Hamilton transversals in random Latin squares

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Joint work with Stephen Gould

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Transversals in Latin squares

The Ryser-Brualdi-Stein conjecture

Every $n \times n$ Latin square has a partial transversal of size $n - 1$.

Moreover, when n is odd, there may be a "full" transveral (i.e. size n).

Latin square: An $n \times n$ array of n symbols such that each row and column contains one instance of each symbol.

Partial transversal: A collection of entries, with at most one from each row and column, containing at most one instance of each symbol.

a	b
b	a

a	b	c
c	a	b
b	c	a

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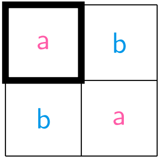

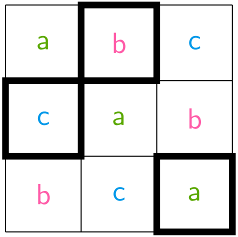

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Kwan (2020): Almost all $n \times n$ Latin squares have a full transversal.

I.e., the Ryser-Brualdi-Stein conjecture holds with high probability for an $n \times n$ Latin square chosen uniformly at random.

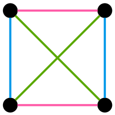
Rainbow paths in properly edge-colored complete graphs

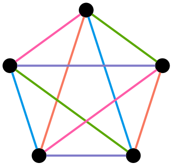
Andersen's conjecture (1989)

Every properly edge-colored K_n has a rainbow path of length $n - 2$.

Proper edge-coloring: an assignment of colors to the edges such that no two edges of the same color share a vertex.

Rainbow: every edge has a distinct color.





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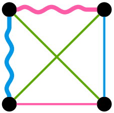

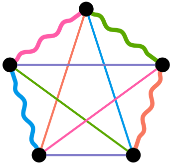

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Gould, K., Kühn, and Osthus (2020+): Almost all optimally (and properly) edge-colored K_n have a rainbow Hamilton path.

I.e., Andersen's conjecture holds with high probability for an optimal edge-coloring chosen uniformly at random.

Cycle-free transversals and the directed setting

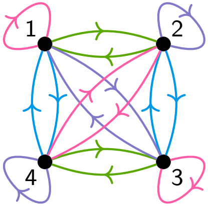
Conjecture (Gyárfás and Sárközy, 2014)

Every $n \times n$ Latin square has a **cycle-free** partial transversal of size $n-2$.

I.e., every properly n-arc-colored $\boldsymbol{K_n^{\leftrightarrow}}$ has a rainbow directed linear forest with $n-2$ arcs.

- $n \times n$ Latin squares correspond to proper n-arc-colorings of $K_n^{\leftrightarrow}$.
- Partial transversals $\approx$ rainbow subgraphs w/ in- and out-degrees ≤ 1.

1	a	b	c	d
2	b	c	d	a
3	c	d	a	b
4	d	a	b	c



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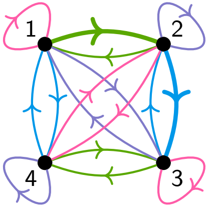

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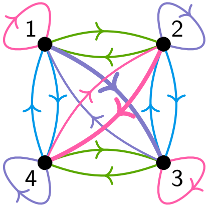

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Benzing, Pokrovskiy, and Sudakov (2020): Every properly arc-colored $K_n^{\leftrightarrow}$ has a rainbow directed linear forest with $n - O(n^{2/3})$ arcs.

Improved the $n - O(n \log\log n / \log n)$ bound of **Gyárfás and Sárközy ('14)**.

Benzing, Pokrovskiy, and Sudakov (2020): Every properly arc-colored $K_n^{\leftrightarrow}$ has a rainbow directed cycle of length $n - O(n^{4/5})$.

Connected transversals: a common generalization?

Conjecture (K. and Gould, 2021+)

Every properly arc-colored $K_n^{\leftrightarrow}$ has a rainbow directed cycle or path of length $n-1$.

Equivalently, every $n \times n$ Latin array has a **connected** partial transversal of size $n-1$.

If true, this conjecture implies:

- the Ryser-Brualdi-Stein conjecture,
- Andersen's conjecture, and
- Gyárfás and Sárközy's conjecture.

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Theorem (K. and Gould, 2021+)

Almost all $n \times n$ Latin squares have a **Hamilton** transversal. I.e., almost all properly n-arc-colored $K_n^{\leftrightarrow}$'s have a rainbow directed Hamilton cycle.

In fact, we prove an asymptotically optimal counting result, finding

$$\left((1-o(1))\frac{n}{e^2}\right)^n \text{ Hamilton transversals / cycles,}$$

which matches Kwan's bound up to lower order terms.

Diagonals of random Latin squares

Let $X(L) :=$ max # of times a symbol appears on the diagonal of a Latin square L.

- If $X(L) = n$ (where L is $n \times n$), then L has no Hamilton transversal.

a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c

$X = 2$

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

$X = 4$

Diagonals of random Latin squares

Let $\mathrm{X}(L) :=$ max # of times a symbol appears on the diagonal of a Latin square L.

- If $\mathrm{X}(L) = n$ (where L is $n \times n$), then L has no Hamilton transversal.

Theorem (K. and Gould, 2021+)

If L is a uniformly random $n \times n$ Latin square, then $\mathrm{X}(L) = O(\log n / \log\log n)$ with high probability.

Matches maximum load of a bin in the classic "balls into bins model".

Question: What is the distribution of X?

Conclusion

Common generalization of the Ryser-Brualdi-Stein conjecture, Andersen's conjecture, and Gyárfás and Sárközy's conjecture:

Conjecture (K. and Gould, 2021+)

Every properly arc-colored $K_n^{\leftrightarrow}$ has a rainbow directed cycle or path of length at least $n-1$.

K. and Gould (2021+): Almost all properly n-arc-colored $K_n^{\leftrightarrow}$'s have a rainbow directed Hamilton cycle.

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Question

What is the distribution of the random variable that counts the number of occurences of the diagonal's "modal symbol" in a random Latin square?

K. and Gould (2021+): The modal symbol occurs at most $O(\log n / \log\log n)$ times with high probability.

Thanks for listening!