible point, and its solution set is unbounded.

$$(6.1)\quad M_1 = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ -2 & -1 & 0 & 0 \\ 4 & 8 & 0 & 0 \end{pmatrix},\ M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix},\ M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 1 \end{pmatrix}.$$

LCP6. This example was given by Fathi [15]. The matrix M_4 is given in (6.2) and the vector $d = -e$.

LCP7. This example was given by Ahn [1]. The vector $d = -e$, and the matrix M_5 is given in (6.2),

$$(6.2)\qquad M_4 = \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 2 & 5 & 6 & \dots & 6 \\ 2 & 6 & 9 & \dots & 10 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 6 & 9 & \dots & 4n-3 \end{pmatrix},\ M_5 = \begin{pmatrix} 4 & -2 & 0 & \dots & 0 \\ 1 & 4 & -2 & \dots & 0 \\ 0 & 1 & 4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 4 \end{pmatrix}.$$

TABLE 6.1
LCPs, ε=1e−14.

Problems	Dim.	No. of Iter.	No. of fun.	θ^k	Residual	CPU (sec.)
LCP1	10	8	9	1.4e−06	2.4e−15	0.00
LCP2	5	9	10	1.9e−12	1.6e−15	0.00
LCP3	4	8	9	1.4e−06	2.1e−16	0.00
LCP4	3	8	9	1.4e−06	5.5e−17	0.00
LCP5	3	8	9	1.4e−06	7.8e−17	0.00
LCP6	300	12	19	1.9e−16	9.9e−16	6.53
LCP6	500	12	19	1.9e−16	8.3e−16	42.75
LCP7	300	8	9	1.4e−06	2.3e−15	4.41
LCP7	500	8	9	1.4e−06	3.0e−15	29.16
LCP8	300	8	9	1.4e−06	2.7e−15	4.54
LCP8	500	8	9	1.4e−06	3.6e−15	28.52
LCP9	300	10	13	1.9e−16	0.0	5.66
LCP9	500	10	13	1.9e−16	0.0	40.12
LCP10	300	10	13	3.3e−13	5.6e−15	5.50
LCP10	500	11	16	2.2e−16	8.1e−15	40.57
LCP11	300	9	10	1.9e−12	4.4e−18	6.08
LCP11	500	9	10	1.9e−12	3.7e−18	35.83
LCP12	300	9	10	2.1e−12	2.4e−16	5.61
LCP12	500	9	10	2.0e−12	5.5e−16	34.08
LCP13	300	10	13	1.9e−16	1.4e−17	6.38
LCP13	500	10	13	1.8e−16	1.3e−17	39.50

LCP8. This example was used by Geiger and Kanzow [16], where $d = -e$ and the matrix M_7 is given as in (6.3).

LCP9. This LCP was given in [29]. The matrix M_8 is given in (6.3) and $d = -e$.

$$(6.3)\quad M_7 = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & ... & 0 \\ -1 & 4 & -1 & 0 & 0 & ... & 0 \\ 0 & -1 & 4 & -1 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & ... & ... & ... & 0 & -1 & 4 \end{pmatrix}, \quad M_8 = \begin{pmatrix} 1 & 2 & 2 & ... & 2 \\ 0 & 1 & 2 & ... & 2 \\ 0 & 0 & 1 & ... & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}.$$

LCP10. This example can be found in [16], where $M = \mathrm{diag}(1/n,\ 2/n,\ ...,\ 1)$ and $d = -e$.

LCP11. The matrix is obtained from M_5 by replacing the first diagonal entry by -4, and the vector $d = (0, 1, ..., 1)$. This LCP has no strictly feasible point.

LCP12. The matrix is obtained from M_7 by replacing the first diagonal entry by -4, and the vector $d = (0, 0, 1, ..., 1)$. This LCP has no strictly feasible point.

LCP13. The matrix is obtained from M_8 by replacing the last diagonal entry by -1, and the vector $d = (-1, ..., -1, 0)$. This LCP has no strictly feasible point.

Nonlinear complementarity problems:

NCP1. (Kojima-Shindo [31]) This is an NCP which is difficult to solve by the conventional Newton-type methods.

NCP2. (Watson's Fourth Problem [37]) This is an NCP representing the KKT conditions for a convex programming problem.

NCP3. (Mathiesen's Walrasian Equilibrium Model [31]) This is a 4-variable equilibrium problem depending on three parameters (α, b_2, b_3). We use two sets of constants: $(\alpha, b_2, b_3) = (0.75, 1, 0.5)$ and $(0.75, 1, 2)$. In Table 6.2, NCP3a and NCP3b denote, respectively, the problems corresponding to the above two cases.

NCP4. (Invariant Capital Stock Model [31]) This is an NCP (see [31]) formulated from an invariant capital stock model described by Hansen and Koopmans.

NCP5. (Nash-Cournot Production Problem [18]) We solve this NCP problem with $\gamma = 1.1$ and the data α_i, L_i, β_i can be found in [18]. The 5 and 10-variable problems were solved in our experiments. We use NCP5a and NCP5b in Table 6.2 to denote the 5 and 10-variable problems, respectively.

For NCPs, the number of evaluations of the Jacobian $\nabla f(x)$ should be recorded. However, by the construction of the algorithm, the total number of evaluations of the Jacobian $\nabla f(x)$ equals to the total number of iterations, and hence it is omitted here.

TABLE 6.2
NCPs, ε =1e−14.

Problems	Dim.	No. of Iter.	No. of fun.	θ^k	Residual	CPU (sec.)
NCP1	4	9	12	3.2e−09	1.3e−15	0.00
NCP2	5	16	35	4.0e−15	3.8e−16	0.00
NCP3a	4	8	9	1.4e−06	2.7e−16	0.00
NCP3b	4	8	9	1.4e−06	1.3e−16	0.00
NCP4	14	9	10	1.9e−12	5.6e−16	0.00
NCP5a	5	8	9	1.3e−06	6.9e−15	0.00
NCP5b	10	14	33	1.0e−17	9.7e−15	0.00

Nonlinear programming problems: We also test the algorithm for some nonlinear programming problems (NLPs). These examples can be found in Hock and Schittkowski [19]. We solve these examples via the KKT conditions for these problems which can be formulated as complementarity problems.

The computational results for LCPs are summarized in Table 6.1, for NCPs are reported in Table 6.2, and for NLPs are summarized in Table 6.3 in which the 'Dim' stands for the dimension of the corresponding complementarity problems. From the experiments, we found that the algorithm can solve all these examples effectively. It should be pointed out that the NCP1 is difficult to solve by conventional Newton-type methods, and as pointed out in [37] none of the standard algebraic techniques can solve the LCP2 easily. However, the proposed algorithm deals with the two problems very easily, and a quick convergence is observed. We also note that the value of β has a close relation to the convergence speed of the algorithm. The convergence speed will be slow if β is too small. In fact, a big value of β enables a large iterative steplength to be taken such that a rapid convergence can be achieved. This is indeed shown from our experiments.

7. Final remarks. A new non-interior-point algorithm is presented for P_0 LCPs. The global convergence of the algorithm is proved under a new condition which is different from previously used ones in the literature. A good feature of this condition is that it does not imply the boundedness of the solution set of the problem. Especially, for P_* LCPs, the algorithm is globally convergent provided that a solution exists. The

TABLE 6.3
NLPs, ε =1e−14.

Problems	Dim.	No. of Iter.	No. of fun.	θ^k	Residual	CPU (sec.)
HS18	7	17	76	5.2e−11	6.5e−16	0.00
HS24	4	7	8	1.5e−06	2.3e−16	0.00
HS33	6	12	19	2.1e−10	9.6e−16	0.00
HS34	8	10	33	7.3e−11	1.8e−15	0.00
HS35	4	8	9	1.3e−06	1.5e−15	0.00
HS36	7	14	90	1.0e−13	8.9e−16	0.00
HS44	10	8	9	1.4e−06	1.5e−15	0.00
HS63	7	9	82	8.1e−08	6.1e−15	0.00
HS66	8	14	56	3.8e−10	1.0e−15	0.00

superlinear convergence of the algorithm is also proved under a standard nondegeneracy assumption and a suitable choice of some parameters. The effectiveness of the algorithm was verified by our numerical experiments.

The essence of our algorithm is to follow a newly introduced regularized central path whose existence and theoretical properties were proved in [43]. Although the discussion in this paper was limited to LCPs, all the analysis of this paper can be extended to nonlinear P_0 complementarity problems as long as the function f is assumed to be continuously differentiable and Lipschitzian.

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