Large Induced Forests in Planar and Subcubic Graphs of Girth 4 and 5

Tom Kelly[1] Chun-Hung Liu[2]

[1]Department of Combinatorics & Optimization
University of Waterloo

[2]Department of Mathematics
Princeton University

SIAM DM 2016

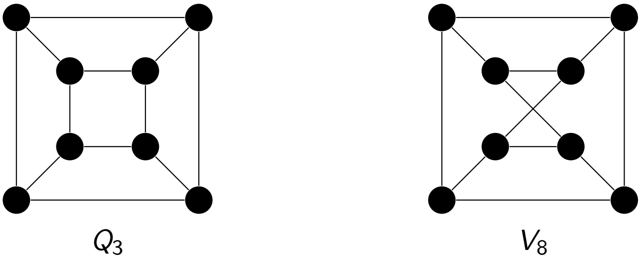
Introduction

- A set $S \subseteq V(G)$ is a *feedback vertex set* (FVS) of G if $G - S$ is a forest.

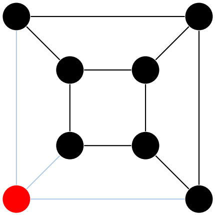
Introduction

- A set $S \subseteq V(G)$ is a *feedback vertex set* (FVS) of G if $G - S$ is a forest.
- Let $\phi(G)$ denote the size of a minimum feedback vertex set of G.

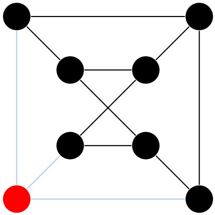
- A set $S \subseteq V(G)$ is a *feedback vertex set* (FVS) of G if $G - S$ is a forest.
- Let $\phi(G)$ denote the size of a minimum feedback vertex set of G.



- A set $S \subseteq V(G)$ is a *feedback vertex set* (FVS) of G if $G - S$ is a forest.
- Let $\phi(G)$ denote the size of a minimum feedback vertex set of G.



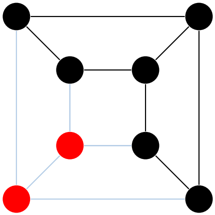
Q_3



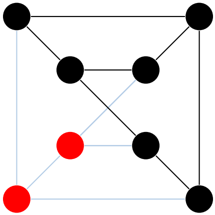
V_8

Introduction

- A set $S \subseteq V(G)$ is a *feedback vertex set* (FVS) of G if $G - S$ is a forest.
- Let $\phi(G)$ denote the size of a minimum feedback vertex set of G.



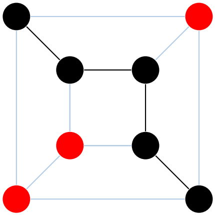
Q_3



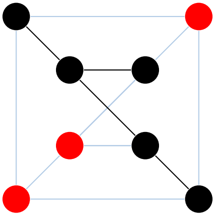
V_8

Introduction

- A set $S \subseteq V(G)$ is a *feedback vertex set* (FVS) of G if $G - S$ is a forest.
- Let $\phi(G)$ denote the size of a minimum feedback vertex set of G.



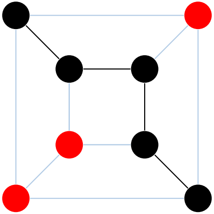
Q_3



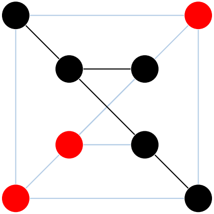
V_8

Introduction

- A set $S \subseteq V(G)$ is a *feedback vertex set* (FVS) of G if $G - S$ is a forest.
- Let $\phi(G)$ denote the size of a minimum feedback vertex set of G.



$\phi(Q_3) = 3$



$\phi(V_8) = 3$

Introduction

- A set $S \subseteq V(G)$ is a *feedback vertex set* (FVS) of G if $G - S$ is a forest.
- Let $\phi(G)$ denote the size of a minimum feedback vertex set of G.

Introduction

- A set $S \subseteq V(G)$ is a *feedback vertex set* (FVS) of G if $G - S$ is a forest.
- Let $\phi(G)$ denote the size of a minimum feedback vertex set of G.
- Given a graph G and $k \in \mathbb{Z}$, deciding if $\phi(G) \leq k$ is NP-hard, even if G is planar with maximum degree four.

Introduction

- A set $S \subseteq V(G)$ is a *feedback vertex set* (FVS) of G if $G - S$ is a forest.
- Let $\phi(G)$ denote the size of a minimum feedback vertex set of G.
- Given a graph G and $k \in \mathbb{Z}$, deciding if $\phi(G) \leq k$ is NP-hard, even if G is planar with maximum degree four.

Question

Can we upper bound $\phi(G)$ for certain classes of graphs?

Subcubic Results (Vertex Bounds)

Theorem (Bondy, Hopkins, and Staton, 1987)

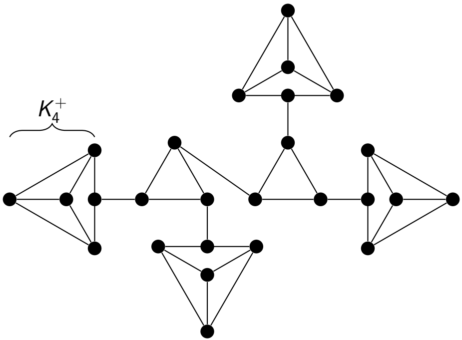
If G is a connected subcubic graph on n vertices and $G \neq K_4$, then $\phi(G) \leq \frac{3n}{8} + \frac{1}{4}$.

Subcubic Results (Vertex Bounds)

Theorem (Bondy, Hopkins, and Staton, 1987)

If G is a connected subcubic graph on n vertices and $G \neq K_4$, then $\phi(G) \leq \frac{3n}{8} + \frac{1}{4}$.

This is tight if G is cubic and every nontrivial block of G is K_3 or K_4^+.



Subcubic Results (Vertex Bounds)

Theorem (Bondy, Hopkins, and Staton, 1987)

If G is a connected subcubic graph on n vertices and $G \neq K_4$, then $\phi(G) \leq \frac{3n}{8} + \frac{1}{4}$.

Theorem (BHS, 1987)

Additionally, if G has girth ≥ 4, then $\phi(G) \leq \frac{n}{3} + \frac{1}{3}$.

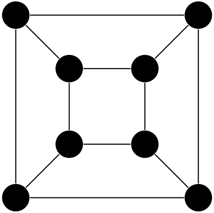
Subcubic Results (Vertex Bounds)

Theorem (Bondy, Hopkins, and Staton, 1987)

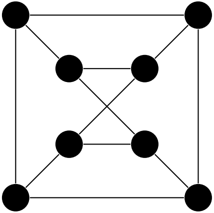
If G is a connected subcubic graph on n vertices and $G \neq K_4$, then $\phi(G) \leq \frac{3n}{8} + \frac{1}{4}$.

Theorem (BHS, 1987)

Additionally, if G has girth ≥ 4, then $\phi(G) \leq \frac{n}{3} + \frac{1}{3}$.



$\phi(Q_3) = 3$



$\phi(V_8) = 3$

Subcubic Results (Edge Bound)

Theorem (Alon, Mubayi, and Thomas, 2001)

If G is a 2-connected subcubic graph with m edges, then

$$\phi(G) \leq \frac{m}{4} + \begin{cases} \frac{1}{2} & \text{if } G = K_4, \\ \frac{1}{4} & \text{if } G = K_3, K_4^+, \\ 0 & \text{otherwise.} \end{cases}$$

Subcubic Results (Edge Bound)

Theorem (Alon, Mubayi, and Thomas, 2001)

If G is a 2-connected subcubic graph with m edges, then

$$\phi(G) \leq \frac{m}{4} + \begin{cases} \frac{1}{2} & \text{if } G = K_4, \\ \frac{1}{4} & \text{if } G = K_3, K_4^+, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary (BHS)

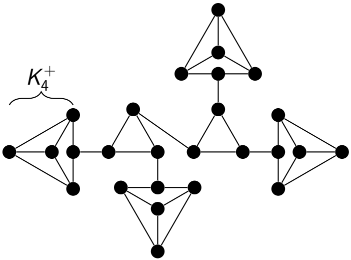
If $G \neq K_4$, then $\phi(G) \leq \frac{m}{4} + \frac{1}{4} \leq \frac{(3n/2)}{4} + \frac{1}{4}$.

Subcubic Results (Edge Bound)

Theorem (Alon, Mubayi, and Thomas, 2001)

If G is a 2-connected subcubic graph with m edges, then

$$\phi(G) \leq \frac{m}{4} + \begin{cases} \frac{1}{2} & \text{if } G = K_4, \\ \frac{1}{4} & \text{if } G = K_3, K_4^+, \\ 0 & \text{otherwise.} \end{cases}$$

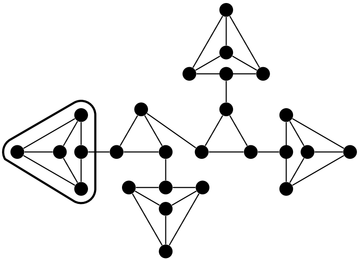


Subcubic Results (Edge Bound)

Theorem (Alon, Mubayi, and Thomas, 2001)

If G is a 2-connected subcubic graph with m edges, then

$$\phi(G) \leq \frac{m}{4} + \begin{cases} \frac{1}{2} & \text{if } G = K_4, \\ \frac{1}{4} & \text{if } G = K_3, K_4^+, \\ 0 & \text{otherwise.} \end{cases}$$

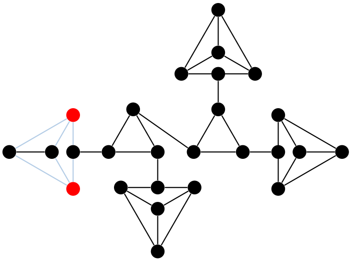


Subcubic Results (Edge Bound)

Theorem (Alon, Mubayi, and Thomas, 2001)

If G is a 2-connected subcubic graph with m edges, then

$$\phi(G) \leq \frac{m}{4} + \begin{cases} \frac{1}{2} & \text{if } G = K_4, \\ \frac{1}{4} & \text{if } G = K_3, K_4^+, \\ 0 & \text{otherwise.} \end{cases}$$

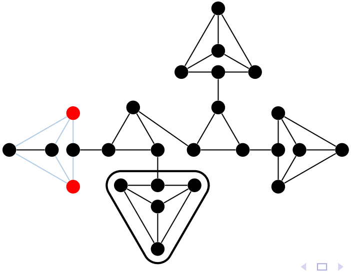


Subcubic Results (Edge Bound)

Theorem (Alon, Mubayi, and Thomas, 2001)

If G is a 2-connected subcubic graph with m edges, then

$$\phi(G) \leq \frac{m}{4} + \begin{cases} \frac{1}{2} & \text{if } G = K_4, \\ \frac{1}{4} & \text{if } G = K_3, K_4^+, \\ 0 & \text{otherwise.} \end{cases}$$

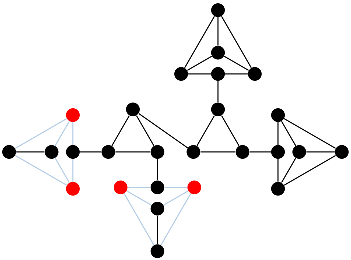


Subcubic Results (Edge Bound)

Theorem (Alon, Mubayi, and Thomas, 2001)

If G is a 2-connected subcubic graph with m edges, then

$$\phi(G) \leq \frac{m}{4} + \begin{cases} \frac{1}{2} & \text{if } G = K_4, \\ \frac{1}{4} & \text{if } G = K_3, K_4^+, \\ 0 & \text{otherwise.} \end{cases}$$

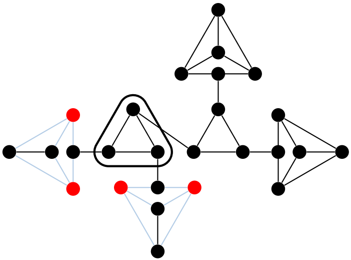


Subcubic Results (Edge Bound)

Theorem (Alon, Mubayi, and Thomas, 2001)

If G is a 2-connected subcubic graph with m edges, then

$$\phi(G) \leq \frac{m}{4} + \begin{cases} \frac{1}{2} & \text{if } G = K_4, \\ \frac{1}{4} & \text{if } G = K_3, K_4^+, \\ 0 & \text{otherwise.} \end{cases}$$

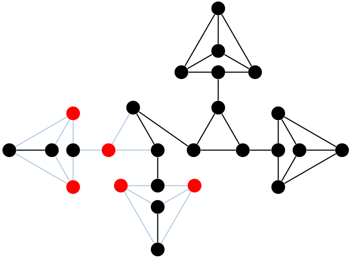


Subcubic Results (Edge Bound)

Theorem (Alon, Mubayi, and Thomas, 2001)

If G is a 2-connected subcubic graph with m edges, then

$$\phi(G) \leq \frac{m}{4} + \begin{cases} \frac{1}{2} & \text{if } G = K_4, \\ \frac{1}{4} & \text{if } G = K_3, K_4^+, \\ 0 & \text{otherwise.} \end{cases}$$



Our Subcubic Results (Girth 4)

Theorem (KL, 2016)

If G is a 2-connected subcubic graph of girth ≥ 4 on m edges, then

$$\phi(G) \leq \frac{2m}{9} + \begin{cases} \frac{1}{3} & \text{if } G = Q_3, V_8, \\ \frac{2}{9} & \text{if } G = K_{3,3}^{-}, \\ \frac{1}{9} & \text{if } G \text{ is one of five graphs}, \\ 0 & \text{otherwise.} \end{cases}$$

Our Subcubic Results (Girth 4)

Theorem (KL, 2016)

If G is a 2-connected subcubic graph of girth ≥ 4 *on m edges, then*

$$\phi(G) \leq \frac{2m}{9} + \begin{cases} \frac{1}{3} & \text{if } G = Q_3, V_8, \\ \frac{2}{9} & \text{if } G = K_{3,3}^-, \\ \frac{1}{9} & \text{if } G \text{ is one of five graphs}, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary (Zheng and Lu 1990)

If $G \notin \{Q_3, V_8\}$, $\phi(G) \leq \frac{n}{3}$.

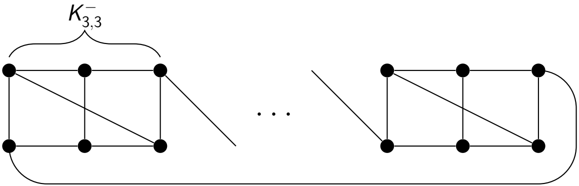
Our Subcubic Results (Girth 4)

Theorem (KL, 2016)

If G is a 2-connected subcubic graph of girth ≥ 4 on m edges, then

$$\phi(G) \leq \frac{2m}{9} + \begin{cases} \frac{1}{3} & \text{if } G = Q_3, V_8, \\ \frac{2}{9} & \text{if } G = K_{3,3}^{-}, \\ \frac{1}{9} & \text{if } G \text{ is one of five graphs}, \\ 0 & \text{otherwise.} \end{cases}$$

This is tight for infinitely many 2-connected graphs.



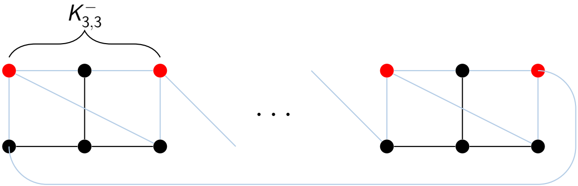
Our Subcubic Results (Girth 4)

Theorem (KL, 2016)

If G is a 2-connected subcubic graph of girth ≥ 4 on m edges, then

$$\phi(G) \leq \frac{2m}{9} + \begin{cases} \frac{1}{3} & \text{if } G = Q_3, V_8, \\ \frac{2}{9} & \text{if } G = K_{3,3}^{-}, \\ \frac{1}{9} & \text{if } G \text{ is one of five graphs}, \\ 0 & \text{otherwise.} \end{cases}$$

This is tight for infinitely many 2-connected graphs.



Our Subcubic Results (Girth 5)

Theorem (KL, 2016)

If G is a 2-connected subcubic graph on m edges of girth ≥ 5, then

$$\phi(G) \leq \frac{m}{5} + \begin{cases} \frac{2}{5} & \text{if } G \text{ is one of two cubic graphs,} \\ \frac{1}{5} & \text{if } G \text{ is one of finitely many graphs,} \\ 0 & \text{otherwise.} \end{cases}$$

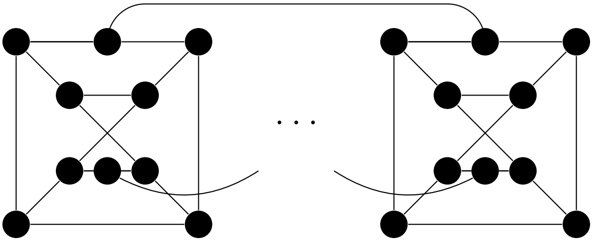
Our Subcubic Results (Girth 5)

Theorem (KL, 2016)

If G is a 2-connected subcubic graph on m edges of girth ≥ 5, then

$$\phi(G) \leq \frac{m}{5} + \begin{cases} \frac{2}{5} & \text{if } G \text{ is one of two cubic graphs,} \\ \frac{1}{5} & \text{if } G \text{ is one of finitely many graphs,} \\ 0 & \text{otherwise.} \end{cases}$$

This is tight for infinitely many 2-connected graphs.



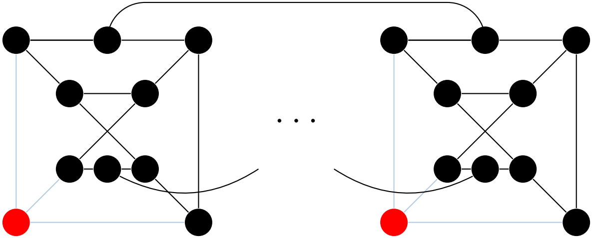
Our Subcubic Results (Girth 5)

Theorem (KL, 2016)

If G is a 2-connected subcubic graph on m edges of girth ≥ 5, then

$$\phi(G) \leq \frac{m}{5} + \begin{cases} \frac{2}{5} & \text{if } G \text{ is one of two cubic graphs,} \\ \frac{1}{5} & \text{if } G \text{ is one of finitely many graphs,} \\ 0 & \text{otherwise.} \end{cases}$$

This is tight for infinitely many 2-connected graphs.



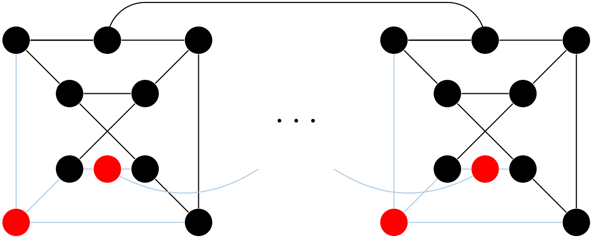
Our Subcubic Results (Girth 5)

Theorem (KL, 2016)

If G is a 2-connected subcubic graph on m edges of girth ≥ 5, then

$$\phi(G) \leq \frac{m}{5} + \begin{cases} \frac{2}{5} & \text{if } G \text{ is one of two cubic graphs,} \\ \frac{1}{5} & \text{if } G \text{ is one of finitely many graphs,} \\ 0 & \text{otherwise.} \end{cases}$$

This is tight for infinitely many 2-connected graphs.



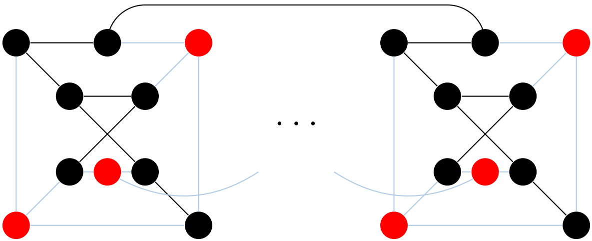
Our Subcubic Results (Girth 5)

Theorem (KL, 2016)

If G is a 2-connected subcubic graph on m edges of girth ≥ 5, then

$$\phi(G) \leq \frac{m}{5} + \begin{cases} \frac{2}{5} & \text{if } G \text{ is one of two cubic graphs,} \\ \frac{1}{5} & \text{if } G \text{ is one of finitely many graphs,} \\ 0 & \text{otherwise.} \end{cases}$$

This is tight for infinitely many 2-connected graphs.



Subcubic Summary

If G is a 2-connected subcubic graph on m edges, then

Girth	$\phi(G) \leq$?	Exceptions
3	$\frac{m}{4}$	K_4, K_3, K_4^+
4	$\frac{2m}{9}$	$Q_3, V_8, K_{3,3}^-$, five other graphs
5	$\frac{m}{5}$	finitely many

Planar Results & Conjectures

If G is a planar graph on n vertices then $\phi(G) \leq$?

Planar Results & Conjectures

If G is a planar graph on n vertices then $\phi(G) \leq$?

Girth	Conjectured	Best Known
3		
Bipartite		
4		
5		

Planar Results & Conjectures

If G is a planar graph on n vertices then $\phi(G) \leq$?

Girth	Conjectured	Best Known
3	$\frac{n}{2}$ Albertson & Berman 1979	
Bipartite		
4		
5		

Planar Results & Conjectures

If G is a planar graph on n vertices then $\phi(G) \leq$?

Girth	Conjectured	Best Known
3	$\frac{n}{2}$ Albertson & Berman 1979	$\frac{3n}{5}$ Borodin 1976
Bipartite		
4		
5		

Planar Results & Conjectures

If G is a planar graph on n vertices then $\phi(G) \leq$?

Girth	Conjectured	Best Known
3	$\frac{n}{2}$ Albertson & Berman 1979	$\frac{3n}{5}$ Borodin 1976
Bipartite	$\frac{3n}{8}$ Akiyama & Watanabe 1987	
4		
5		

Planar Results & Conjectures

If G is a planar graph on n vertices then $\phi(G) \leq$?

Girth	Conjectured	Best Known
3	$\frac{n}{2}$ Albertson & Berman 1979	$\frac{3n}{5}$ Borodin 1976
Bipartite	$\frac{3n}{8}$ Akiyama & Watanabe 1987	$\frac{3n}{7}$ Wang, Xie, & Yu 2016
4		
5		

Planar Results & Conjectures

If G is a planar graph on n vertices then $\phi(G) \leq$?

Girth	Conjectured	Best Known
3	$\frac{n}{2}$ Albertson & Berman 1979	$\frac{3n}{5}$ Borodin 1976
Bipartite	$\frac{3n}{8}$ Akiyama & Watanabe 1987	$\frac{3n}{7}$ Wang, Xie, & Yu 2016
4	$\frac{3n}{8}$?	
5		

Planar Results & Conjectures

If G is a planar graph on n vertices then $\phi(G) \leq$?

Girth	Conjectured	Best Known
3	$\frac{n}{2}$ Albertson & Berman 1979	$\frac{3n}{5}$ Borodin 1976
Bipartite	$\frac{3n}{8}$ Akiyama & Watanabe 1987	$\frac{3n}{7}$ Wang, Xie, & Yu 2016
4	$\frac{3n}{8}$?	$\frac{5n}{11}$ Dross, Montassier, Pinlou 2014
5		

Planar Results & Conjectures

If G is a planar graph on n vertices then $\phi(G) \leq$?

Girth	Conjectured	Best Known
3	$\frac{n}{2}$ Albertson & Berman 1979	$\frac{3n}{5}$ Borodin 1976
Bipartite	$\frac{3n}{8}$ Akiyama & Watanabe 1987	$\frac{3n}{7}$ Wang, Xie, & Yu 2016
4	$\frac{3n}{8}$?	$\frac{5n}{11}$ Dross, Montassier, Pinlou 2014
5	$\frac{3n}{10}$ Kowalik, Lužar, Škrekovski 2010	

Planar Results & Conjectures

If G is a planar graph on n vertices then $\phi(G) \leq$?

Girth	Conjectured	Best Known
3	$\frac{n}{2}$ Albertson & Berman 1979	$\frac{3n}{5}$ Borodin 1976
Bipartite	$\frac{3n}{8}$ Akiyama & Watanabe 1987	$\frac{3n}{7}$ Wang, Xie, & Yu 2016
4	$\frac{3n}{8}$?	$\frac{5n}{11}$ Dross, Montassier, Pinlou 2014
5	$\frac{3n}{10}$ Kowalik, Lužar, Škrekovski 2010	$\frac{25n}{69}$ ($\approx .362n$) DMP 2014

Planar Results & Conjectures

If G is a planar graph on n vertices then $\phi(G) \leq$?

Girth	Conjectured	Best Known
3	$\frac{n}{2}$ Albertson & Berman 1979	$\frac{3n}{5}$ Borodin 1976
Bipartite	$\frac{3n}{8}$ Akiyama & Watanabe 1987	$\frac{3n}{7}$ Wang, Xie, & Yu 2016
4	$\frac{3n}{8}$?	$\frac{5n}{11}$ Dross, Montassier, Pinlou 2014
5	$\frac{3n}{10}$ Kowalik, Lužar, Škrekovski 2010	$\frac{n}{3}$ KL 2016

Our Planar Results

Theorem (KL, 2016)

If G is a connected planar graph on n vertices and m edges of girth ≥ 5, then $\phi(G) \leq \frac{2m-n+2}{7}$.

Our Planar Results

Theorem (KL, 2016)

If G is a connected planar graph on n vertices and m edges of girth ≥ 5, then $\phi(G) \leq \frac{2m-n+2}{7}$.

By Euler's formula,

Corollary

If G is a planar graph on n vertices and m edges of girth ≥ 5, then $\phi(G) \leq \frac{2m-n+2}{7} \leq \frac{m}{5} \leq \frac{n-2}{3}$.

Our Planar Results

Theorem (KL, 2016)

If G is a connected planar graph on n vertices and m edges of girth ≥ 5, then $\phi(G) \leq \frac{2m-n+2}{7}$.

By Euler's formula,

Corollary

If G is a planar graph on n vertices and m edges of girth ≥ 5, then $\phi(G) \leq \frac{2m-n+2}{7} \leq \frac{m}{5} \leq \frac{n-2}{3}$.

Conjecture (DMP, 2015)

If G is a planar graph on m edges of girth $\geq g$, then $\phi(G) \leq \frac{g}{5}$.

The real theorems

Theorem (Girth 4 Subcubic)

If G is a 2-connected subcubic graph of girth ≥ 4 *on m edges, then* $\phi(G) \leq \frac{2m}{9} + \frac{1}{3}$

The real theorems

Theorem (Real Girth 4 Subcubic)

If G is a 2-connected subcubic graph with no disjoint triangles on m edges, then $\phi(G) \leq \frac{2m}{9} + \frac{2}{3}$

The real theorems

Theorem (Real Girth 4 Subcubic)

If G is a 2-connected subcubic graph with no disjoint triangles on m edges, then $\phi(G) \leq \frac{2m}{9} + \frac{2}{3}$

Theorem (Girth 5 Subcubic)

If G is a 2-connected subcubic graph on m edges of girth ≥ 5, then $\phi(G) \leq \frac{m}{5} + \frac{2}{5}$

The real theorems

Theorem (Real Girth 4 Subcubic)

If G is a 2-connected subcubic graph with no disjoint triangles on m edges, then $\phi(G) \leq \frac{2m}{9} + \frac{2}{3}$

Theorem (Real Girth 5 Subcubic)

If G is a 2-connected subcubic graph on m edges with no disjoint triangles or 4-cycles, then $\phi(G) \leq \frac{m}{5} + \frac{4}{5}$

The real theorems

Theorem (Real Girth 4 Subcubic)

If G is a 2-connected subcubic graph with no disjoint triangles on m edges, then $\phi(G) \leq \frac{2m}{9} + \frac{2}{3}$

Theorem (Real Girth 5 Subcubic)

If G is a 2-connected subcubic graph on m edges with no disjoint triangles or 4-cycles, then $\phi(G) \leq \frac{m}{5} + \frac{4}{5}$

Theorem (Real Planar)

If G is a connected planar graph on n vertices and m edges of girth ≥ 5, then $\phi(G) \leq \frac{2m-n+2}{7}$.

Theorem (Real Girth 4 Subcubic)

If G is a 2-connected subcubic graph with no disjoint triangles on m edges, then $\phi(G) \leq \frac{2m}{9} + \frac{2}{3}$

Theorem (Real Girth 5 Subcubic)

If G is a 2-connected subcubic graph on m edges with no disjoint triangles or 4-cycles, then $\phi(G) \leq \frac{m}{5} + \frac{4}{5}$

Theorem (Real Planar)

If G is a 2-connected subcubic planar graph on n vertices and m edges with no disjoint triangles or 4-cycles, then $\phi(G) \leq \frac{2m-n+6}{7}$.

Proof Outline of the Real Theorems

Let G be a minimum counterexample to the real girth 5 subcubic theorem.

Proof Outline of the Real Theorems

Let G be a minimum counterexample to the real girth 5 subcubic theorem. Then

- G is internally 3-edge-connected

Proof Outline of the Real Theorems

Let G be a minimum counterexample to the real girth 5 subcubic theorem. Then

- G is internally 3-edge-connected
- G has no triangle

Proof Outline of the Real Theorems

Let G be a minimum counterexample to the real girth 5 subcubic theorem. Then

- G is internally 3-edge-connected
- G has no triangle
- G has girth at least five

Proof Outline of the Real Theorems

Let G be a minimum counterexample to the real girth 5 subcubic theorem. Then

- G is internally 3-edge-connected
- G has no triangle
- G has girth at least five
- G is cubic

Proof Outline of the Real Theorems

Let G be a minimum counterexample to the real girth 5 subcubic theorem. Then

- G is internally 3-edge-connected
- G has no triangle
- G has girth at least five
- G is cubic
- For every $v \in V(G)$ and $av, bv \in E(G)$, $G - v$ contains two disjoint 5-cycles, one containing a and the other containing b

Proof Outline of the Real Theorems

Let G be a minimum counterexample to the real girth 5 subcubic theorem. Then

- G is internally 3-edge-connected
- G has no triangle
- G has girth at least five
- G is cubic
- For every $v \in V(G)$ and $av, bv \in E(G)$, $G - v$ contains two disjoint 5-cycles, one containing a and the other containing b
- G is the dodecahedron.

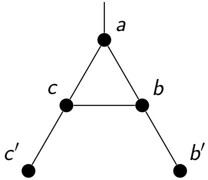
Proof Outline of the Real Theorems

Let G be a minimum counterexample to the real girth 5 subcubic theorem. Then

- G is internally 3-edge-connected
- G has no triangle
- G has girth at least five
- G is cubic
- For every $v \in V(G)$ and $av, bv \in E(G)$, $G - v$ contains two disjoint 5-cycles, one containing a and the other containing b
- G is the dodecahedron.

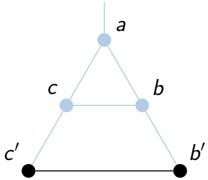
Proving G has no triangle

- Suppose G contains a triangle abc with $bb', cc' \in E(G)$ and $b'c' \notin E(G)$.



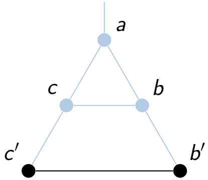
Proving G has no triangle

- Suppose G contains a triangle abc with $bb', cc' \in E(G)$ and $b'c' \notin E(G)$.
- Let $G' = G - \{a, b, c\} + b'c'$.



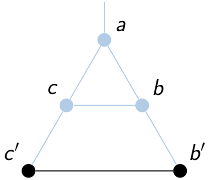
Proving G has no triangle

- Suppose G contains a triangle abc with $bb', cc' \in E(G)$ and $b'c' \notin E(G)$.
- Let $G' = G - \{a, b, c\} + b'c'$. Note $|V(G')| = |V(G)| - 3$, $|E(G')| = |E(G)| - 5$, and G contains no disjoint short cycles.



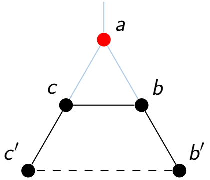
Proving G has no triangle

- Suppose G contains a triangle abc with $bb', cc' \in E(G)$ and $b'c' \notin E(G)$.
- Let $G' = G - \{a, b, c\} + b'c'$. Note $|V(G')| = |V(G)| - 3$, $|E(G')| = |E(G)| - 5$, and G contains no disjoint short cycles.
- By minimality, there is a FVS S of G' of size at most $\frac{m-5}{5} + \epsilon_5(G') = \frac{m}{5} + \epsilon_5(G') - 1$



Proving G has no triangle

- Suppose G contains a triangle abc with $bb', cc' \in E(G)$ and $b'c' \notin E(G)$.
- Let $G' = G - \{a, b, c\} + b'c'$. Note $|V(G')| = |V(G)| - 3$, $|E(G')| = |E(G)| - 5$, and G contains no disjoint short cycles.
- By minimality, there is a FVS S of G' of size at most $\frac{m-5}{5} + \epsilon_5(G') = \frac{m}{5} + \epsilon_5(G') - 1$



- Then $S \cup \{a\}$ is a FVS of G of size at most $\frac{m}{5} + \epsilon_5(G')$.

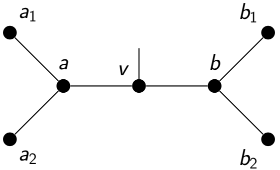
Proof Outline of the Real Theorems

Let G be a minimum counterexample to the real girth 5 subcubic theorem. Then

- G is internally 3-edge-connected
- G has no triangle
- G has girth at least five
- G is cubic
- For every $v \in V(G)$ and $av, bv \in E(G)$, $G - v$ contains two disjoint 5-cycles, one containing a and the other containing b
- G is the dodecahedron.

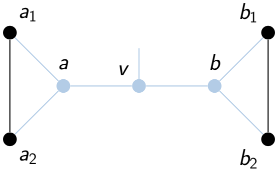
Proving the 5-cycle property

- Let $v \in V(G)$ and $av, bv \in E(G)$. Say $aa_1, aa_2, bb_1, bb_2 \in E(G)$ and $a_1a_2, b_1b_2 \notin E(G)$.



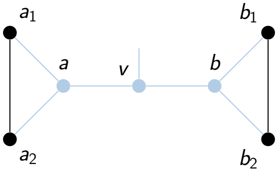
Proving the 5-cycle property

- Let $v \in V(G)$ and $av, bv \in E(G)$. Say $aa_1, aa_2, bb_1, bb_2 \in E(G)$ and $a_1a_2, b_1b_2 \notin E(G)$.
- Let $G' = G - \{a, b, v\} + a_1a_2 + b_1b_2$.



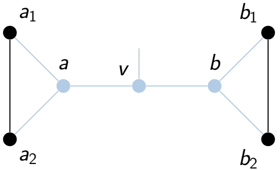
Proving the 5-cycle property

- Let $v \in V(G)$ and $av, bv \in E(G)$. Say $aa_1, aa_2, bb_1, bb_2 \in E(G)$ and $a_1a_2, b_1b_2 \notin E(G)$.
- Let $G' = G - \{a, b, v\} + a_1a_2 + b_1b_2$. Note $|V(G')| = |V(G)| - 3$, and $|E(G')| = |E(G)| - 5$.



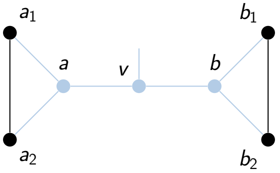
Proving the 5-cycle property

- Let $v \in V(G)$ and $av, bv \in E(G)$. Say $aa_1, aa_2, bb_1, bb_2 \in E(G)$ and $a_1a_2, b_1b_2 \notin E(G)$.
- Let $G' = G - \{a, b, v\} + a_1a_2 + b_1b_2$. Note $|V(G')| = |V(G)| - 3$, and $|E(G')| = |E(G)| - 5$.
- Suppose for contradiction $G - v$ does not contain disjoint 4-cycles.



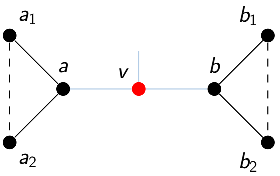
Proving the 5-cycle property

- Let $v \in V(G)$ and $av, bv \in E(G)$. Say $aa_1, aa_2, bb_1, bb_2 \in E(G)$ and $a_1a_2, b_1b_2 \notin E(G)$.
- Let $G' = G - \{a, b, v\} + a_1a_2 + b_1b_2$. Note $|V(G')| = |V(G)| - 3$, and $|E(G')| = |E(G)| - 5$.
- Suppose for contradiction $G - v$ does not contain disjoint 4-cycles.
- By minimality, there is a FVS S of G' of size at most $\frac{m-5}{5} + \epsilon_5(G') = \frac{m}{5} + \epsilon_5(G') - 1$.



Proving the 5-cycle property

- Let $v \in V(G)$ and $av, bv \in E(G)$. Say $aa_1, aa_2, bb_1, bb_2 \in E(G)$ and $a_1a_2, b_1b_2 \notin E(G)$.
- Let $G' = G - \{a, b, v\} + a_1a_2 + b_1b_2$. Note $|V(G')| = |V(G)| - 3$, and $|E(G')| = |E(G)| - 5$.
- Suppose for contradiction $G - v$ does not contain disjoint 4-cycles.
- By minimality, there is a FVS S of G' of size at most $\frac{m-5}{5} + \epsilon_5(G') = \frac{m}{5} + \epsilon_5(G') - 1$.



- Then $S \cup \{v\}$ is a FVS of G of size at most $\frac{m}{5} + \epsilon_5(G')$.

Thanks for listening!

References I

J. Akiyama and M. Watanabe,
Maximum induced forests of planar graphs,
Graphs Combin., 3 (1987), 201–202.

M. O. Albertson and D. M. Berman,
A conjecture on planar graphs.
Graph Theory and Related Topics, Academic Press 1979.

N. Alon, D. Mubayi, and R. Thomas,
Large induced forests in sparse graphs,
J. Graph Theory 38 (2001), 113–123.

J. A. Bondy, G. Hopkins, and W. Staton,
Lower bounds for induced forests in cubic graphs,
Canad. Math. Bull. 30 (1987), 193–199.

References II

O. V. Borodin,
A proof of B. Grünbaum's conjecture on the acyclic 5-colorability of planar graphs,
Dokl. Akad. Nauk SSSR, 231 (1976) 18–20.

F. Dross, M. Montassier, and A. Pinlou,
Large induced forests in planar graphs with girth 4 or 5,
arXiv:1409.1348

F. Dross, M. Montassier, and A. Pinlou,
A lower bound on the order of the largest induced forest in planar graphs with high girth,
arXiv:1504.01949

References III

L. Kowalik, B. Lužar, and R. Škrekovski,
An improved bound on the largest induced forests for triangle-free planar graphs,
Discrete Math. Theor. Comput. Sci., 12 (2010) 87–100.

J. Liu and C. Zhao,
A new bound on the feedback vertex sets in cubic graphs,
Discrete Math., 148 (1996), 119–131.

M. L. Zheng and X. Lu,
On the maximum induced forests of a connected cubic graph without triangles,
Discrete Math., 85 (1990), 89–96.