Fractional Coloring with Local Demands

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If G has no simplicial vertices (i.e. $N(v)$ is not a clique), then

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- no $n/(d+1-\varepsilon)$-bound known for $\varepsilon > 0$; $\varepsilon = 1/2$ is tight for 5-cycle.
- actually holds if less than half of each clique's vertices are simplicial.

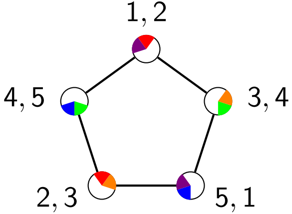
Fractional coloring

Multicoloring: a map

$$\psi : V(G) \to \text{subsets of } \mathbb{N}$$

such that $\psi(u) \cap \psi(v) = \varnothing$ for all $uv \in E(G)$.

Fractional chromatic number: denoted $\chi_f(G)$ – the min $k \in \mathbb{Q}$ such that G has a multicoloring ψ using N colors for some N such that $|\psi(v)| \geq N/k$ for all $v \in V(G)$.



$$\chi_f(C_5) \leq 5/2$$

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Conventions introduced by Dvořák, Sereni, and Volec:

Demand function: a map $f : V(G) \to [0, 1]$.

(f, N)-coloring: a multicoloring ψ using N colors s.t. $|\psi(v)| \geq f(v) \cdot N$ for all $v \in V(G)$, i.e. v receives "at least $f(v)$ fraction of the colors."

f-colorable: there exists an (f, N)-coloring.

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Proposition: If G is f-colorable, then $\alpha(G) \geq \sum_{v \in V(G)} f(v)$.

Greedy coloring

Proposition (Local Fractional Greedy): If $f(v) \leq 1/(d(v)+1)$ for all $v \in V(G)$, then G is f-colorable.

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Proof.

- Let G be a min counterexample, and let $v \in V(G)$ have min degree.
- $G - v$ has an (f, N)-coloring, ψ, where $d(u) + 1 \mid N \; \forall u \in V(G)$.
- Let $\psi(v) = [N] \setminus (\cup_{u \in N(v)} \psi(u))$, i.e. color v what it doesn't "see."
- v "sees" at most $\sum_{u \in N(v)} f(u) \cdot N \leq d(v) \cdot N/(d(v)+1)$ colors, so
- $|\phi(v)| \geq N/(d(v)+1)$, as required.

□

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"Local Fractional Greedy" simultaneously generalizes:

Corollary (Frac. Relaxation of Greedy Bound): $\chi_f(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the max degree.

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- Obvious necessary condition: every clique K satisfies $\sum_{v \in K} f(v) \leq 1$.
- Local Fractional Brooks': we prove this condition is also sufficient.

Local Fractional Brooks'

Theorem (Local Fractional Brooks', K.-Postle '18+)

If $f(v) \leq 1/(d(v) + 1/2)$ *for all* $v \in V(G)$ *and every clique* K *satisfies* $\sum_{v \in K} f(v) \leq 1$*, then* G *is* f*-colorable.*

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Corollary (K.-Postle)

Let $\sigma \leq 1/2$. If each clique $K \subseteq V(G)$ has at most $(1 - \sigma)(|K| - \sigma)$ simplicial vertices, then

$$\alpha(G) \geq \sum_{v \in V(G)} 1/(d(v) + 1 - \sigma).$$

Local Demands

Coloring $\chi \leq$	Local demands $f(v) \leq$	deg-seq ind. # $\alpha \geq \sum_v$

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$(1 + o(1)) \frac{\Delta}{\ln \Delta}$ (if $\omega \leq 2$, Molloy)	**??**	$(1 - o(1)) \frac{d(v)}{\ln d(v)}$ (if $\omega \leq 2$, Shearer)

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Conjecture ("Local Independence Reed's," Brause et al. '16):

$\alpha(G) \geq \sum_{v \in V(G)} \frac{2}{d(v)+1+\omega(v)}$, where $\omega(v)$ is the size of the largest clique containing v.

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If $f(v) \leq 2/(d(v) + 1 + \omega(v))$ for all $v \in V(G)$, then G is f-colorable.

If true, local fractional Reed's implies:

- local independence Reed's,
- the fractional relaxation of Reed's Conjecture, and
- local fractional Brooks'.

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Borodin-Kostochka Conj: If $\omega \leq \Delta - 1$ and $\Delta \geq 9$, then $\chi \leq \Delta - 1$.

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Local demands analogue of large Δ or χ: large min degree/small demands.

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For every $\sigma < 3/4$ there exists $\delta \in \mathbb{N}$ such that the following holds. If $f(v) \leq 1/(d(v) + 1 - \sigma)$ and $d(v) \geq \delta$ for all $v \in V(G)$ and every clique K satisfies $\sum_{v \in K} f(v) \leq 1$, then G is f-colorable.

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Farzad-Molloy-Reed '05: Characterized the $(\Delta + 1 - k)$-critical graphs for small k and large Δ.

Problem: Determine the "fractionally $1/(d(v) + 1 - \sigma)$-critical" graphs of large min degree for small σ.

Triangle-free graphs

If G is triangle-free then...

Shearer '91: ... $\alpha(G) \geq \sum_{v \in V(G)} (1 - o(1)) \log d(v)/d(v)$.

Molloy '17+: ... $\chi(G) \leq (1 + o(1))\Delta(G)/\log(\Delta(G))$.

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All three imply:

Corollary: The Ramsey number $R(3, k) \leq (1 + o(1))k^2/\log k$.

Equivalently, every triangle-free graph on n vertices satisfies

$$n/\alpha(G) \leq (\sqrt{2} + o(1))\sqrt{n/\log n}.$$

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Conjecture (Cames van Batenburg et al. '18): If G is triangle-free, then $\chi_f(G) \leq (\sqrt{2} + o(1))\sqrt{n/\log n}$.

Triangle-free graphs

If G is triangle-free then...

Shearer '91: ... $\alpha(G) \geq \sum_{v \in V(G)} (1 - o(1)) \log d(v)/d(v)$.

Molloy '17+: ... $\chi(G) \leq (1 + o(1))\Delta(G)/\log(\Delta(G))$.

Conjecture (Local fractional Shearer/Molloy)

If $f(v) \leq (1 - o(1)) \log d(v)/d(v)$ for all $v \in V(G)$ and G is triangle-free, then G is f-colorable.

All three imply:

Corollary: The Ramsey number $R(3, k) \leq (1 + o(1))k^2/\log k$.

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If true, local fractional S/M implies this conjecture.

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An approximate version of local fractional S/M:

Theorem (K.-Postle '19++)

If $f(v) \leq (2e - o(1)) \log d(v)/(d(v) \log \log d(v))$ and G is triangle-free, then G is f-colorable.

Any improvement over local fractional greedy for triangle-free is nontrivial and was not previously known.

K_r-free graphs

Let G be a K_r-free graph on n vertices of avg degree d and max degree Δ.

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Conjecture (Local fractional A-E-K-S)

For every r, there exists some constant c such that the following holds. If $f(v) \le c\log d(v)/d(v)$ for each $v \in V(G)$ and G is a K_r-free, then G is f-colorable.

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If $f(v) = O\left(\frac{\log d(v)}{d(v)(\log\log d(v))^2}\right)$ and G is K_r-free, then G is f-colorable.

Corollary: $\alpha(G) \geq c \sum_{v \in V(G)} \frac{\log d(v)}{d(v)(\log\log d(v))^2}$.

Conclusion

Local Fractional...	hypotheses	demands: $f(v) \leq$
Brooks'	$\sum_{v \in K} f(v) \leq 1, \forall K$	$1/(d(v) + 1/2)$
Borodin-Kostochka	$\sigma < 3/4$, $\delta(G)$ large, $\sum_{v \in K} f(v) \leq 1, \forall K$	$1/(d(v) + 1 - \sigma)$
Reed's		$2/(d(v) + 1 + \omega(v))$

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Shearer/Molloy	triangle-free	$(1 - o(1)) \ln d(v)/d(v)$
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A-E-K-S	K_r-free	$O(\ln d(v)/d(v))$
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Vizing's	line-graph	$1/(\omega(v) + 1)$
TCC	total-graph	$1/(\omega(v) + 1)$
Perfect graph Thm	perfect graph	$1/\omega(v)$

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proved/conjectured

Thanks for listening!