A proof of the Erdős-Faber-Lovász conjecture

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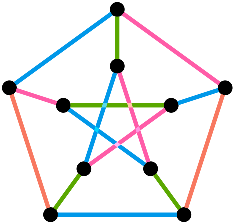
Matchings and edge-coloring

matching: a set of disjoint edges

(proper) edge-coloring: no two edges of same color share a vertex

chromatic index: min # colors used in proper edge-coloring, denoted χ'



$\chi'(\text{Petersen graph}) = 4$

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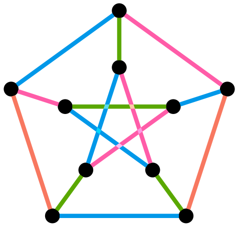

χ'(Petersen graph) $= 4$

Classical graph theory results:

- matchings: **Hall ('35)**, **Tutte ('47)**, **Edmonds ('65)**
- edge-coloring: **Vizing ('64):** $\chi' \in \{\Delta, \Delta + 1\}$, where $\Delta =$ max degree

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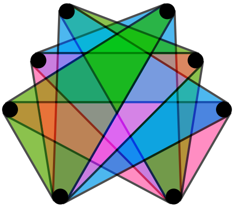

$\chi'(\mathcal{H}) = 3$

More complex for hypergraphs: e.g.

- 3-dimensional matching: one of Karp's original NP-complete problems
- block designs $\cong$ perfect matchings in a highly symmetric hypergraph

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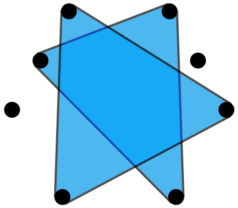

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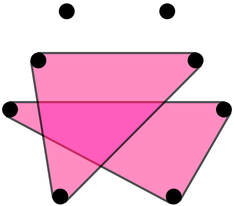

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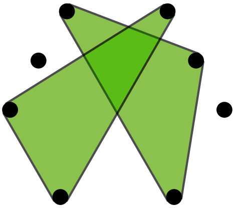

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The Erdős-Faber-Lovász conjecture

linear hypergraph: every pair of vertices contained in at most one edge

Erdős-Faber-Lovász conjecture (1972)

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

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> *An innocent looking problem often gives no hint as to its true nature. It might be like a 'marshmallow', serving as a tasty tidbit supplying a few moments of fleeting enjoyment. Or it might be like an 'acorn', requiring deep and subtle new insights from which a mighty oak can develop.* –Paul Erdős

One of Erdős' three favorite problems:

- formulated at a tea party in Boulder, CO.
- Erdős first offered \$50 for a solution, raised to \$500.

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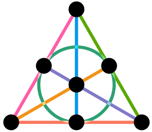
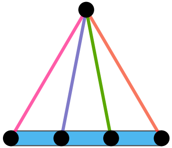
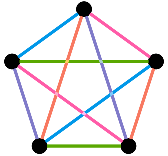
Extremal examples:







Finite projective plane of order k: $(k+1)$-uniform intersecting linear hypergraph with $n = k^2 + k + 1$ vertices and edges

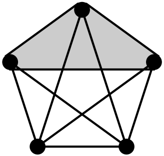
Degenerate plane / near pencil: intersecting linear hypergraph with $n-1$ size-two edges and one size-$(n-1)$ edge

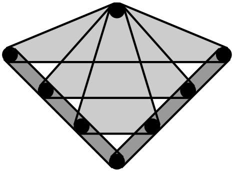
Complete graph: $\binom{n}{2}$ size-two edges; if $\chi' < n$, then color classes are perfect matchings $\Rightarrow$ n is even

Dual versions

Erdős-Faber-Lovász conjecture (dual)

If $\mathcal{H}$ is an n-uniform, n-edge, linear hypergraph, then the vertices of $\mathcal{H}$ can be n-colored such that every edge contains a vertex of every color.





Hypergraph duality:

- edges $\rightarrow$ vertices and vertices $\rightarrow$ edges
- linearity is preserved
- proper edge-coloring $\leftrightarrow$ vertex-coloring where no edge contains two vertices of same color

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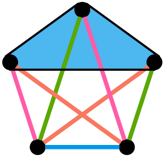

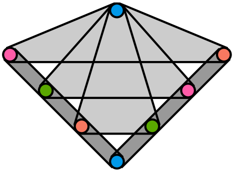

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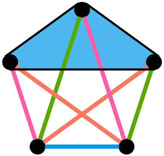
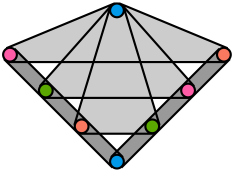




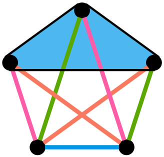
Equivalent "set theoretic" formulation:

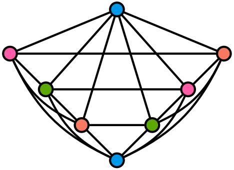
If $A_1, \ldots, A_n$ are sets of size n such that $|A_i \cap A_j| \leq 1 \ \forall \{i,j\} \in \binom{[n]}{2}$, then $\bigcup_{i=1}^{n} A_i$ can be colored with n colors so that all colors appear in each A_i.

Dual versions

Erdős-Faber-Lovász conjecture ("graphic")

If G is the union of n complete graphs, each on at most n vertices, such that every pair shares at most one vertex, then $\chi(G) \leq n$.





Line graph:

- edges $\rightarrow$ vertices: edges that share a vertex are adjacent
- proper edge-coloring $\rightarrow$ proper vertex-coloring (no monochromatic edge)

Previous results

Erdős-Faber-Lovász conjecture (1972)

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

Direct approaches:

Trivial: $\chi'(\mathcal{H}) \leq 2n - 3$ (color greedily, in order of size)

Chang-Lawler (1989): $\chi'(\mathcal{H}) \leq \lceil 3n/2 - 2 \rceil$

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Relaxed parameters:

de Bruijn-Erdős (1948): true for intersecting hypergraphs

Seymour (1982): $\exists$ a matching of size at least $|\mathcal{H}|/n$

Kahn-Seymour (1992): fractional chromatic index is at most n

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Probabilistic "nibble" approach:

Faber-Harris (2019): EFL is true if $|e| \in [3, c\sqrt{n}]$ $\forall e \in \mathcal{H}$ $(c \ll 1)$

Kahn (1992): $\chi'(\mathcal{H}) \leq (1 + o(1))n$

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Both use "list coloring" generalization (proved by Kahn) of:

Pippenger-Spencer theorem (1989)

If $\mathcal{H}$ is a linear hypergraph with bounded edge-sizes and maximum degree at most Δ, then $\chi'(\mathcal{H}) \leq \Delta + o(\Delta)$.

- implies EFL if $|e| \in [3, k]$ $\forall e \in \mathcal{H}$ and $n \gg k$
- implies EFL "asymptotically" if $|e| \leq k$ $\forall e \in \mathcal{H}$ and $n \gg k$.

Our results

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

For sufficiently large n, every n-vertex linear hypergraph has chromatic index at most n.

I.e., we confirm the EFL conjecture for all but finitely many hypergraphs.

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For sufficiently large n, every n-vertex linear hypergraph has chromatic index at most n.

I.e., we confirm the EFL conjecture for all but finitely many hypergraphs.
We also prove a stability result, predicted by Kahn:

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

$\forall \delta > 0$, $\exists \sigma > 0$ *such that the following holds for n sufficiently large.*
If $\mathcal{H}$ is an n-vertex linear hypergraph such that

- $\Delta(\mathcal{H}) \leq (1 - \delta)n$ *and*
- *at most* $(1 - \delta)n$ *edges have size* $(1 \pm \delta)\sqrt{n}$,

then $\chi'(\mathcal{H}) \leq (1 - \sigma)n$.

Overview of the proof

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Upshot: Reduce to the "right" **graph** coloring problem in each case.

"Dream proof" for bounded edge-sizes

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \in \{2, 3\}$ $\forall e \in \mathcal{H}$.

Proof (dream) of $\chi'(\mathcal{H}) \leq n$:

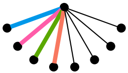
Using $k = \lfloor n/2 \rfloor$ colors, (partially) color $\mathcal{H}$ such that

- all size-3 edges are colored and
- for each vertex, $\geq 1/2$ of the graph edges containing it are colored.

Uncolored edges comprise a **graph** of max degree $< n - k$. $(\star)$

Finish with Vizing's theorem!

Low degree: more flexibility



High degree: more graph-like

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Approach more amenable to probabilistic method:
Fix $0 < \gamma \ll \varepsilon \ll 1$, and let $U := \{v \in V(\mathcal{H}) : d(v) > (1 - \varepsilon)n\}$.
Aim: Using $k = (1/2 + \gamma)n$ colors, color $\mathcal{H}$ such that:

- all size-3 edges are colored;
- for each vertex, nearly half of graph edges containing it are colored;
- every color class covers U (**perfect coverage** of U).

Simplified proof for $\chi' \leq n + 1$

Randomized "dream" proof strategy

Put each graph edge in a "reservoir" R independently with probability $1/2$;

- with high probability $\Delta(\mathcal{H} \setminus R) \leq (1/2 + o(1))n$, so $\chi'(\mathcal{H} \setminus R) \leq (1/2 + \gamma)n$ by the Pippenger-Spencer theorem.

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Nibble + absorption: using $k = (1/2 + \gamma)n$ colors, color some $\mathcal{H}' \supseteq \mathcal{H} \setminus R$ with **perfect coverage** of U:

- vertices in U have leftover degree $\leq (n - 1) - k < n - k$;
- vertices not in U have leftover degree $\leq (1 - \varepsilon)n/2 + o(n) < n - k$.

Thus $\mathcal{H} \setminus \mathcal{H}'$ is a graph and $\Delta(\mathcal{H} \setminus \mathcal{H}') < n - k$, so by Vizing's thm

$$\chi'(\mathcal{H}) \leq \chi'(\mathcal{H}') + \chi'(\mathcal{H} \setminus \mathcal{H}') \leq k + (n - k) = n. \qquad \square$$

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Perfect coverage of $U := \{v \in V(\mathcal{H}) : d(v) > (1-\varepsilon)n\}$ not possible (e.g. K_n for n odd); Instead, find coloring with **nearly perfect coverage:**

- every color class covers all but one vertex of U and
- each vertex of U is covered by all but one color class.

Simplified proof for $\chi' \leq n+1$

Proof (sketch) of $\chi' \leq n+1$

Put each graph edge in a "reservoir" R independently with probability $1/2$;

- with high probability $\Delta(\mathcal{H} \setminus R) \leq (1/2 + o(1))n$, so $\chi'(\mathcal{H} \setminus R) \leq (1/2 + \gamma)n$ by the Pippenger-Spencer theorem.

Nibble + absorption: using $k = (1/2 + \gamma)n$ colors, color some $\mathcal{H}' \supseteq \mathcal{H} \setminus R$ with **nearly perfect coverage** of U:

- vertices in U have leftover degree $\leq (n-1) - (k-1) \leq n - k$;
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Thus $\mathcal{H} \setminus \mathcal{H}'$ is a graph and $\Delta(\mathcal{H} \setminus \mathcal{H}') \leq n - k$, so by Vizing's thm

$$\chi'(\mathcal{H}) \leq \chi'(\mathcal{H}') + \chi'(\mathcal{H} \setminus \mathcal{H}') \leq k + (n - k + 1) = n + 1. \qquad \square$$

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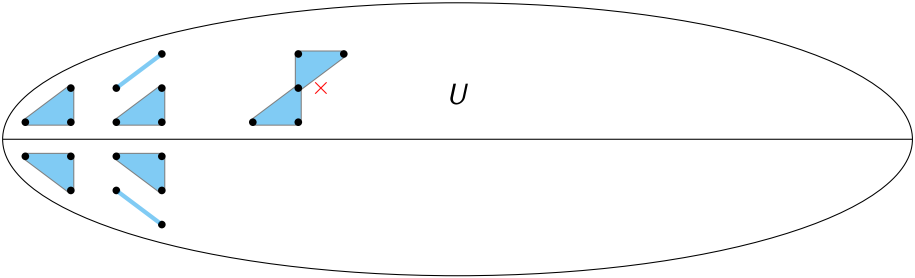
Nibble + absorption

- $U = \{v \in V(\mathcal{H}) : d(v) > (1-\varepsilon)n\}$ $(0 < \gamma \ll \varepsilon \ll 1)$
- R = random "reservoir" – graph edges included with prob $1/2$

Nibble: One-by-one, randomly select each color class in $\mathcal{H} \setminus R$, in small "bites", until $(1-\gamma)n$ vertices are covered.



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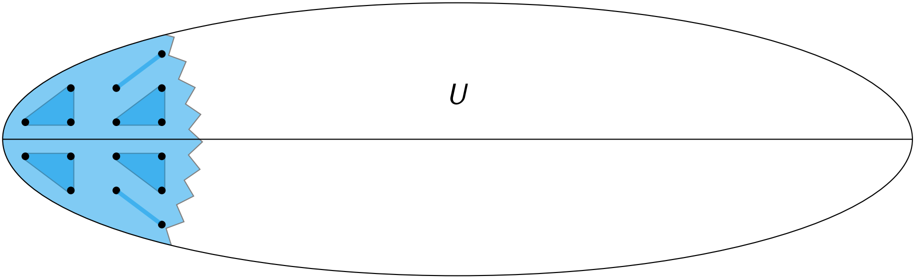

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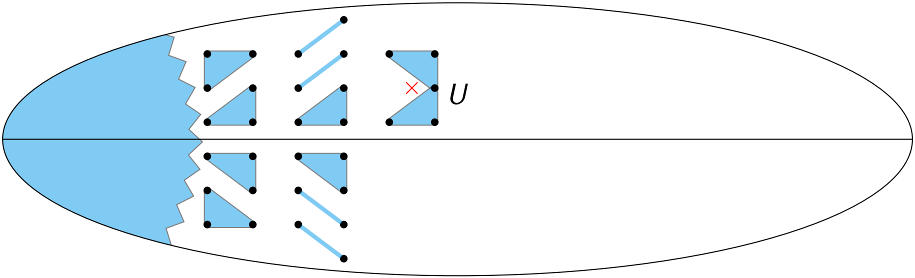

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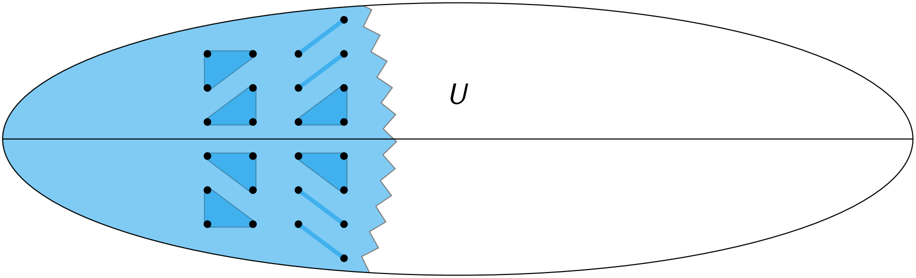

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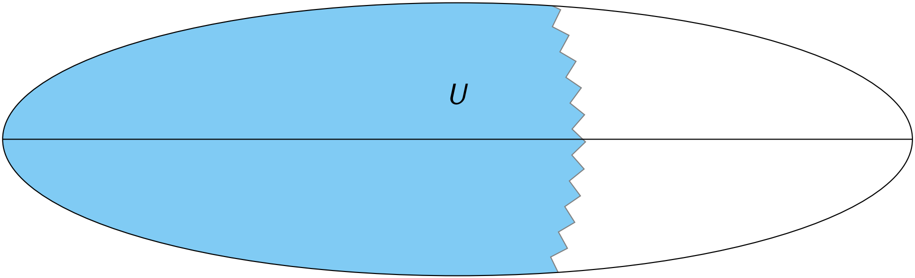

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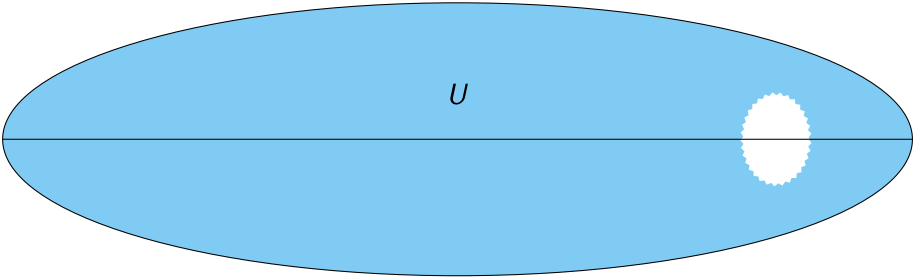
Vertices uncovered $\approx$ independently with probability γ



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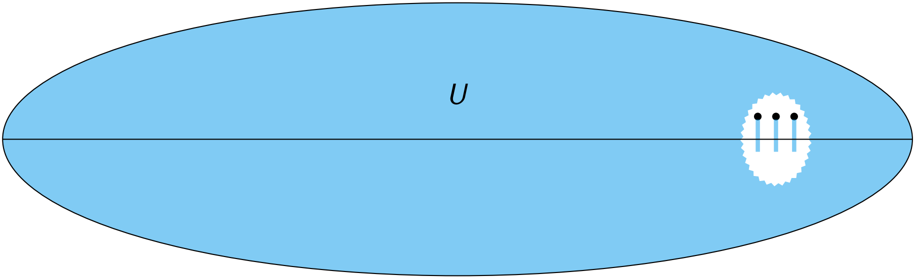

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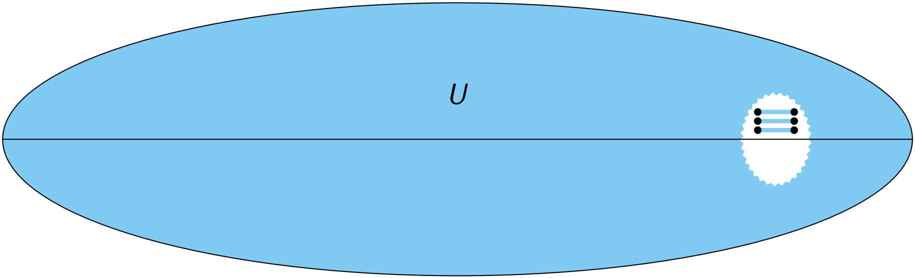

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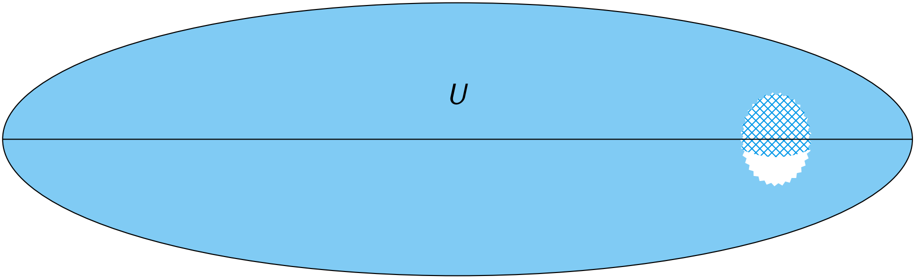

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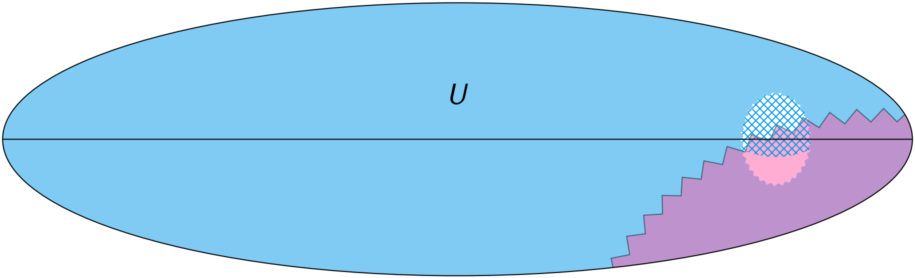

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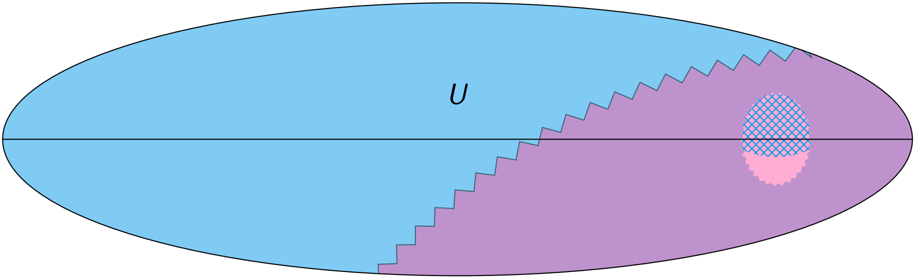

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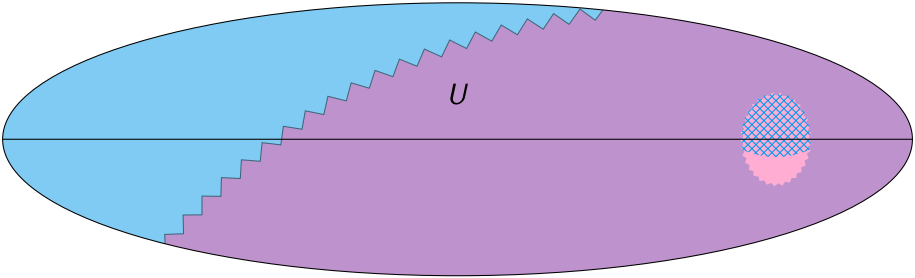

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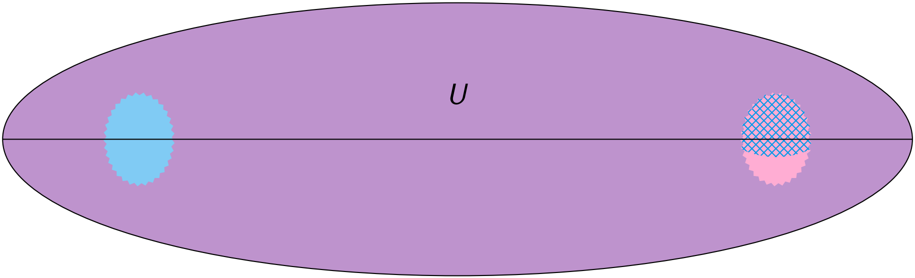

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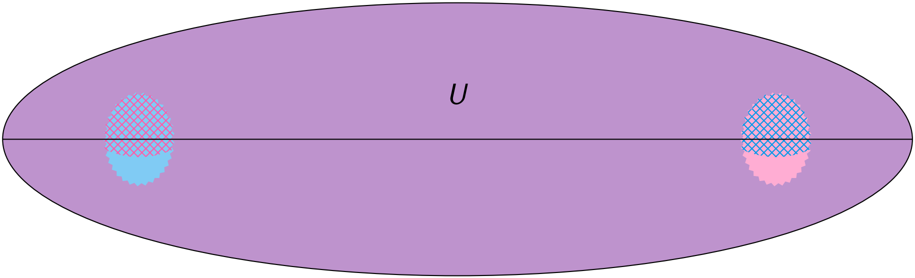
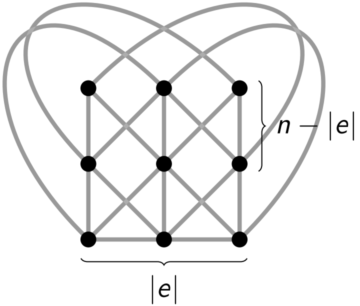


Large edges: reordering

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \geq r \ \forall e \in \mathcal{H}$, where $r \gg 1$.

Trivial: $\forall e \in \mathcal{H}$, at most $|e|(n - |e|)/(|e| - 1) \leq n + o(n)$ edges of size at least $|e|$ intersect e.



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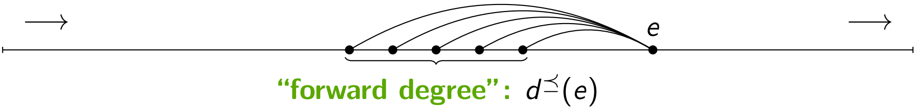
Corollary: $\chi'(\mathcal{H}) \leq n + o(n)$: color greedily.



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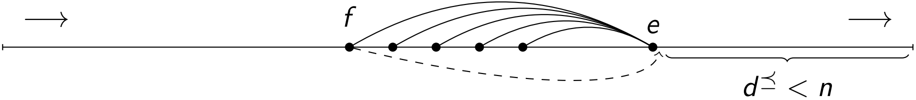
Reordering: Let e be the last edge with $d^{\preceq}(e) \geq n$. If f intersects e and $< n$ edges preceding e intersect f, then move f immediately after e.



If reordering "finishes', then $d^{\preceq}(e) < n \ \forall e \in \mathcal{H}$, so $\chi'(\mathcal{H}) \leq n$.

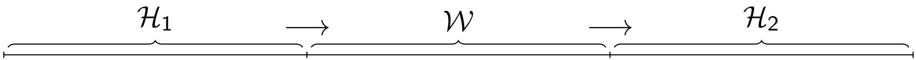
Reordering lemma (informal)

If reordering "gets stuck", then there is a highly structured $\mathcal{W} \subseteq \mathcal{H}$.

Proof when all edges are large

For $0 < \delta \ll 1$ and $\zeta < 1$: $(1/r \ll \delta)$

- $\mathcal{W}$ covers $(1-\delta)\binom{n}{2}$ pairs of vertices, and $|e| \sim (1-\zeta)\sqrt{n}$ $\forall e \in \mathcal{W}$.
- If $e \in \mathcal{H}_2$, then $d^{\preceq}(e) < n$.
- If $e \in \mathcal{H}_1$, then $|e| \geq (1-\zeta)\sqrt{n}$



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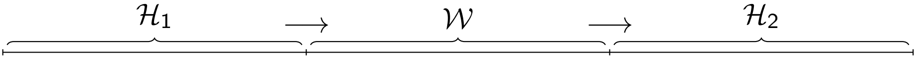

Case 1: $\zeta < \sqrt{\delta}$ ($\mathcal{W} \approx$ projective plane)

Proof (sketch)

Find $|\mathcal{H}_1 \cup \mathcal{W}| - n$ pairs of disjoint edges in $\mathcal{H}_1 \cup \mathcal{W}$:

- assign edges of each pair the same color;
- assign remaining edges (of $\mathcal{H}_1 \cup \mathcal{W}$) distinct colors.

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$$\underbrace{\qquad\qquad}_{\mathcal{H}_1} \longrightarrow \underbrace{\qquad\qquad}_{\mathcal{W}} \longrightarrow \underbrace{\qquad\qquad}_{\mathcal{H}_2}$$

Case 2: $\zeta \geq \sqrt{\delta}$ ("non-extremal case")

Proof (sketch)

Line graph of $\mathcal{W}$ has max degree $\leq (1+o(1))n$ and is **locally sparse**, i.e. $\leq (1-\zeta/2)\binom{n}{2}$ edges in the neighborhood of every vertex:

- randomly color $\mathcal{W}$; thm of Molloy & Reed $\Rightarrow \chi'(\mathcal{W}) \leq (1-2^{-10}\zeta)n$;

Apply "reordering" argument to edges preceding $\mathcal{W}$:

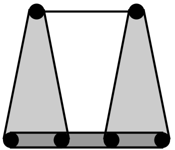
- If $e \in \mathcal{H}_1$, then $d^{\preceq}(e) \leq 2^{-10}\zeta n - 1 \Rightarrow \chi'(\mathcal{H}_1) \leq 2^{-10}\zeta n$.

Open problems

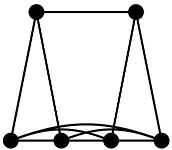
Conjecture (Berge '89, Füredi '86, Meyniel (unpublished))

If $\mathcal{H}$ is a linear hypergraph, then $\chi'(\mathcal{H}) \leq \max_{v \in V(\mathcal{H})} |\bigcup_{e \ni v} e|$.

- common generalization of Vizing's theorem and EFL



$\max_v |\bigcup_{e \ni v} e| = 5$



$\Delta(\text{"shadow"}) + 1 = 5$

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List EFL

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\mathcal{H}$ has list chromatic index $\leq n$.

I.e. if $C(e)$ is a "list of colors" such that $|C(e)| \geq n \; \forall e \in \mathcal{H}$, then $\mathcal{H}$ can be properly edge-colored s.t. every e is assigned a color from $C(e)$.

- Implies EFL if $C(e) = \{1, \ldots, n\} \; \forall e \in \mathcal{H}$.

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