# Coloring hypergraphs of small codegree, and a proof of the Erdős–Faber–Lovász conjecture


Tom Kelly

Joint work with:
Dong Yeap Kang, Daniela Kühn, Abhishek Methuku, and Deryk Osthus

UNIVERSITY OF BIRMINGHAM


Graphs & Matroids Seminar
November 23rd, 2021

# Part I

## Coloring a nearly disjoint union of complete graphs

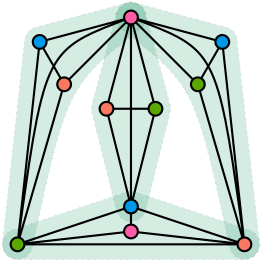
# The Erdős–Faber–Lovász conjecture

**proper coloring:** adjacent vertices assigned different colors

**chromatic number:** min # colors used in proper coloring, denoted by $\chi$

**The Erdős–Faber–Lovász conjecture (1972)**

If $G_1, \ldots, G_n$ are complete graphs, each on at most $n$ vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^{n} G_i) \leq n$.



# The Erdős–Faber–Lovász conjecture

**proper coloring:** adjacent vertices assigned different colors

**chromatic number:** min # colors used in proper coloring, denoted by $\chi$

**The Erdős–Faber–Lovász conjecture (1972)**

If $G_1, \ldots, G_n$ are complete graphs, each on at most $n$ vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^n G_i) \leq n$.

One of Erdős' "three most favorite combinatorial problems":

- Erdős initially offered \$50 for a solution, raised to \$500.

*Faber, Lovász and I made this harmless looking conjecture at a party in Boulder Colorado in September 1972. Its difficulty was realised only slowly. I now offer 500 dollars for a proof or disproof. (Not long ago I only offered 50; the increase is not due to inflation but to the fact that I now think the problem is very difficult. Perhaps I am wrong.)* –*Paul Erdős, 1981*

# The Erdős–Faber–Lovász conjecture

**proper coloring:** adjacent vertices assigned different colors

**chromatic number:** min # colors used in proper coloring, denoted by $\chi$

**The Erdős–Faber–Lovász conjecture (1972)**

If $G_1, \ldots, G_n$ are complete graphs, each on at most $n$ vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^{n} G_i) \leq n$.

**Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)**

The Erdős–Faber–Lovász conjecture is true for sufficiently large $n$.

# A more general question of Erdős

**Question (Erdős, 1977)**

If $G_1, \ldots, G_n$ are complete graphs, each on at most $n$ vtcs, such that every pair shares at most $t$ vtcs, what is the max possible value of $\chi(\bigcup_{i=1}^{n} G_i)$?

- The EFL conjecture asserts that the answer for $t = 1$ is $n$.

# A more general question of Erdős

**Question (Erdős, 1977)**

If $G_1, \ldots, G_n$ are complete graphs, each on at most $n$ vtcs, such that every pair shares at most $t$ vtcs, what is the max possible value of $\chi(\bigcup_{i=1}^n G_i)$?

- The EFL conjecture asserts that the answer for $t = 1$ is $n$.

We prove that for $2 \leq t < \sqrt{n}$ and $n$ sufficiently large, the answer is $tn$:

**Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)**

For $t \geq 2$, $n$ sufficiently large, and $G_1, \ldots, G_n$ as above, we have

$$\chi\left(\bigcup_{i=1}^n G_i\right) \leq tn.$$

Moreover, for infinitely many $k \in \mathbb{N}$, if $n = k^2 + k + 1$ and $t \leq k$, then there exist such $G_1, \ldots, G_n$ such that $\bigcup_{i=1}^n G_i$ has $tn$ vtcs and is complete.

# A more general question of Erdős

**Question (Erdős, 1977)**

If $G_1, \dots, G_n$ are complete graphs, each on at most $n$ vtcs, such that every pair shares at most $t$ vtcs, what is the max possible value of $\chi(\bigcup_{i=1}^n G_i)$?

- The EFL conjecture asserts that the answer for $t = 1$ is $n$.

We prove that for $2 \le t < \sqrt{n}$ and $n$ sufficiently large, the answer is $tn$:

**Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)**

For $t \ge 2$, $n$ sufficiently large, and $G_1, \dots, G_n$ as above, we have

$$\chi\left(\bigcup_{i=1}^n G_i\right) \le tn.$$

Moreover, for infinitely many $k \in \mathbb{N}$, if $n = k^2 + k + 1$ and $t \le k$, then there exist such $G_1, \dots, G_n$ such that $\bigcup_{i=1}^n G_i$ has $tn$ vtcs and is complete.

**Horák and Tuza (1990):** $\chi(\bigcup_{i=1}^n G_i) \le n^{3/2}$; covers range $t > \sqrt{n}$.

# Part II

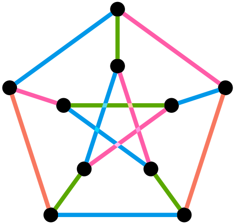
## Hypergraph edge-coloring

# Matchings and edge-coloring

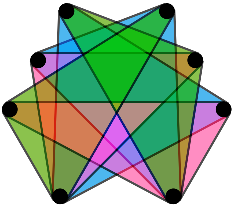
**matching:** a set of disjoint edges

**(proper) edge-coloring:** no two edges of same color share a vertex

**chromatic index:** min # colors used in proper edge-coloring, denoted $\chi'$



$\chi'(\text{Petersen graph}) = 4$



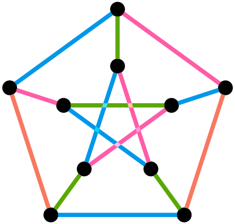
$\chi'(\mathcal{H}) = 3$

# Matchings and edge-coloring

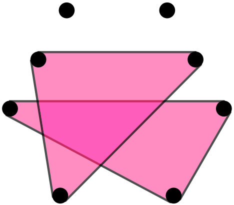
**matching:** a set of disjoint edges

**(proper) edge-coloring:** no two edges of same color share a vertex

**chromatic index:** min # colors used in proper edge-coloring, denoted $\chi'$



$\chi'$(Petersen graph) $= 4$



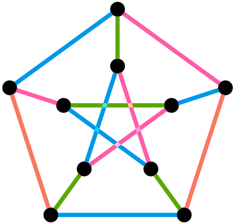
$\chi'(\mathcal{H}) = 3$

# Matchings and edge-coloring

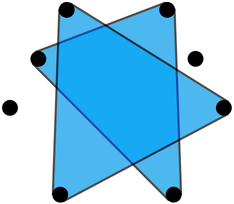
**matching:** a set of disjoint edges

**(proper) edge-coloring:** no two edges of same color share a vertex

**chromatic index:** min # colors used in proper edge-coloring, denoted $\chi'$



$\chi'(\text{Petersen graph}) = 4$



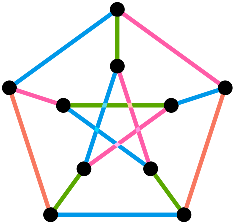
$\chi'(\mathcal{H}) = 3$

# Matchings and edge-coloring

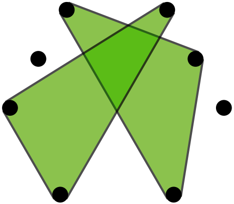
**matching:** a set of disjoint edges

**(proper) edge-coloring:** no two edges of same color share a vertex

**chromatic index:** min # colors used in proper edge-coloring, denoted $\chi'$



$\chi'(\text{Petersen graph}) = 4$



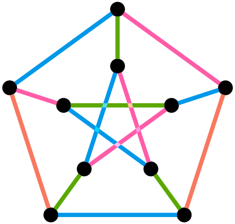
$\chi'(\mathcal{H}) = 3$

# Matchings and edge-coloring

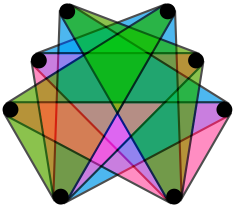
**matching:** a set of disjoint edges

**(proper) edge-coloring:** no two edges of same color share a vertex

**chromatic index:** min # colors used in proper edge-coloring, denoted $\chi'$



$\chi'(\text{Petersen graph}) = 4$



$\chi'(\mathcal{H}) = 3$

**degree:** # edges containing a vertex

**Vizing's theorem (1964)**

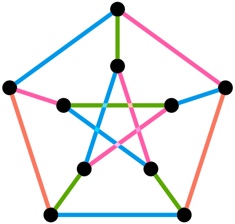
Every graph of maximum degree $\leq \Delta$ has chromatic index $\leq \Delta + 1$.

# Matchings and edge-coloring

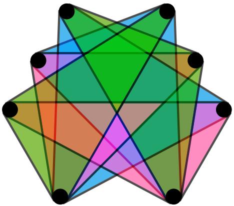
**matching:** a set of disjoint edges

**(proper) edge-coloring:** no two edges of same color share a vertex

**chromatic index:** min # colors used in proper edge-coloring, denoted $\chi'$



$\chi'(\text{Petersen graph}) = 4$



$\chi'(\mathcal{H}) = 3$

More complex for hypergraphs: e.g.

- 3-dimensional matching: one of Karp's original NP-complete problems

**Question:** Which hypergraphs have large matchings or small $\chi'$?

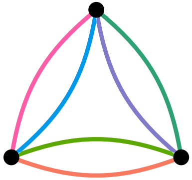
# Hypergraph basics

In this talk, hypergraphs can have repeated edges but no size-one edges.

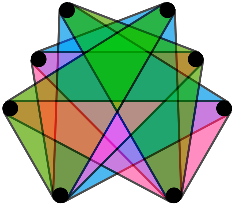
**codegree:** max # edges containing any given pair of vertices

**linear:** every pair of vertices contained in at most one edge

*k*-**uniform:** every edge has size $k$



A 2-uniform hypergraph of codegree 2



A linear 3-uniform hypergraph

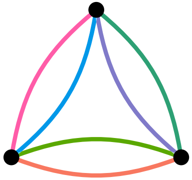
# Hypergraph basics

In this talk, hypergraphs can have repeated edges but no size-one edges.

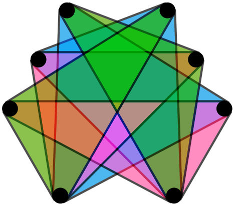
**codegree:** max # edges containing any given pair of vertices

**linear:** every pair of vertices contained in at most one edge

***k*-uniform:** every edge has size $k$



A 2-uniform hypergraph of codegree 2



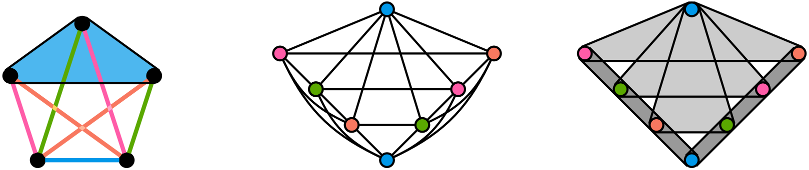
A linear 3-uniform hypergraph

- multigraphs are 2-uniform hypergraphs
- graphs are 2-uniform linear hypergraphs

# Erdős–Faber–Lovász conjecture (reformulated)

**The Erdős–Faber–Lovász conjecture (1972)**

If $\mathcal{H}$ is an $n$-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.



**Line graph:**

- edges $\to$ vertices: edges that share a vertex are adjacent
- proper edge-coloring $\to$ proper vertex-coloring

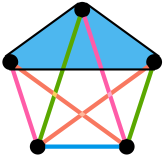
The previous formulation is equivalent:

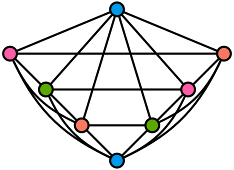
If $G_1, \ldots, G_n$ are complete graphs, each on at most $n$ vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^{n} G_i) \leq n$.

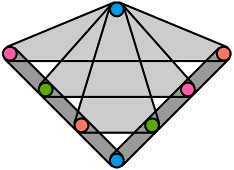
# Erdős–Faber–Lovász conjecture (reformulated)

**The Erdős–Faber–Lovász conjecture (1972)**

If $\mathcal{H}$ is an $n$-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.







**Hypergraph duality**:

- edges $\to$ vertices and vertices $\to$ edges
- linearity is preserved

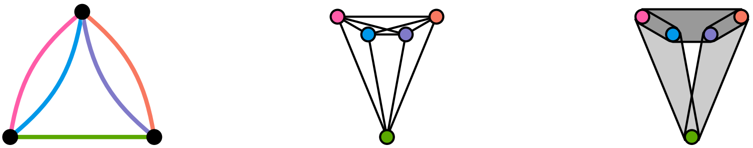
The previous formulation is equivalent:

If $G_1, \ldots, G_n$ are complete graphs, each on at most $n$ vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^{n} G_i) \leq n$.

# The dual of Erdős' question

**Question (Erdős, 1977)**

If $\mathcal{H}$ is an $n$-vertex hypergraph of maximum degree at most $n$ and codegree at most $t$, what is the max possible value of $\chi'(\mathcal{H})$?



- max degree of $\mathcal{H} = \max |V(G_i)|$
- codegree of $\mathcal{H} = \max_{i \neq j} |V(G_i) \cap V(G_j)|$

The previous formulation is equivalent:

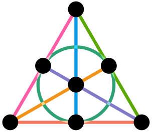
If $G_1, \ldots, G_n$ are complete graphs, each on at most $n$ vtcs, such that every pair shares at most $t$ vtcs, what is the max possible value of $\chi(\bigcup_{i=1}^{n} G_i)$?

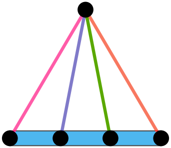
# Extremal examples for EFL

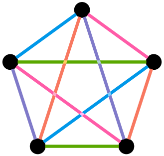
**The Erdős–Faber–Lovász conjecture (1972)**

If $\mathcal{H}$ is an $n$-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

Extremal examples:







**Finite projective plane of order $k$:** $(k+1)$-uniform intersecting linear hypergraph with $n = k^2 + k + 1$ vertices and edges

**Degenerate plane / near pencil:** intersecting linear hypergraph with $n - 1$ size-two edges and one size-$(n-1)$ edge

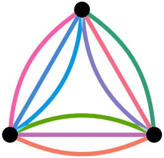
**Complete graph:** $\binom{n}{2}$ size-two edges; if $\chi' < n$, then color classes are perfect matchings $\Rightarrow$ $n$ is even

# Extremal examples for $t \geq 2$

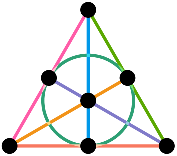
**The "$t$-EFL" conjecture**

If $\mathcal{H}$ is an $n$-vertex codegree-$t$ hypergraph of max degree $\leq n$, then

$$\chi'(\mathcal{H}) \leq tn.$$



3-fold order-1 projective plane



1-fold Fano plane

**$t$-fold projective plane:** replace each edge with $t$ repeated edges

Extremal examples: $t$-fold projective planes of order $k$ for $t \leq k$:

- codegree $t$
- max degree $t(k+1)$ (and $t(k+1) \leq n$ if $t \leq k$)

# Part III

# Results

# Previous results

**The Erdős–Faber–Lovász conjecture (1972)**

If $\mathcal{H}$ is an $n$-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

Direct approaches:

**Trivial:** $\chi'(\mathcal{H}) \leq 2n - 3$ (color greedily, in order of size)

**Chang–Lawler (1989):** $\chi'(\mathcal{H}) \leq \lceil 3n/2 - 2 \rceil$

# Previous results

**The Erdős–Faber–Lovász conjecture (1972)**

If $\mathcal{H}$ is an $n$-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

Relaxed parameters:

**de Bruijn–Erdős (1948):** true for intersecting hypergraphs

**Seymour (1982):** $\exists$ a matching of size at least $|\mathcal{H}|/n$

**Kahn–Seymour (1992):** fractional chromatic index is at most $n$

# Previous results

**The Erdős–Faber–Lovász conjecture (1972)**

If $\mathcal{H}$ is an $n$-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

Probabilistic "nibble" approach:

**Faber–Harris (2019):** EFL is true if $|e| \in [3, c\sqrt{n}]$ $\forall e \in \mathcal{H}$ $(c \ll 1)$

**Kahn (1992):** $\chi'(\mathcal{H}) \leq (1 + o(1))n$

# Previous results

**The Erdős–Faber–Lovász conjecture (1972)**

If $\mathcal{H}$ is an $n$-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

Probabilistic "nibble" approach:

**Faber–Harris (2019):** EFL is true if $|e| \in [3, c\sqrt{n}]$ $\forall e \in \mathcal{H}$ $(c \ll 1)$

**Kahn (1992):** $\chi'(\mathcal{H}) \leq (1 + o(1))n$

Both use "list coloring" generalization (proved by Kahn) of:

**Pippenger–Spencer theorem (1989)**

If $\mathcal{H}$ is a hypergraph with bounded edge-sizes with maximum degree at most $\Delta$ and codegree $o(\Delta)$, then $\chi'(\mathcal{H}) \leq \Delta + o(\Delta)$.

- $\Rightarrow$ EFL if $|e| \in [3, k]$ $\forall e \in \mathcal{H}$ and $n \gg k$ (since $\Delta(\mathcal{H}) \leq n/2$)
- $\Rightarrow$ EFL "asymptotically" if $|e| \leq k$ $\forall e \in \mathcal{H}$ and $n \gg k$ $(\Delta(\mathcal{H}) \leq n)$
- $\Rightarrow$ "$t$-EFL" for $t \geq 2$ if $|e| \leq k$ $\forall e \in \mathcal{H}$ and $n \gg k$

# Our results

We confirm the EFL conjecture for all but finitely many hypergraphs:

**Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)**

For sufficiently large $n$, every $n$-vertex linear hypergraph has chromatic index at most $n$.

# Our results

We confirm the EFL conjecture for all but finitely many hypergraphs:

**Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)**

For sufficiently large $n$, every $n$-vertex linear hypergraph has chromatic index at most $n$.

We also prove a stability result, predicted by Kahn:

**Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)**

$\forall \delta > 0$, $\exists \sigma > 0$ such that the following holds for $n$ sufficiently large.
If $\mathcal{H}$ is an $n$-vertex linear hypergraph such that

- $\Delta(\mathcal{H}) \leq (1-\delta)n$ and
- at most $(1-\delta)n$ edges have size $(1 \pm \delta)\sqrt{n}$,

then $\chi'(\mathcal{H}) \leq (1-\sigma)n$.

# Our results II

We confirm $t$-EFL for $t \geq 2$ for all but finitely many hypergraphs:

**Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)**

$\forall \varepsilon > 0$, the following holds for $n$ sufficiently large and $t \in \mathbb{N}$.
If $\mathcal{H}$ is an $n$-vertex hypergraph with codegree at most $t$ and maximum degree at most $(1-\varepsilon)tn$, then $\chi'_\ell(\mathcal{H}) \leq tn$. Moreover, if $\chi'_\ell(\mathcal{H}) = tn$, then $\mathcal{H}$ is a $t$-fold projective plane.

Strengthens answer to Erdős question in three ways:

- allows relaxed maximum degree assumption (except when $t = 1$)
- characterizes extremal examples
- holds for list coloring

# Our results II

We confirm $t$-EFL for $t \geq 2$ for all but finitely many hypergraphs:

**Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)**

$\forall \varepsilon > 0$, the following holds for $n$ sufficiently large and $t \in \mathbb{N}$.
If $\mathcal{H}$ is an $n$-vertex hypergraph with codegree at most $t$ and maximum degree at most $(1-\varepsilon)tn$, then $\chi'_\ell(\mathcal{H}) \leq tn$. Moreover, if $\chi'_\ell(\mathcal{H}) = tn$, then $\mathcal{H}$ is a $t$-fold projective plane.

We also generalize our stability result and the de Bruijn–Erdős theorem:

**Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)**

If $\mathcal{H}$ is an $n$-vertex intersecting hypergraph with codegree at most $t$, then $\mathcal{H}$ has at most $tn$ edges, and if equality holds, then $\mathcal{H}$ is either

- a $t$-fold projective plane or
- a $t$-fold near-pencil.

# Part IV

## Proof ideas

# Roadmap to the proofs

**KKKMO (2021+):** If $\mathcal{H}$ is an $n$-vertex hypergraph of maximum degree at most $n$ and codegree at most $t$, then $\chi'(\mathcal{H}) \leq tn$. $(n \gg 1)$

# Roadmap to the proofs

**KKKMO (2021+):** If $\mathcal{H}$ is an $n$-vertex hypergraph of maximum degree at most $n$ and codegree at most $t$, then $\chi'(\mathcal{H}) \leq tn$. $(n \gg 1)$

**1: "Small" edge case:** $|e| \leq k \; \forall e \in \mathcal{H}$ (Kahn asked in '94 for $k = 3$)

- The Pippenger–Spencer theorem (i.e. "nibble") implies the case $t \geq 2$ and implies $\chi'(\mathcal{H}) \leq n + o(n)$ for $t = 1$
- Using absorption, reduce $t = 1$ case to a graph coloring problem

# Roadmap to the proofs

**KKKMO (2021+):** If $\mathcal{H}$ is an $n$-vertex hypergraph of maximum degree at most $n$ and codegree at most $t$, then $\chi'(\mathcal{H}) \leq tn$. $(n \gg 1)$

**1: "Small" edge case:** $|e| \leq k \; \forall e \in \mathcal{H}$ (Kahn asked in '94 for $k = 3$)

- The Pippenger–Spencer theorem (i.e. "nibble") implies the case $t \geq 2$ and implies $\chi'(\mathcal{H}) \leq n + o(n)$ for $t = 1$
- Using absorption, reduce $t = 1$ case to a graph coloring problem

**2: "FPP-extremal case":** $|e| \geq (1 - \delta)\sqrt{n} \; \forall e \in \mathcal{H}$ (for $\delta \ll 1$)

- Delicate argument – includes when $\mathcal{H} \approx t$-fold proj. plane
- Can also prove $\chi'_\ell(\mathcal{H}) < tn$ unless $\mathcal{H}$ is intersecting

# Roadmap to the proofs

**KKKMO (2021+):** If $\mathcal{H}$ is an $n$-vertex hypergraph of maximum degree at most $n$ and codegree at most $t$, then $\chi'(\mathcal{H}) \leq tn$. $(n \gg 1)$

**1: "Small" edge case:** $|e| \leq k \ \forall e \in \mathcal{H}$ (Kahn asked in '94 for $k = 3$)

- The Pippenger–Spencer theorem (i.e. "nibble") implies the case $t \geq 2$ and implies $\chi'(\mathcal{H}) \leq n + o(n)$ for $t = 1$
- Using absorption, reduce $t = 1$ case to a graph coloring problem

**2: "FPP-extremal case":** $|e| \geq (1 - \delta)\sqrt{n} \ \forall e \in \mathcal{H}$ (for $\delta \ll 1$)

- Delicate argument – includes when $\mathcal{H} \approx t$-fold proj. plane
- Can also prove $\chi'_\ell(\mathcal{H}) < tn$ unless $\mathcal{H}$ is intersecting

**3: "Large" edge case:** $|e| \geq r \ \forall e \in \mathcal{H}$ (for $r \gg 1$)

- Greedy coloring in order of size $\Rightarrow \chi'(\mathcal{H}) \leq (1 + 2/r)tn$.
- "Reordering lemma" finds highly structured $\mathcal{W} \subseteq \mathcal{H}$ – either $\mathcal{W} \approx t$-fold proj. plane or line graph of $\mathcal{W}$ is "locally sparse"

# Roadmap to the proofs

**KKKMO (2021+):** If $\mathcal{H}$ is an $n$-vertex hypergraph of maximum degree at most $n$ and codegree at most $t$, then $\chi'(\mathcal{H}) \leq tn$. $(n \gg 1)$

**1: "Small" edge case:** $|e| \leq k \; \forall e \in \mathcal{H}$ (Kahn asked in '94 for $k = 3$)

- The Pippenger–Spencer theorem (i.e. "nibble") implies the case $t \geq 2$ and implies $\chi'(\mathcal{H}) \leq n + o(n)$ for $t = 1$
- Using absorption, reduce $t = 1$ case to a graph coloring problem

**2: "FPP-extremal case":** $|e| \geq (1 - \delta)\sqrt{n} \; \forall e \in \mathcal{H}$ (for $\delta \ll 1$)

- Delicate argument – includes when $\mathcal{H} \approx t$-fold proj. plane
- Can also prove $\chi'_\ell(\mathcal{H}) < tn$ unless $\mathcal{H}$ is intersecting

**3: "Large" edge case:** $|e| \geq r \; \forall e \in \mathcal{H}$ (for $r \gg 1$)

- Greedy coloring in order of size $\Rightarrow \chi'(\mathcal{H}) \leq (1 + 2/r)tn$.
- "Reordering lemma" finds highly structured $\mathcal{W} \subseteq \mathcal{H}$ – either $\mathcal{W} \approx t$-fold proj. plane or line graph of $\mathcal{W}$ is "locally sparse"

**4:** Merge cases

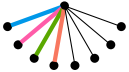
- Color large edges first, with special properties
- Stability result: can use fewer colors in non-FPP-extremal case

# Vizing-reduction strategy for bounded edge-sizes

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \in \{2, 3\}$ $\forall e \in \mathcal{H}$.

- Fix $0 < \gamma \ll \varepsilon \ll 1$, and let $U := \{v \in V(\mathcal{H}) : d(v) > (1 - \varepsilon)n\}$.

**Low degree:** more flexibility



**High degree:** more graph-like

# Vizing-reduction strategy for bounded edge-sizes

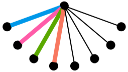
Let $\mathcal{H}$ be a linear hypergraph such that $|e| \in \{2, 3\}\ \forall e \in \mathcal{H}$.

- Fix $0 < \gamma \ll \varepsilon \ll 1$, and let $U := \{v \in V(\mathcal{H}) : d(v) > (1 - \varepsilon)n\}$.

**Vizing-reduction:** Using $k := \lfloor (1/2 + \gamma)n \rfloor$ colors, color $\mathcal{H}$ such that:

- all size-3 edges are colored;
- $\geq (1/2 - \gamma)$-proportion of graph edges at each vtx are colored;
- every color class covers $U$ (**perfect coverage** of $U$).

**Low degree:** more flexibility



**High degree:** more graph-like

# Vizing-reduction strategy for bounded edge-sizes

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \in \{2, 3\}$ $\forall e \in \mathcal{H}$.

- Fix $0 < \gamma \ll \varepsilon \ll 1$, and let $U := \{v \in V(\mathcal{H}) : d(v) > (1 - \varepsilon)n\}$.

**Vizing-reduction:** Using $k := \lfloor (1/2 + \gamma)n \rfloor$ colors, color $\mathcal{H}$ such that:

- all size-3 edges are colored;
- $\geq (1/2 - \gamma)$-proportion of graph edges at each vtx are colored;
- every color class covers $U$ (**perfect coverage** of $U$).

**Proof that $\chi'(\mathcal{H}) \leq n$ (assuming Vizing-reduction)**

- vertices in $U$ have leftover degree $\leq (n-1) - k < n - k$;
- vertices not in $U$ have leftover degree $\leq (1/2 + \gamma)(1 - \varepsilon)n < n - k$.

Uncolored edges comprise a **graph** of max degree $< n - k$. $(\star)$

Finish with Vizing's theorem! □

# Vizing-reduction strategy for bounded edge-sizes

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \in \{2, 3\}$ $\forall e \in \mathcal{H}$.

- Fix $0 < \gamma \ll \varepsilon \ll 1$, and let $U := \{v \in V(\mathcal{H}) : d(v) > (1 - \varepsilon)n\}$.

**Vizing-reduction:** Using $k := \lfloor (1/2 + \gamma)n \rfloor$ colors, color $\mathcal{H}$ such that:

- all size-3 edges are colored;
- $\geq (1/2 - \gamma)$-proportion of graph edges at each vtx are colored;
- every color class covers $U$ (**perfect coverage** of $U$).

Perfect coverage of $U$ not always possible (e.g. $K_n$ for $n$ odd).
Instead, find coloring with **nearly perfect coverage:**

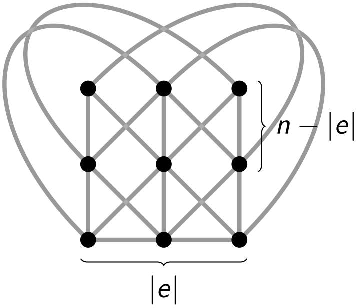
- every color class covers all but one vertex of $U$ and
- each vertex of $U$ is covered by all but one color class.

Works with one extra color; additional ideas needed to prove $\chi' \leq n$.

# Coloring the large edges

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \geq r$ $\forall e \in \mathcal{H}$, where $r \gg 1$.

**Trivial:** $\forall e \in \mathcal{H}$, at most $|e|(n - |e|)/(|e| - 1) \leq n + 2n/r$ edges of size at least $|e|$ intersect $e$.

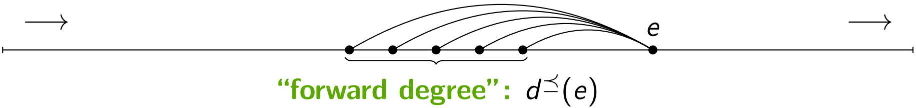


# Coloring the large edges

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \geq r\ \forall e \in \mathcal{H}$, where $r \gg 1$.

**Trivial:** $\forall e \in \mathcal{H}$, at most $|e|(n - |e|)/(|e| - 1) \leq n + 2n/r$ edges of size at least $|e|$ intersect $e$. I.e. $d^{\preceq}(e) \leq n + 2n/r\ \forall e \in \mathcal{H}$ if $\preceq$ is a **size-monotone decreasing** ordering of the line graph.

**Corollary:** $\chi'(\mathcal{H}) \leq n + o(n)$: color greedily.



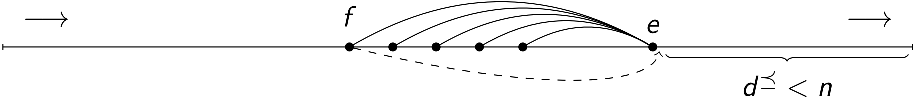
# Coloring the large edges

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \geq r$ $\forall e \in \mathcal{H}$, where $r \gg 1$.

**Trivial:** $\forall e \in \mathcal{H}$, at most $|e|(n - |e|)/(|e| - 1) \leq n + 2n/r$ edges of size at least $|e|$ intersect $e$. I.e. $d^{\preceq}(e) \leq n + 2n/r$ $\forall e \in \mathcal{H}$ if $\preceq$ is a **size-monotone decreasing** ordering of the line graph.

**Corollary:** $\chi'(\mathcal{H}) \leq n + o(n)$: color greedily.

**Reordering:** Let $e$ be the last edge with $d^{\preceq}(e) \geq n$. If $f$ intersects $e$ and $< n$ edges preceding $e$ intersect $f$, then move $f$ immediately after $e$.



# Coloring the large edges

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \geq r\ \forall e \in \mathcal{H}$, where $r \gg 1$.

**Trivial:** $\forall e \in \mathcal{H}$, at most $|e|(n - |e|)/(|e| - 1) \leq n + 2n/r$ edges of size at least $|e|$ intersect $e$. I.e. $d^{\preceq}(e) \leq n + 2n/r\ \forall e \in \mathcal{H}$ if $\preceq$ is a **size-monotone decreasing** ordering of the line graph.

**Corollary:** $\chi'(\mathcal{H}) \leq n + o(n)$: color greedily.

**Reordering:** Let $e$ be the last edge with $d^{\preceq}(e) \geq n$. If $f$ intersects $e$ and $< n$ edges preceding $e$ intersect $f$, then move $f$ immediately after $e$.

If reordering "finishes", then $d^{\preceq}(e) < n\ \forall e \in \mathcal{H}$, so $\chi'(\mathcal{H}) \leq n$.

**"Reordering lemma" (informal)**

If reordering "gets stuck", then there is a highly structured $\mathcal{W} \subseteq \mathcal{H}$: either

- $\mathcal{W} \approx$ projective plane (i.e. its line graph is close to complete), or
- line graph of $\mathcal{W}$ is **locally sparse** (i.e. nbrhoods far from complete).

Use structure to color $\mathcal{H}$ with $\leq n$ colors (via graph theoretical techniques)

# Part V

# Conclusion

# Summary

**The "$t$-EFL" conjecture**

If $\mathcal{H}$ is an $n$-vertex codegree-$t$ hypergraph of max degree $\leq n$, then

$$\chi'(\mathcal{H}) \leq tn.$$

Combining our results resolves the "$t$-EFL" conjecture for large $n$:

**Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)**

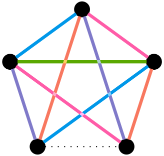
For sufficiently large $n$, every $n$-vertex hypergraph of maximum degree at most $n$ and codegree at most $t$ has chromatic index at most $tn$.

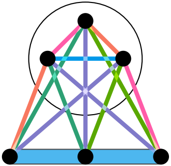
- The case $t = 1$ confirms the Erdős–Faber–Lovász conjecture for all but finitely many hypergraphs
- For $t \geq 2$, we characterize extremal examples and prove bounds hold for list coloring and with relaxed max degree assumption
- We also prove stability results and a generalization of the de Bruijn–Erdős theorem

# More extremal examples

**Overfull graph:** $> \Delta \lfloor n/2 \rfloor$ edges, where $\Delta$ =max degree and $n = \#$ vtcs

**"Blowup" of degenerate plane:** replace pencil point with a clique





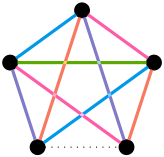
Additional extremal examples for EFL:

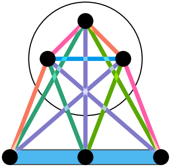
- overfull graphs with maximum degree $n-1$
- "odd blowups" of a degenerate plane

# More extremal examples

**Overfull graph:** $> \Delta \lfloor n/2 \rfloor$ edges, where $\Delta =$max degree and $n = \#$ vtcs

**"Blowup" of degenerate plane:** replace pencil point with a clique





Additional extremal examples for EFL:

- overfull graphs with maximum degree $n - 1$
- "odd blowups" of a degenerate plane

**Conjecture**

If $\mathcal{H}$ is an $n$-vertex linear hypergraph of chromatic index $n$, then either

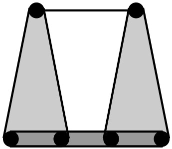
- $\mathcal{H}$ has more than $(n-1)^2/2$ size-two edges and $n$ is odd,
- $\mathcal{H}$ is a finite projective plane (of order $k$, where $n = k^2 + k + 1$), or
- $\mathcal{H}$ is an odd blowup of a degenerate plane.

# Open problems

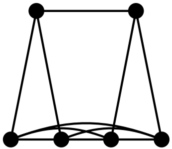
**Conjecture (Berge, 1989; Füredi, 1986; Meyniel)**

If $\mathcal{H}$ is a linear hypergraph, then $\chi'(\mathcal{H}) \le \max_{v \in V(\mathcal{H})} |\bigcup_{e \ni v} e|$.

- common generalization of Vizing's theorem and EFL



$\max_v |\bigcup_{e \ni v} e| = 5$



$\Delta(\text{“shadow”}) + 1 = 5$

# Open problems

**Conjecture (Berge, 1989; Füredi, 1986; Meyniel)**

If $\mathcal{H}$ is a linear hypergraph, then $\chi'(\mathcal{H}) \le \max_{v \in V(\mathcal{H})} |\bigcup_{e \ni v} e|$.

- common generalization of Vizing's theorem and EFL

**The List EFL conjecture (Faber, 2017)**

If $\mathcal{H}$ is an $n$-vertex linear hypergraph, then $\mathcal{H}$ has list chromatic index $\le n$.

I.e. if $C(e)$ is a "list of colors" such that $|C(e)| \ge n \ \forall e \in \mathcal{H}$, then $\mathcal{H}$ can be properly edge-colored s.t. every $e$ is assigned a color from $C(e)$.

- Implies EFL if $C(e) = \{1, \ldots, n\} \ \forall e \in \mathcal{H}$.

# Open problems

**Conjecture (Berge, 1989; Füredi, 1986; Meyniel)**

If $\mathcal{H}$ is a linear hypergraph, then $\chi'(\mathcal{H}) \leq \max_{v \in V(\mathcal{H})} |\bigcup_{e \ni v} e|$.

- common generalization of Vizing's theorem and EFL

**The List EFL conjecture (Faber, 2017)**

If $\mathcal{H}$ is an $n$-vertex linear hypergraph, then $\mathcal{H}$ has list chromatic index $\leq n$.

I.e. if $C(e)$ is a "list of colors" such that $|C(e)| \geq n \ \forall e \in \mathcal{H}$, then $\mathcal{H}$ can be properly edge-colored s.t. every $e$ is assigned a color from $C(e)$.

- Implies EFL if $C(e) = \{1, \ldots, n\} \ \forall e \in \mathcal{H}$.

**"Restricted" Larman's conjecture, 1981**

If $\mathcal{H}$ is an $n$-vertex **intersecting** hypergraph, then $\mathcal{H}$ can be decomposed into $\mathcal{F}_1, \ldots, \mathcal{F}_n \subseteq \mathcal{H}$ such that $|F \cap F'| \geq \mathbf{2} \ \forall \ F, F' \in \mathcal{F}_i$ and $i \in [n]$.

# Open problems

**Conjecture (Berge, 1989; Füredi, 1986; Meyniel)**

If $\mathcal{H}$ is a linear hypergraph, then $\chi'(\mathcal{H}) \leq \max_{v \in V(\mathcal{H})} |\bigcup_{e \ni v} e|$.

- common generalization of Vizing's theorem and EFL

**The List EFL conjecture (Faber, 2017)**

If $\mathcal{H}$ is an $n$-vertex linear hypergraph, then $\mathcal{H}$ has list chromatic index $\leq n$.

I.e. if $C(e)$ is a "list of colors" such that $|C(e)| \geq n \ \forall e \in \mathcal{H}$, then $\mathcal{H}$ can be properly edge-colored s.t. every $e$ is assigned a color from $C(e)$.

- Implies EFL if $C(e) = \{1, \ldots, n\} \ \forall e \in \mathcal{H}$.

**"Restricted" Larman's conjecture, 1981**

If $\mathcal{H}$ is an $n$-vertex **intersecting** hypergraph, then $\mathcal{H}$ can be decomposed into $\mathcal{F}_1, \ldots, \mathcal{F}_n \subseteq \mathcal{H}$ such that $|F \cap F'| \geq \mathbf{2} \ \forall \ F, F' \in \mathcal{F}_i$ and $i \in [n]$.

# Thanks for listening!