A proof of the Erdős–Faber–Lovász conjecture

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Joint work with:
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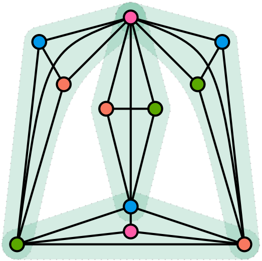
The Erdős–Faber–Lovász conjecture

proper coloring: adjacent vertices assigned different colors

chromatic number: min # colors used in proper coloring, denoted by χ

The Erdős–Faber–Lovász conjecture (1972)

If $G_1, \ldots, G_n$ are complete graphs, each on at most n vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^{n} G_i) \leq n$.



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One of Erdős' "three most favorite combinatorial problems":

- Erdős initially offered \$50 for a solution, raised to \$500.

  Faber, Lovász and I made this harmless looking conjecture at a party in Boulder Colorado in September 1972. Its difficulty was realised only slowly. I now offer 500 dollars for a proof or disproof. (Not long ago I only offered 50; the increase is not due to inflation but to the fact that I now think the problem is very difficult. Perhaps I am wrong.) *–Paul Erdős, 1981*

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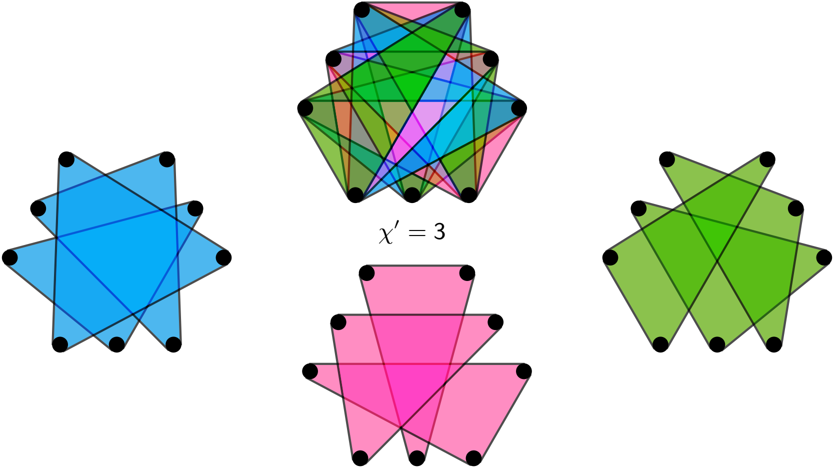
Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

The Erdős–Faber–Lovász conjecture is true for sufficiently large n.

Hypergraph edge-coloring

(proper) edge-coloring: no two edges of same color share a vertex

chromatic index: min # colors used in proper edge-coloring, denoted χ'



Erdős–Faber–Lovász conjecture (reformulated)

linear hypergraph: every pair of vertices contained in at most one edge

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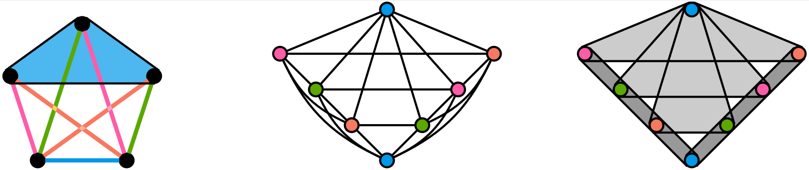
If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.



Line graph:

- edges $\rightarrow$ vertices: edges that share a vertex are adjacent
- proper edge-coloring $\rightarrow$ proper vertex-coloring

The previous formulation is equivalent:

If $G_1, \ldots, G_n$ are complete graphs, each on at most n vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^{n} G_i) \leq n$.

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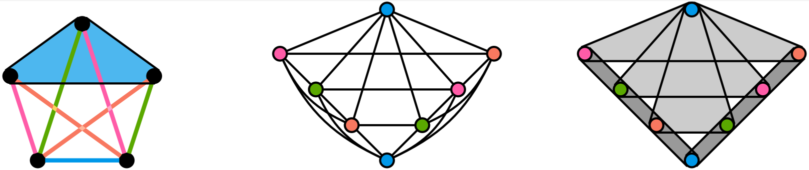

Hypergraph duality:

- edges $\rightarrow$ vertices and vertices $\rightarrow$ edges
- linearity is preserved

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Basic background

The Erdős–Faber–Lovász conjecture (1972)

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The Erdős–Faber–Lovász conjecture (1972)

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

- Graphs are linear hypergraphs
- Linear hypergraphs with n vertices have maximum degree $\leq n - 1$

Vizing's theorem (1964)

If G is a graph of maximum degree at most Δ, then $\chi'(G) \leq \Delta + 1$.

Corollary: EFL is true for graphs

Basic background

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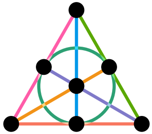
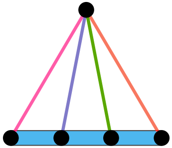
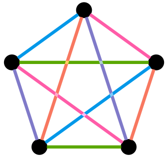
Extremal examples:







Finite projective plane of order k: $(k+1)$-uniform intersecting linear hypergraph with $n = k^2 + k + 1$ vertices and edges

Degenerate plane / near pencil: intersecting linear hypergraph with $n - 1$ size-two edges and one size-$(n-1)$ edge

Complete graph: $\binom{n}{2}$ size-two edges; if $\chi' < n$, then color classes are perfect matchings $\Rightarrow$ n is even

Basic background

The Erdős–Faber–Lovász conjecture (1972)

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

Direct approaches:

Trivial: $\chi'(\mathcal{H}) \leq 2n - 3$ (color greedily, in order of size)

Chang–Lawler (1989): $\chi'(\mathcal{H}) \leq \lceil 3n/2 - 2 \rceil$

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Relaxed parameters:

de Bruijn–Erdős (1948): true for intersecting hypergraphs

Seymour (1982): $\exists$ a matching of size at least $|\mathcal{H}|/n$

Kahn–Seymour (1992): fractional chromatic index is at most n

Basic background

The Erdős–Faber–Lovász conjecture (1972)

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

Probabilistic approach:

Faber–Harris (2019): EFL is true if $|e| \in [3, c\sqrt{n}]$ $\forall e \in \mathcal{H}$ $(c \ll 1)$

Kahn (1992): $\chi'(\mathcal{H}) \leq n + o(n)$

Our results

We confirm the EFL conjecture for all but finitely many hypergraphs:

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

If $\mathcal{H}$ is an n-vertex linear hypergraph where n is sufficiently large, then

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If $\mathcal{H}$ is an n-vertex linear hypergraph where n is sufficiently large, then

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We also prove a stability result, predicted by Kahn:

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

$\forall \delta > 0$, $\exists \sigma > 0$ such that the following holds for n sufficiently large.
If $\mathcal{H}$ is an n-vertex linear hypergraph such that

- $\Delta(\mathcal{H}) \leq (1-\delta)n$ and
- at most $(1-\delta)n$ edges have size $(1 \pm \delta)\sqrt{n}$,

then $\chi'(\mathcal{H}) \leq (1-\sigma)n$.

The nibble method

nibble: probabilistic approach for coloring or finding matchings

Pippenger–Spencer theorem (1989)

If $\mathcal{H}$ is a linear hypergraph with bounded edge-sizes with maximum degree at most Δ, then $\chi'(\mathcal{H}) \leq \Delta + o(\Delta)$.

An n-vtx linear hypergraph $\mathcal{H}$ has max degree at most $n / \min_{e\in\mathcal{H}}(|e|-1)$.

Corollary 1: EFL holds if $|e| \in [3, k]$ $\forall e \in \mathcal{H}$ and $n \gg k$

Corollary 2: EFL holds "asymptotically" if $|e| \leq k$ $\forall e \in \mathcal{H}$ and $n \gg k$

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Kahn (1996): The Pippenger–Spencer theorem holds for **list coloring**

- Kahn used an intermediate result to generalize Corollary 2 for all linear hypergraphs in 1992.

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Our proof also uses nibble to color "small" edges and moreover exploits **quasirandomness** properties of the resulting coloring.

Coloring locally sparse graphs

Theorem (Alon, Krivelevich, and Sudakov, 1999)

Let G be a graph of maximum degree $\leq \Delta$. If every $v \in V(G)$ satisfies $|E(G[N(v)])| \leq \Delta^2/f$ for $f \leq \Delta^2 + 1$, then $\chi(G) = O(\Delta/\log\sqrt{f})$.

Corollary: Johansson's theorem for triangle-free graphs

Davies, Kang, Pirot, & Sereni (2020+): $\chi(G) \leq (1 + o(1))\Delta/\log\sqrt{f}$

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Corollary: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. the following holds for $1/\delta \leq k \leq \delta\sqrt{n}$:
If $\mathcal{H}$ is a k-uniform, n-vtx, linear hypergraph, then $\chi'(\mathcal{H}) \leq \varepsilon n$

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Theorem (Molloy and Reed, 2002)

Let G be a graph of maximum degree $\leq \Delta$. If every $v \in V(G)$ satisfies $|E(G[N(v)])| \leq (1-\sigma)\binom{\Delta}{2}$ for $1/\Delta \ll \sigma$, then $\chi(G) \leq (1 - \sigma/e^6)\Delta$.

Improved by **Bruhn and Joos (2018)**, **Bonamy, Perrett, and Postle (2018+)**, and **Hurley, de Joannis de Verclos, and Kang (2020+)**

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Corollary: $\forall \delta \in (0,1)$, the following holds for $k = (1-\delta)\sqrt{n}$ and $n \gg 1$:
If $\mathcal{H}$ is a k-uniform, n-vtx, linear hypergraph, then $\chi'(\mathcal{H}) \leq (1-\delta/2^9)n$

Roadmap to the proof

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

If $\mathcal{H}$ is an n-vertex linear hypergraph where n is sufficiently large, then

$$\chi'(\mathcal{H}) \leq n.$$

Roadmap to the proof

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1. **"Small" edge case:** $|e| \leq k \ \forall e \in \mathcal{H}$ (Kahn asked in '94 for $k = 3$)
   - Pippenger–Spencer theorem (i.e. **nibble**) $\Rightarrow \chi'(\mathcal{H}) \leq n + o(n)$
   - Using **absorption**, reduce to a graph coloring problem

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   - "**Reordering lemma**" finds structure in line graph – a large **nearly complete** or **locally sparse** induced subgraph

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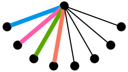
3. **Merge cases**
   - Color large edges first, with special properties
   - Extend to small edges, avoiding conflicts

Vizing-reduction strategy for bounded edge-sizes

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \in \{2,3\}$ $\forall e \in \mathcal{H}$.

- Fix $0 < \gamma \ll \varepsilon \ll 1$, and let $U := \{v \in V(\mathcal{H}) : d(v) > (1-\varepsilon)n\}$.

Low degree: more flexibility



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Vizing-reduction: Using $k := \lfloor (1/2 + \gamma)n \rfloor$ colors, color $\mathcal{H}$ such that:

- all size-3 edges are colored;
- $\geq (1/2 - \gamma)$-proportion of graph edges at each vtx are colored;
- every color class covers U (**perfect coverage** of U).

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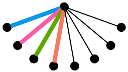

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- every color class covers U (**perfect coverage** of U).

Proof that $\chi'(\mathcal{H}) \leq n$ (assuming Vizing-reduction)

- vertices in U have leftover degree $\leq (n - 1) - k < n - k$;
- vertices not in U have leftover degree $\leq (1/2 + \gamma)(1 - \varepsilon)n < n - k$.

Uncolored edges comprise a **graph** of max degree $< n - k$. $(\star)$

Finish with Vizing's theorem! □

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- every color class covers U (**perfect coverage** of U).

Perfect coverage of U not always possible (e.g. K_n for n odd).
Instead, find coloring with **nearly perfect coverage:**

- every color class covers all but one vertex of U and
- each vertex of U is covered by all but one color class.

Works with one extra color; additional ideas needed to prove $\chi' \leq n$.

Simplified proof with one extra color

Recall: $U = \{v \in V(\mathcal{H}) : d(v) > (1 - \varepsilon)n\}$ $(0 < \gamma \ll \varepsilon \ll 1)$

Aim: Using $k = \lfloor (1/2 + \gamma)n \rfloor$ colors, color $\mathcal{H}$ such that:

- all size-3 edges are colored;
- for each vertex, nearly half of graph edges containing it are colored;
- the color classes have **nearly perfect coverage** of U.

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- for each vertex, nearly half of graph edges containing it are colored; ✓
- the color classes have **nearly perfect coverage** of U. ✗

Proof (sketch) of $\chi' \leq n+1$

Put each graph edge in a "reservoir" R independently with probability $1/2$;

- with high probability $\Delta(\mathcal{H} \setminus R) \leq (1/2 + o(1))n$, so $\chi'(\mathcal{H} \setminus R) \leq (1/2+\gamma)n$ by the Pippenger-Spencer theorem.

To obtain nearly perfect coverage, "re-run" Pippenger-Spencer proof (**nibble**) but apply **absorption** for each color class.

Nibble: Randomly construct matching in $\mathcal{H} \setminus R$ covering $\approx (1-\gamma)n$ vtcs.

Absorption: Augment with matching in R covering remaining U-vtcs.

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Proof (sketch) of $\chi' \leq n+1$

Put each graph edge in a "reservoir" R independently with probability 1/2;
Nibble + absorption: using $k = (1/2+\gamma)n$ colors, color some $\mathcal{H}' \supseteq \mathcal{H} \setminus R$ with **nearly perfect coverage** of U:

- vertices in U have leftover degree $\leq (n-1)-(k-1) \leq n-k$;
- vertices not in U have leftover degree $\leq (1-\varepsilon)n/2 + o(n) < n-k$.

Thus $\mathcal{H} \setminus \mathcal{H}'$ is a **graph** and $\Delta(\mathcal{H} \setminus \mathcal{H}') \leq n-k$, so by Vizing's thm

$$\chi'(\mathcal{H}) \leq \chi'(\mathcal{H}') + \chi'(\mathcal{H} \setminus \mathcal{H}') \leq k + (n-k+1) = n+1. \qquad \square$$

Nibble + absorption

- $U = \{v \in V(\mathcal{H}) : d(v) > (1 - \varepsilon)n\}$ $(0 < \gamma \ll \varepsilon \ll 1)$
- R = random "reservoir" – graph edges included with prob $1/2$

Alternate applications of "nibble" & "absorption"; construct k matchings

Nibble + absorption

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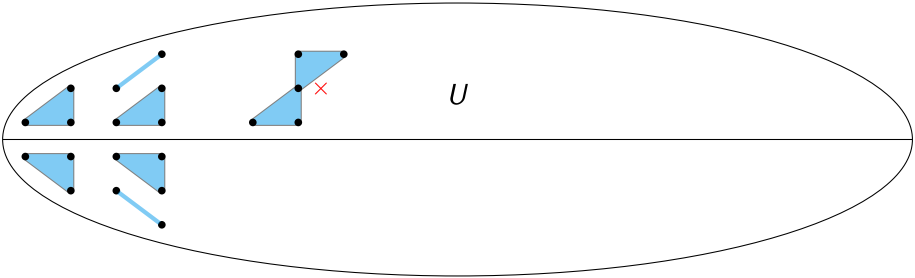
Alternate applications of "nibble" & "absorption"; construct k matchings

Nibble: Randomly select each color class in $\mathcal{H} \setminus R$, in small "bites", until $(1-\gamma)n$ vertices are covered.



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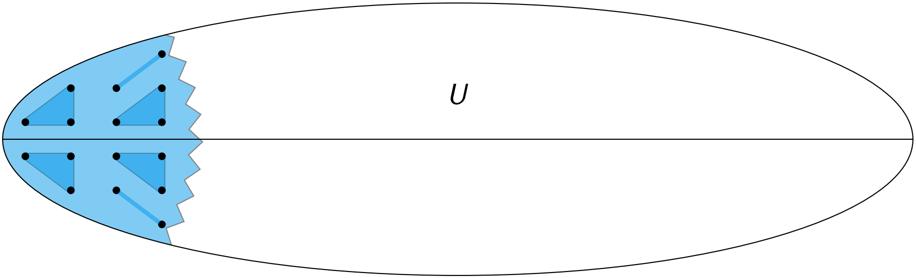

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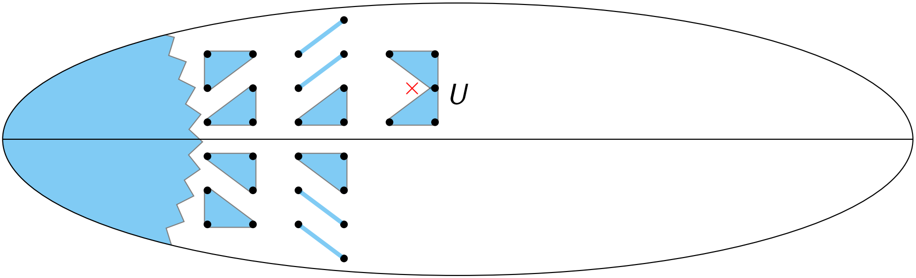

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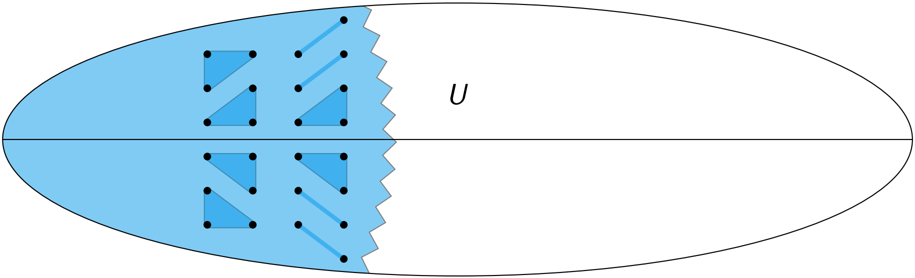

Nibble + absorption

- $U = \{v \in V(\mathcal{H}) : d(v) > (1 - \varepsilon)n\}$ $(0 < \gamma \ll \varepsilon \ll 1)$
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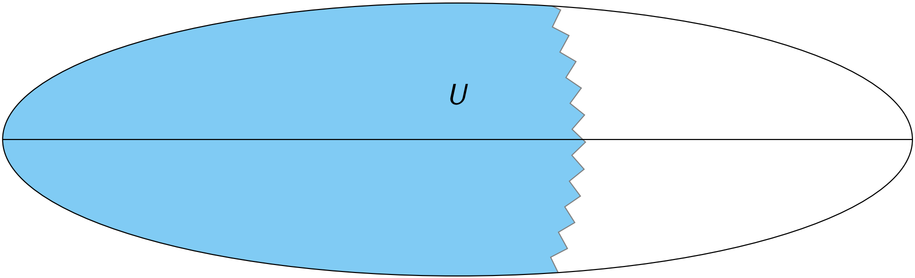

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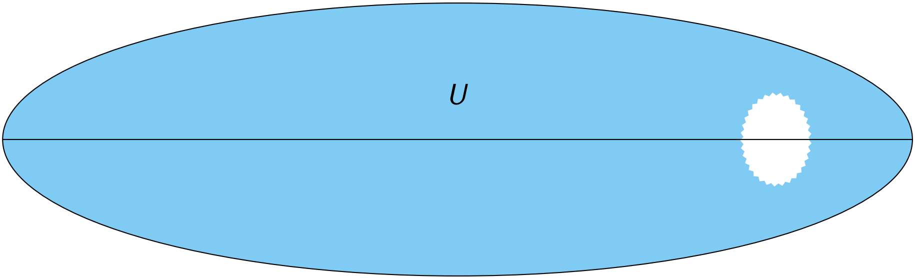

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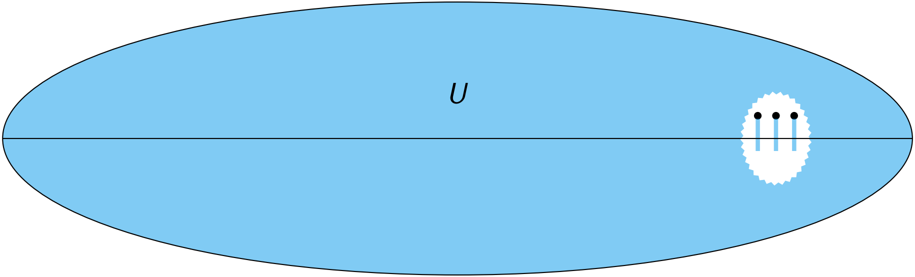

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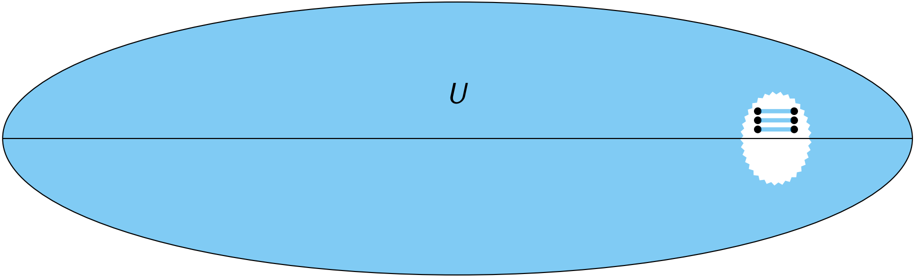

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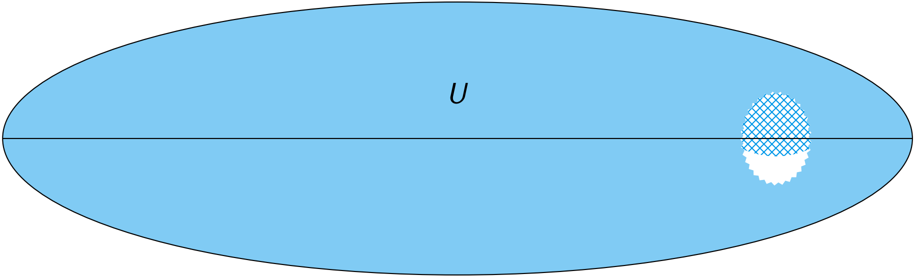

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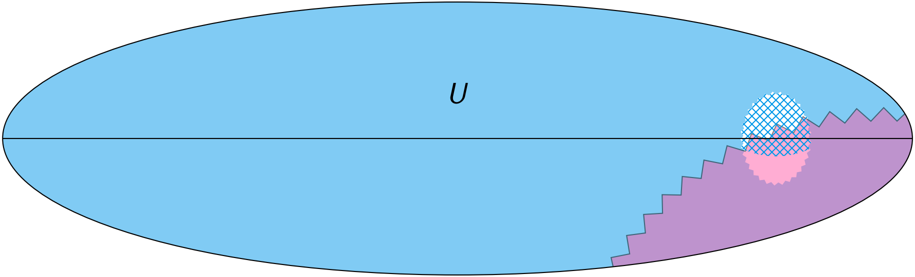

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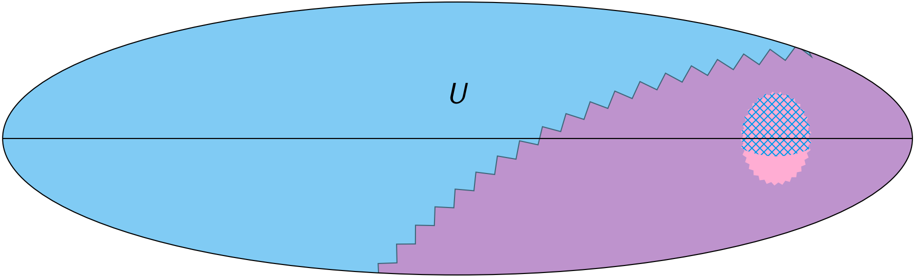

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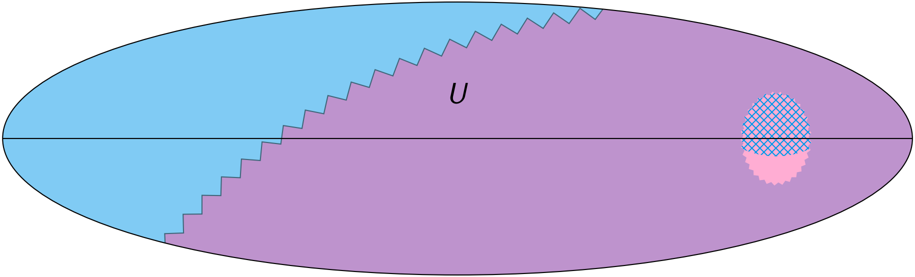

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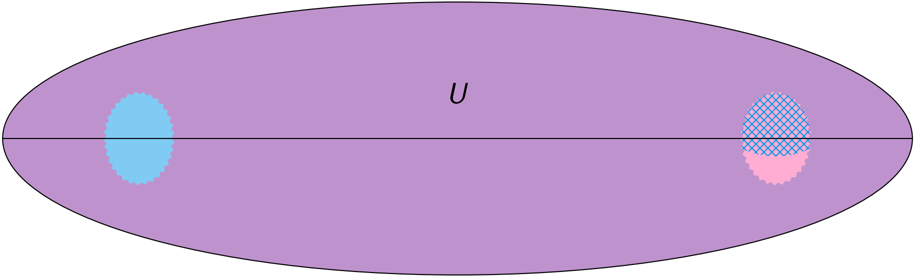

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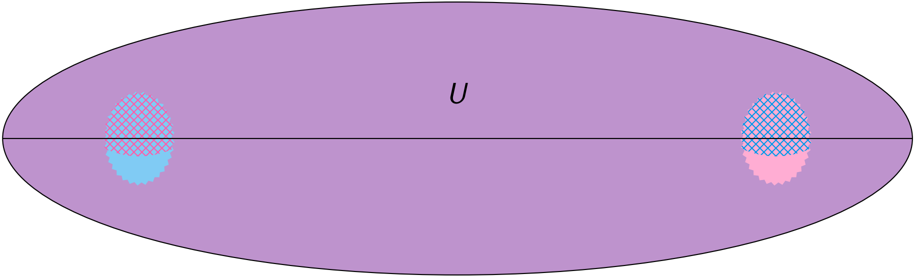
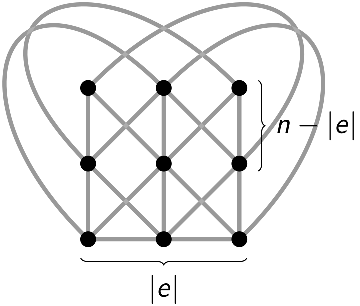


Large edges: reordering

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \geq r \; \forall e \in \mathcal{H}$, where $r \gg 1$.

Trivial: $\forall e \in \mathcal{H}$, at most $|e|(n - |e|)/(|e| - 1) \leq n + 2n/r$ edges of size at least $|e|$ intersect e.



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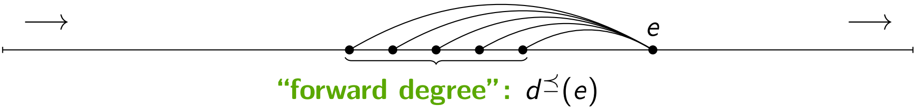
Corollary: $\chi'(\mathcal{H}) \leq n + o(n)$: color greedily.



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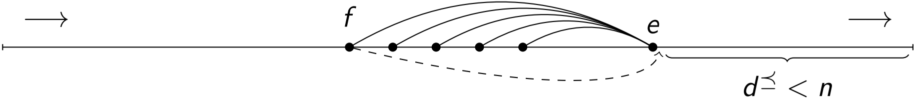
Reordering: Let e be the last edge with $d^{\preceq}(e) \geq n$. If f intersects e and $< n$ edges preceding e intersect f, then move f immediately after e.



If reordering "finishes', then $d^{\preceq}(e) < n \; \forall e \in \mathcal{H}$, so $\chi'(\mathcal{H}) \leq n$.

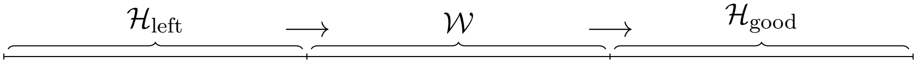
Reordering lemma (informal)

If reordering "gets stuck", then there is a highly structured $\mathcal{W} \subseteq \mathcal{H}$.

Proof when all edges are large

For $0 < \delta \ll 1$ and $\zeta < 1$: $(1/r \ll \delta)$

- $\mathcal{W}$ covers $(1-\delta)\binom{n}{2}$ pairs of vertices, and $|e| \sim (1-\zeta)\sqrt{n}$ $\forall e \in \mathcal{W}$.
- If $e \in \mathcal{H}_{\mathrm{good}}$, then $d^{\preceq}(e) < n$.
- If $e \in \mathcal{H}_{\mathrm{left}}$, then $|e| \geq (1-\zeta)\sqrt{n}$.



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$$\underbrace{\qquad\qquad}_{\mathcal{H}_{\mathrm{left}}} \longrightarrow \underbrace{\qquad\qquad}_{\mathcal{W}} \longrightarrow \underbrace{\qquad\qquad}_{\mathcal{H}_{\mathrm{good}}}$$

Case 1: $\zeta < \sqrt{\delta}$ ($\mathcal{W} \approx$ projective plane)

Proof (sketch)

Find $|\mathcal{H}_{\mathrm{left}} \cup \mathcal{W}| - n$ pairs of disjoint edges in $\mathcal{H}_{\mathrm{left}} \cup \mathcal{W}$:

- assign edges of each pair the same color;
- assign remaining edges (of $\mathcal{H}_{\mathrm{left}} \cup \mathcal{W}$) distinct colors.

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Case 2: $\zeta \geq \sqrt{\delta}$ ("non-extremal case")

Proof (sketch)

Line graph of $\mathcal{W}$ has max degree $\leq (1 + o(1))n$ and is **locally sparse**, i.e. $\leq (1 - \zeta/2)\binom{n}{2}$ edges in the neighborhood of every vertex:

- thm of Molloy & Reed $\Rightarrow \chi'(\mathcal{W}) \leq (1 - 2^{-10}\zeta)n$;

Apply "reordering" argument to edges preceding $\mathcal{W}$:

- If $e \in \mathcal{H}_{\mathrm{left}}$, then $d^{\preceq}(e) \leq 2^{-10}\zeta n - 1 \Rightarrow \chi'(\mathcal{H}_{\mathrm{left}}) \leq 2^{-10}\zeta n$.

Subsequent work

Question (Erdős, 1977)

If $\mathcal{H}$ is an n-vertex hypergraph of maximum degree at most n and **codegree** at most t, what is the maximum possible value of $\chi'(\mathcal{H})$?

- The EFL conjecture asserts that the answer for $t = 1$ is n.

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We prove that for $2 \leq t < \sqrt{n}$ and n sufficiently large, the answer is tn:

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

$\forall \varepsilon > 0$, the following holds for n sufficiently large and $t \in \mathbb{N}$.
If $\mathcal{H}$ is an n-vertex hypergraph with codegree at most t and maximum degree at most $(1-\varepsilon)tn$, then $\chi'_\ell(\mathcal{H}) \leq tn$. Moreover, if $\chi'_\ell(\mathcal{H}) = tn$, then $\mathcal{H}$ is a **t-fold projective plane**.

Strengthens answer to Erdős' question in three ways:

- allows relaxed maximum degree assumption (except when $t = 1$)
- characterizes extremal examples
- holds for list coloring

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When $t \geq \sqrt{n}$, a t-fold projective plane has max degree $> n$

Horák and Tuza (1990): $\chi'(\mathcal{H}) \leq n^{3/2}$; covers range $t > \sqrt{n}$.

Open problems

Conjecture (Berge, 1989; Füredi, 1986; Meyniel)

If $\mathcal{H}$ is a linear hypergraph, then $\chi'(\mathcal{H}) \leq \max_{v \in V(\mathcal{H})} |\bigcup_{e \ni v} e|$.

- common generalization of Vizing's theorem and EFL

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Thm: $|\bigcup_{e \ni v} e| \leq D \ \forall v \in V(\mathcal{H})$ and $D \geq \log^2 n \Rightarrow \chi'_\ell(\mathcal{H}) \leq D + o(D)$

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Two more tractable problems?

- Full "asymptotic" List Berge–Füredi–Meyniel
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