Rainbow Hamilton paths in random 1-factorizations of K_n

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Joint work with Stephen Gould, Daniela Kühn, and Deryk Osthus

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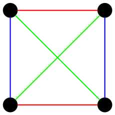
Andersen's Conjecture

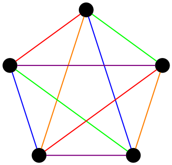
Andersen's Conjecture (1989)

Every properly edge-colored K_n has a rainbow path of length $n-2$.

Proper edge-coloring: no two edges of the same color share a vertex.

Rainbow: every edge has a distinct color.





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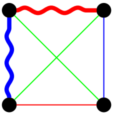

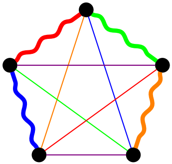

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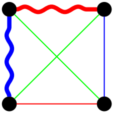

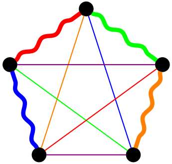

1-factorization: proper edge-coloring where every color is assigned to a perfect matching.

For 1-factorizations, Andersen's Conjecture says there is a rainbow path using all but one vertex and color.

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Glock-Kühn-Montgomery-Osthus (2020): For large enough n, every 1-factorization of K_n can be decomposed into isomorphic rainbow spanning trees.

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Conj (Hahn, 1980): Every "globally $n/2$-bounded" edge-colored K_n has a rainbow Hamilton path.

Conj (Hahn-Thomassen, 1986): Every globally $(n/2-1)$-bounded edge-colored K_n has a rainbow Hamilton path.

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Conj (Hahn-Thomassen, 1986): Every globally $(n/2 - 1)$-bounded edge-colored K_n has a rainbow Hamilton path.

Pokrovskiy-Sudakov (2019): Both are false: $\exists$ globally $n/2$-bounded edge-colorings of K_n with no rainbow $(n - o(\ln n))$-length path.

Thus, Andersen's Conj does not generalize to "sub-Ramsey" setting.

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We also prove:

- Almost all 1-factorizations have a rainbow cycle using all the colors.
- For n odd, almost all n-edge colorings have a rainbow Hamilton cycle.

Latin squares and transversals

Latin square: An $n \times n$ array of n symbols such that each row and each column contains one instance of each symbol.

Transversal: A collection of n cells, one from each row and column, containing one instance of each symbol.

- Latin squares correspond to 1-factorizations of $K_{n,n}$.
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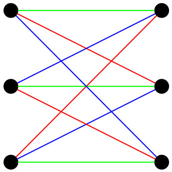

1	2	3
3	1	2
2	3	1

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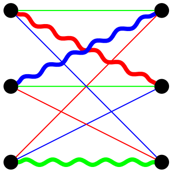

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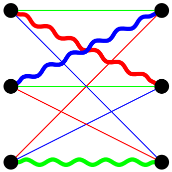

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Ryser-Brualdi-Stein conj: Every LS has a partial transversal of size $n-1$.

Kwan (2016+): Almost all Latin squares have a full transversal – "partite analogue" of our result.

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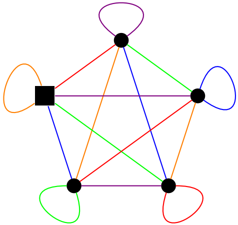

3	1	4	2	5
1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2

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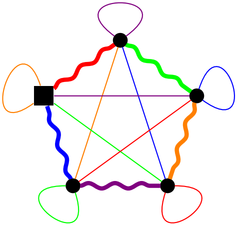

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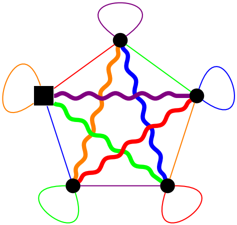

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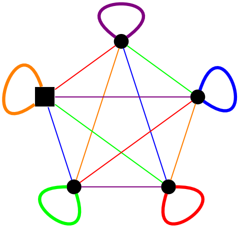

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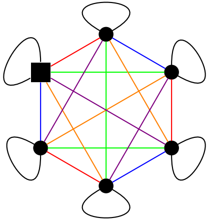

6	1	3	4	5	2
1	6	2	5	3	4
3	2	6	1	4	5
4	5	1	6	2	3
5	3	4	2	6	1
2	4	5	3	1	6

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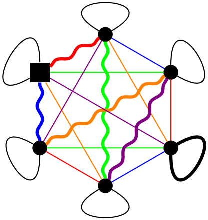

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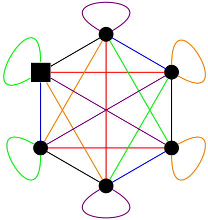

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Our result does not apply for n even if more than one symbol appears on the diagonal.

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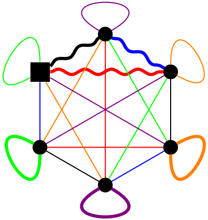

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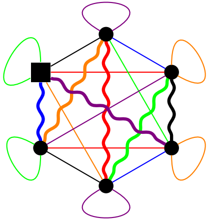

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The "big picture"

Proper edge-coloring of K_n	Latin square
Andersen: rainbow path of length $(n-2)$?	**Ryser-Brualdi-Stein:** order $n-1$ "partial" transversal?
Balogh-Molla: $n - O(\log n\sqrt{n})$	Keevash-Pokrosvkiy-Sudakov-Yepreman: $n - O(\log n/\log\log n)$
G-K-K-O: Almost all 1-factorizations have rainbow Hamilton path	Kwan: Almost all Latin squares have transversal

Proof strategy

Theorem (Gould, K., Kühn, Osthus, 2020++)

A (uniform) random 1-factorization of K_n has a rainbow Ham path ***whp****.*

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Definition: An edge-colored graph G is robustly rainbow-Hamiltonian (with respect to "flexible" sets V_{flex} and C_{flex} of vtcs and colors) if

($\star$) for any pair of equal-sized subsets $X \subseteq V_{\mathrm{flex}}$ and $Y \subseteq C_{\mathrm{flex}}$ of size at most $|V_{\mathrm{flex}}|/2$ and $|C_{\mathrm{flex}}|/2$, the graph $G - X$ contains a rainbow Hamilton path not containing a color in Y.

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Aim: Find robustly rainbow-Hamiltonian subgraph G in a "random slice" of vtcs and colors with ηn-sized flexible sets ($\eta \ll 1$) to use for **absorption**.

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Use G to find rainbow Hamilton path in 3 steps:

1: Find rainbow path P vertex- and color-disjoint from G with only $o(n)$ vertices and colors not in G or P.

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Definition: An edge-colored graph G is robustly rainbow-Hamiltonian (with respect to "flexible" sets $V_{\rm flex}$ and $C_{\rm flex}$ of vtcs and colors) if

($\star$) for any pair of equal-sized subsets $X \subseteq V_{\rm flex}$ and $Y \subseteq C_{\rm flex}$ of size at most $|V_{\rm flex}|/2$ and $|C_{\rm flex}|/2$, the graph $G - X$ contains a rainbow Hamilton path not containing a color in Y.

Aim: Find robustly rainbow-Hamiltonian subgraph G in a "random slice" of vtcs and colors with ηn-sized flexible sets ($\eta \ll 1$) to use for **absorption**.

Use G to find rainbow Hamilton path in 3 steps:

1: Find rainbow path P vertex- and color-disjoint from G with only $o(n)$ vertices and colors not in G or P.

2: Extend P to "cover" leftover vertices and colors using flexible sets.

Proof strategy

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3: "Absorb" remaining vertices of G into P using ($\star$).

The robustly rainbow-Hamilton subgraph

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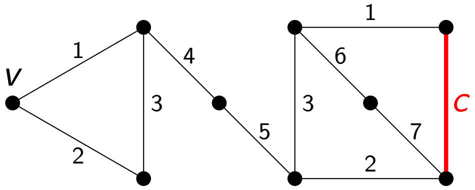
Building the absorber: Use combination of "greedy" and "nibble" to find robustly rainbow-Hamiltonian subgraph in random slice.

Absorbing gadgets

Let v be a vertex and c be a color.

(v, c)-absorbing gadget: Disjoint union of a triangle containing v and 4-cycle containing a c-edge – with colors "corresponding" as shown – "completed" by two rainbow paths.



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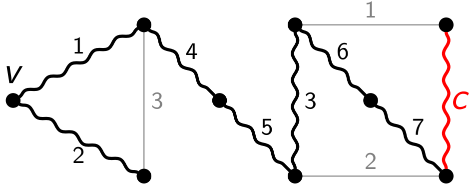

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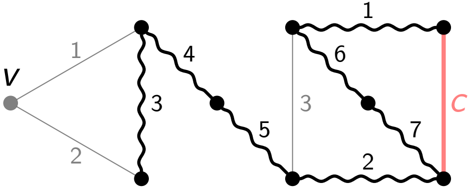

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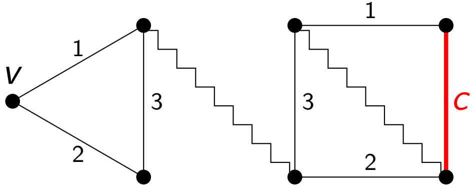

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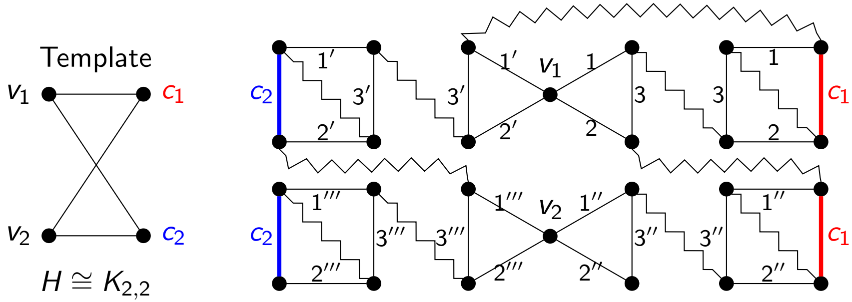
Convention: "Zigzags" form rainbow path forest "color-disjoint" from any drawn colors.

H-absorbers

Use auxiliary bipartite graph H – where one part is vtcs and one part is colors – as a "template" to build absorber from gadgets.

***H*-absorber:** $\forall\ e = vc \in E(H)$, there is a (v, c)-absorbing gadget s.t.:

- gadgets are vertex and color disjoint (except at v and c) and
- gadgets are connected in "path-like" way (by zigzags).



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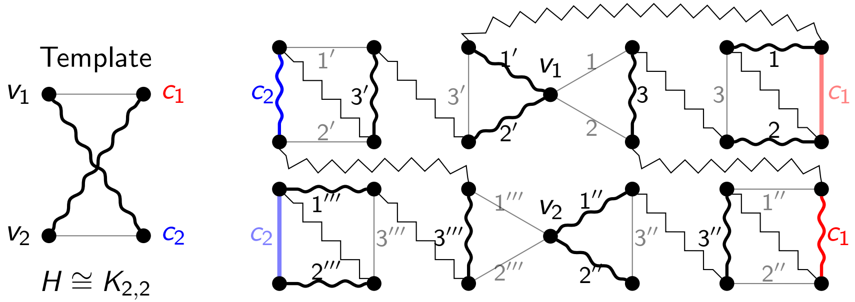

Key fact: Matchings in H corresponds to rainbow paths in H-absorber!

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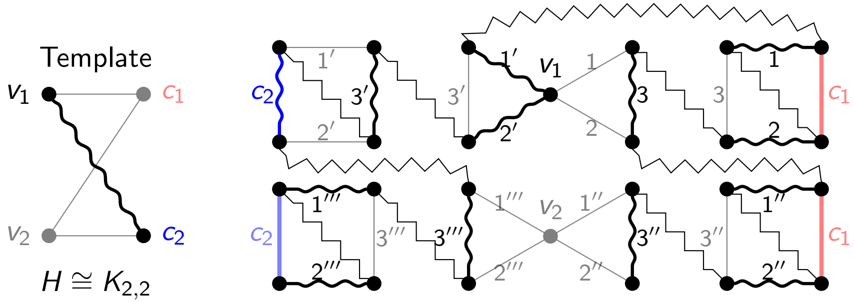

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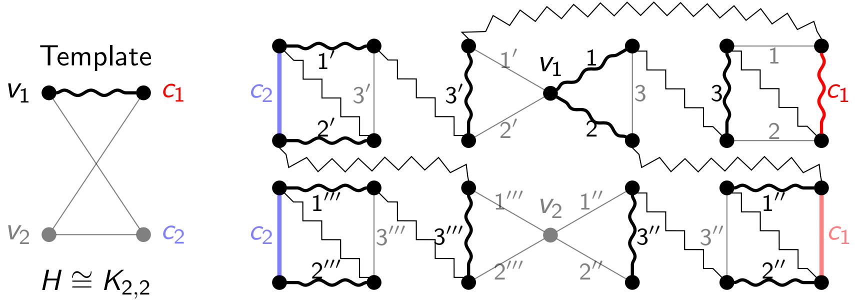

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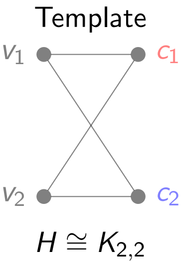

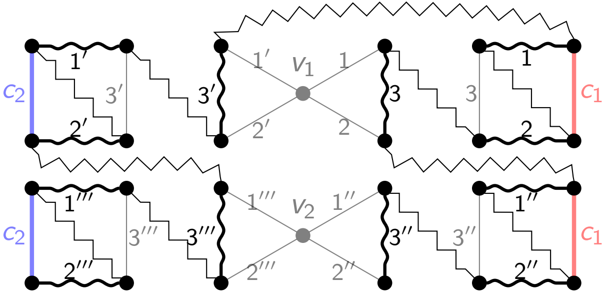

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Robustly matchable bipartite graphs

H bipartite with bipartition (A, B), where $|A| = |B|$.

Robustly matchable bipartite graph (RMBG): H is robustly matchable with respect to "flexible sets" $A_{\mathrm{flex}} \subseteq A$ and $B_{\mathrm{flex}} \subseteq B$ if

($\star$)' for any pair of equal-sized subsets $X \subseteq A_{\mathrm{flex}}$ and $Y \subseteq B_{\mathrm{flex}}$ of size at most $|A_{\mathrm{flex}}|/2$ and $|B_{\mathrm{flex}}|/2$, $H - (X \cup Y)$ has a perfect matching.

Distributive absorption: If H is robustly matchable, then an H-absorber is robustly rainbow-Hamiltonian wrt the same flexible sets.

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Lemma (Montgomery '18): For m large, $\exists$ RMBGs with $7m$ vertices, flexible sets of size m, and max degree at most 256.

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Our absorber is an H-absorber where H is one of these RMBGs, with $m = \eta n$.

Proof (overview) by picture

Randomly slice μn vtcs and colors ($\eta \ll \mu \ll 1$) – let H be RMBG.

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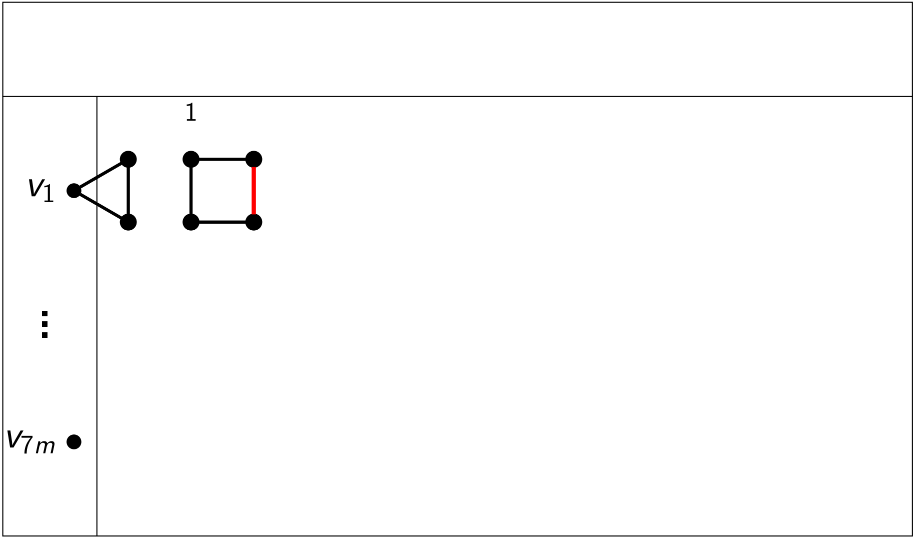

Find (v, c)-absorbing gadgets one by one, using H as template.

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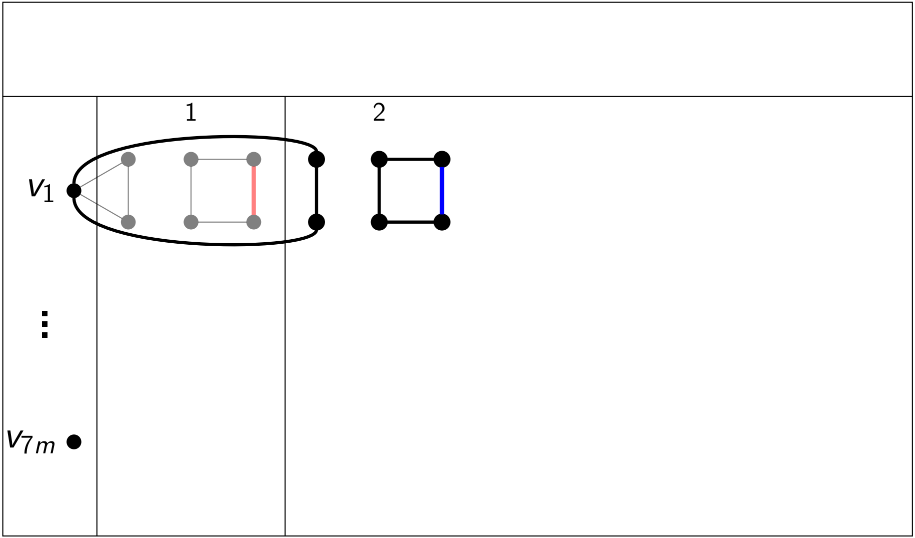

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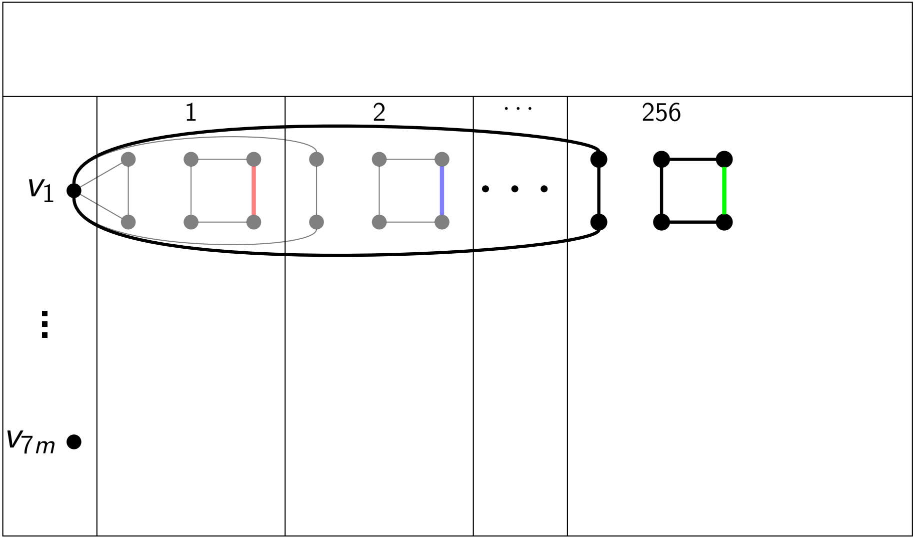

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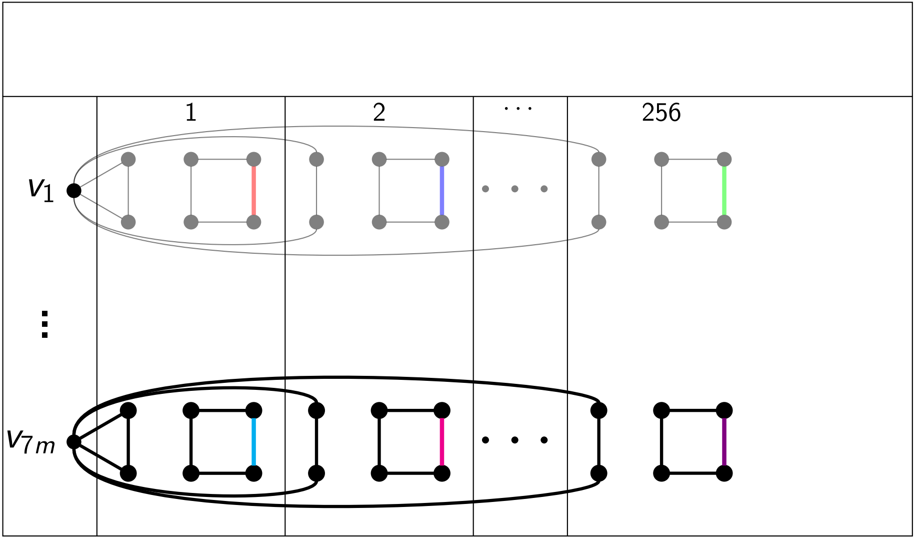

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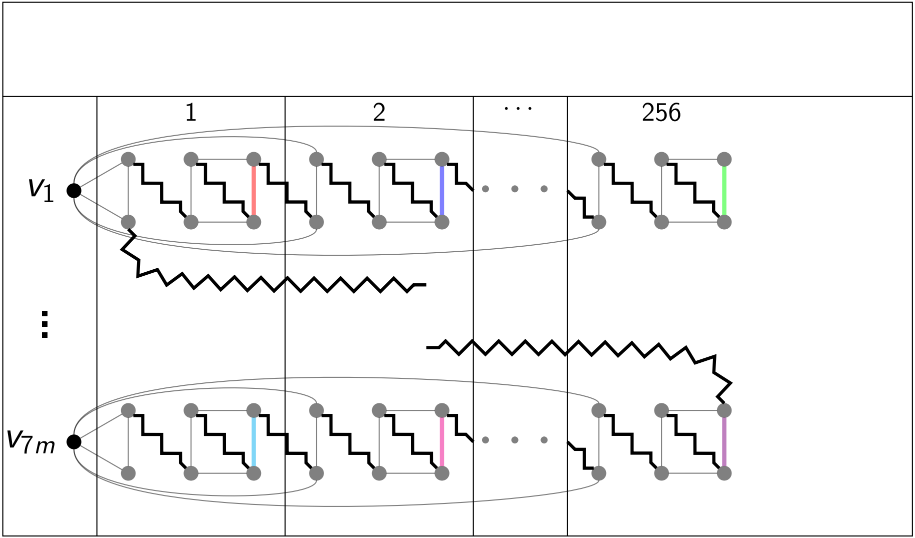

Complete/connect gadgets to obtain H-absorber...

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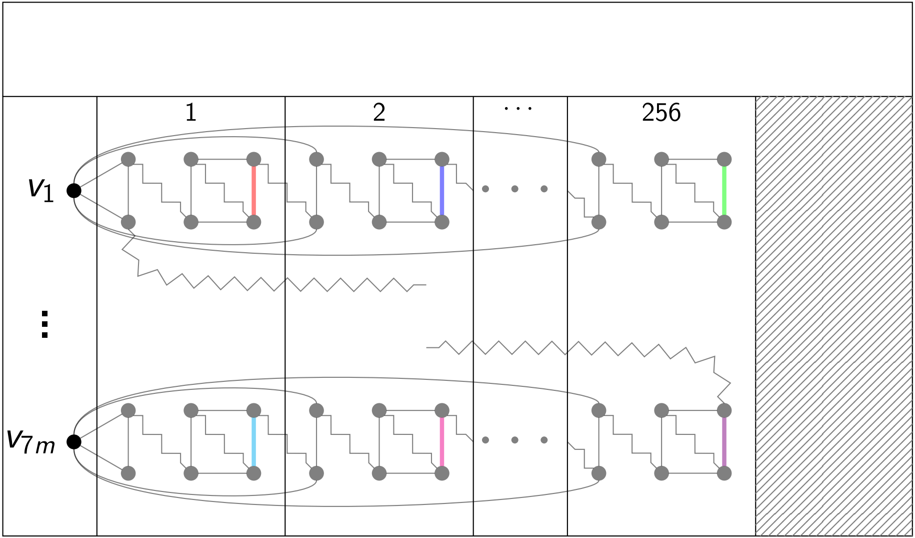

Complete/connect gadgets to obtain H-absorber... but too much leftover.

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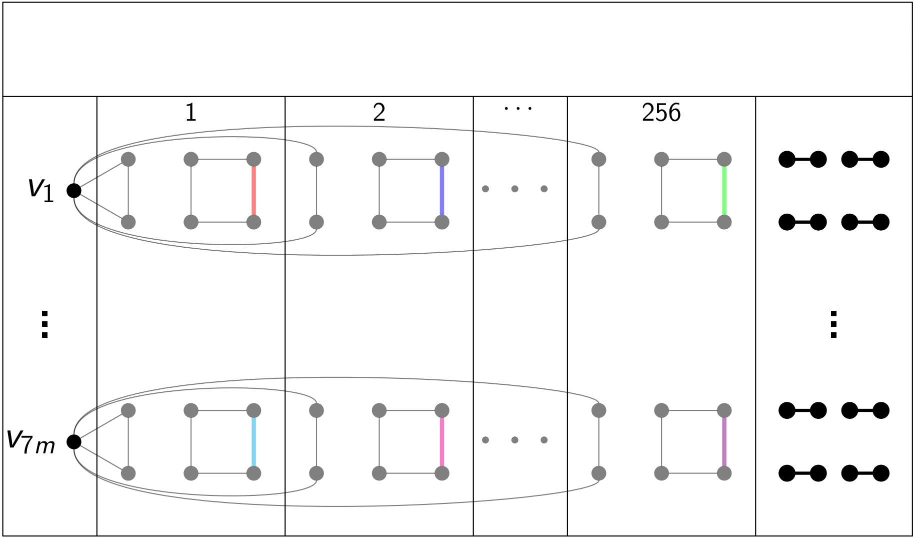

Instead, find rainbow matching w/ almost all unused vtcs and cols in slice.

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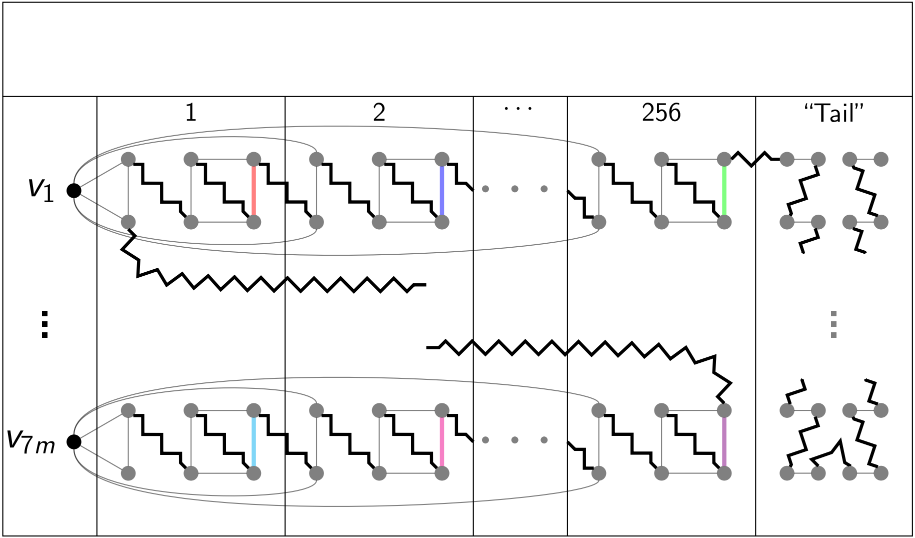

Complete/connect to obtain H-absorber and simultaneously a "tail".

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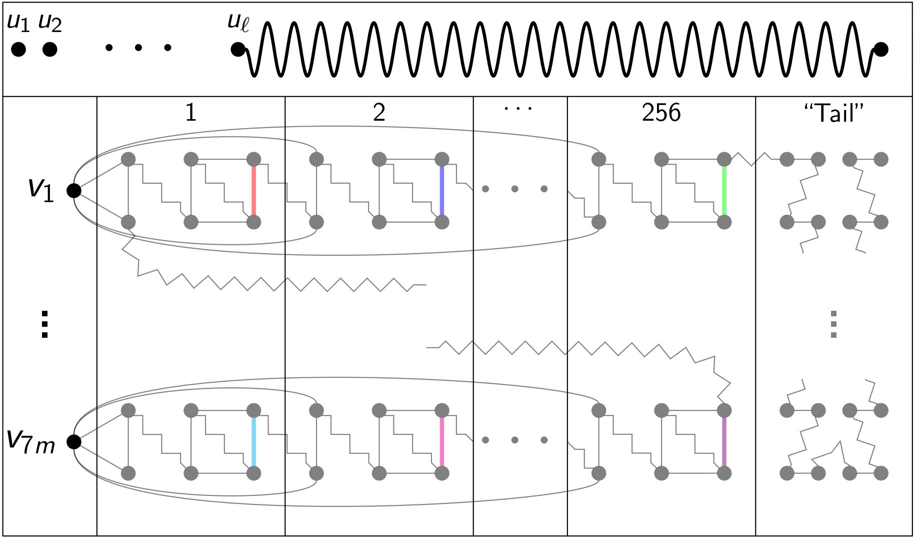

Find almost spanning rainbow path outside slice (leftover $\ell \ll m$).

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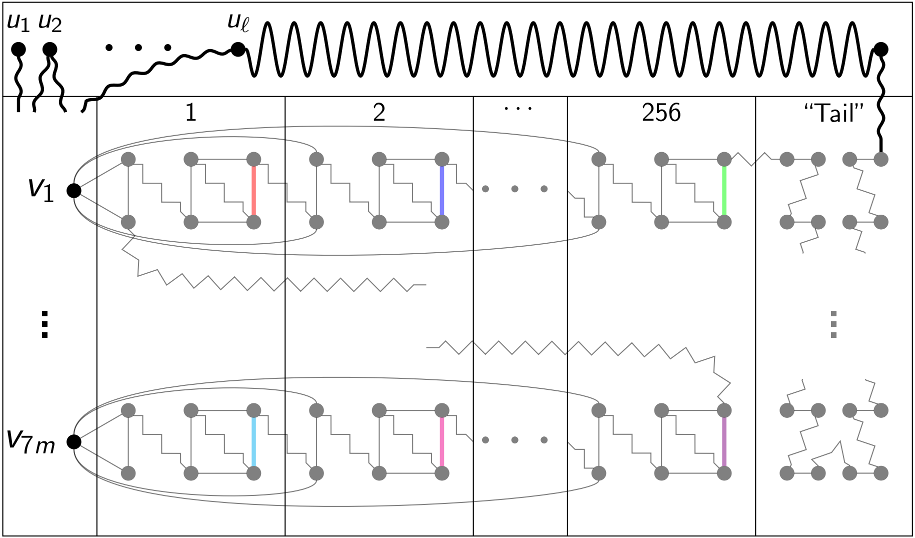

"Cover" unused vtcs and colors with flexible vtcs and colors.

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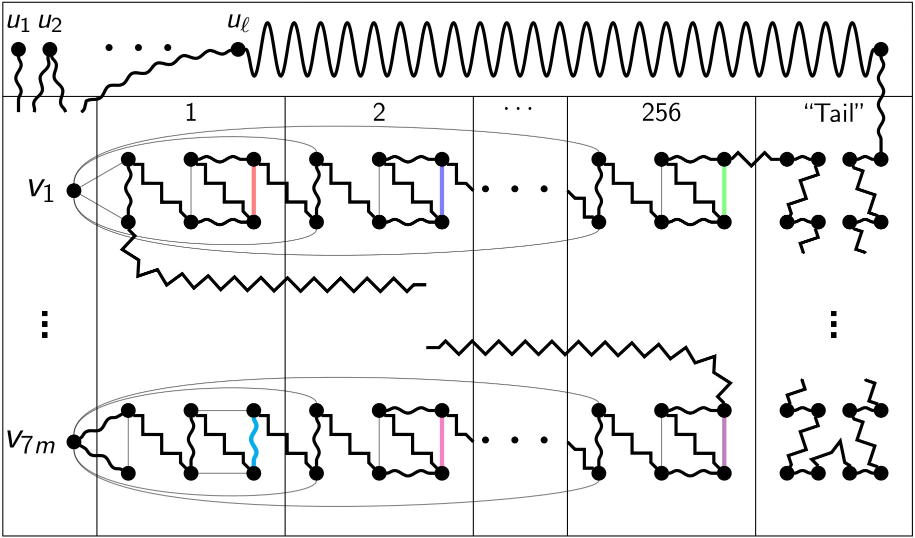

Absorb!

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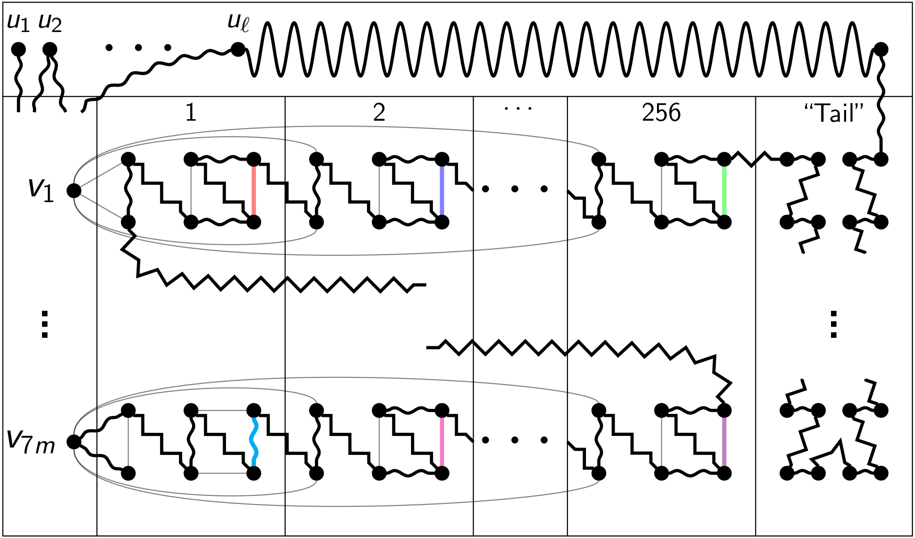

Thanks for listening!