Coloring hypergraphs of small codegree, and a proof of the Erdős–Faber–Lovász conjecture

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Joint work with:
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UNIVERSITY OF BIRMINGHAM

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Part I

Coloring a nearly disjoint union of complete graphs

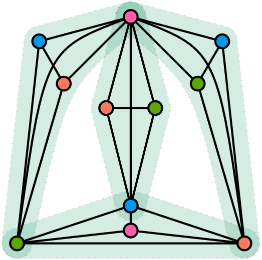
The Erdős–Faber–Lovász conjecture

proper coloring: adjacent vertices assigned different colors

chromatic number: min # colors used in proper coloring, denoted by χ

The Erdős–Faber–Lovász conjecture (1972)

If $G_1, \ldots, G_n$ are complete graphs, each on at most n vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^{n} G_i) \leq n$.



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One of Erdős' "three most favorite combinatorial problems":

- Erdős initially offered \$50 for a solution, raised to \$500.

  Faber, Lovász and I made this harmless looking conjecture at a party in Boulder Colorado in September 1972. Its difficulty was realised only slowly. I now offer 500 dollars for a proof or disproof. (Not long ago I only offered 50; the increase is not due to inflation but to the fact that I now think the problem is very difficult. Perhaps I am wrong.) *–Paul Erdős, 1981*

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Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

The Erdős–Faber–Lovász conjecture is true for sufficiently large n.

A more general question of Erdős

Question (Erdős, 1977)

If $G_1, \ldots, G_n$ are complete graphs, each on at most n vtcs, such that every pair shares at most t vtcs, what is the max possible value of $\chi(\bigcup_{i=1}^{n} G_i)$?

- The EFL conjecture asserts that the answer for $t = 1$ is n.

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We prove that for $2 \leq t < \sqrt{n}$ and n sufficiently large, the answer is tn:

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

For $t \geq 2$, n sufficiently large, and $G_1, \ldots, G_n$ as above, we have

$$\chi\left(\bigcup_{i=1}^{n} G_i\right) \leq tn.$$

Moreover, for infinitely many $k \in \mathbb{N}$, if $n = k^2 + k + 1$ and $t \leq k$, then there exist such $G_1, \ldots, G_n$ such that $\bigcup_{i=1}^{n} G_i$ has tn vtcs and is complete.

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Horák and Tuza (1990): $\chi(\bigcup_{i=1}^n G_i) \leq n^{3/2}$; covers range $t > \sqrt{n}$.

Part II

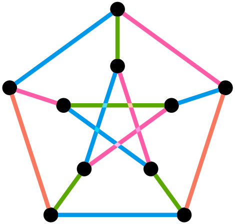
Hypergraph edge-coloring

Matchings and edge-coloring

matching: a set of disjoint edges

(proper) edge-coloring: no two edges of same color share a vertex

chromatic index: min # colors used in proper edge-coloring, denoted χ'



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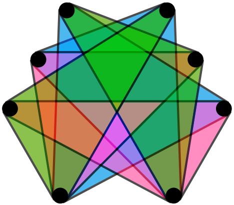

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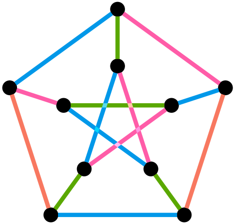

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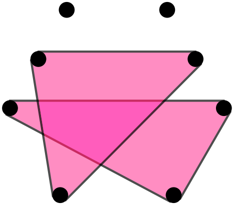

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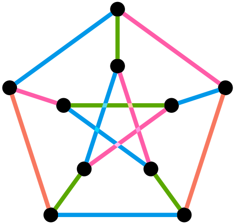

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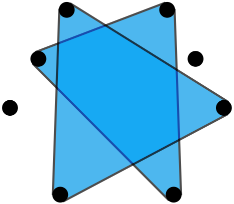

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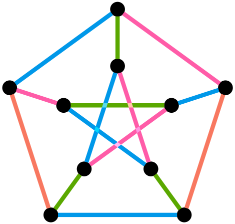

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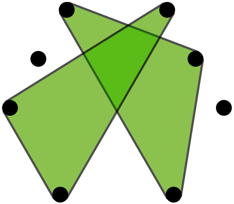

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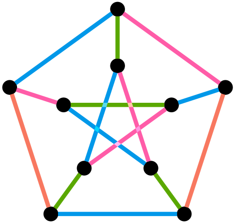

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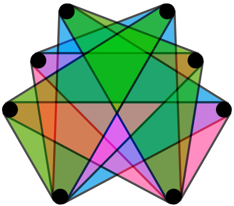

$\chi'(\mathcal{H}) = 3$

degree: # edges containing a vertex

Vizing's theorem (1964)

Every graph of maximum degree $\leq \Delta$ has chromatic index $\leq \Delta + 1$.

Matchings and edge-coloring

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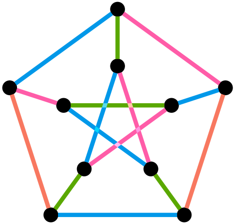

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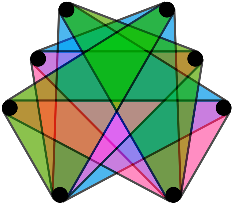


$\chi'(\mathcal{H}) = 3$

More complex for hypergraphs: e.g.

- 3-dimensional matching: one of Karp's original NP-complete problems

Question: Which hypergraphs have large matchings or small χ'?

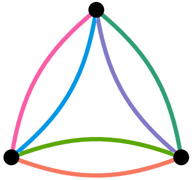
Hypergraph basics

In this talk, hypergraphs can have repeated edges but no size-one edges.

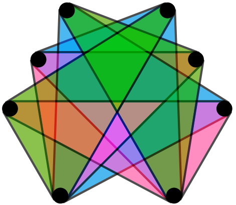
codegree: max # edges containing any given pair of vertices

linear: every pair of vertices contained in at most one edge

***k*-uniform:** every edge has size k



A 2-uniform hypergraph of codegree 2



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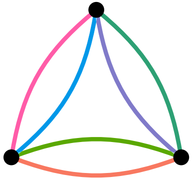

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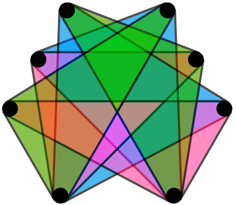

A linear 3-uniform hypergraph

- multigraphs are 2-uniform hypergraphs
- graphs are 2-uniform linear hypergraphs

The “nibble” method

Pippenger–Spencer theorem (1989)

If $\mathcal{H}$ is a hypergraph with bounded edge-sizes of maximum degree at most Δ and codegree $o(\Delta)$, then $\chi'(\mathcal{H}) \leq \Delta + o(\Delta)$.

Corollary (Pippenger's theorem): k-uniform Δ-regular hypergraphs of codegree $o(\Delta)$ have nearly perfect matchings

Corollary (Rödl, 1985): approximate combinatorial designs exist

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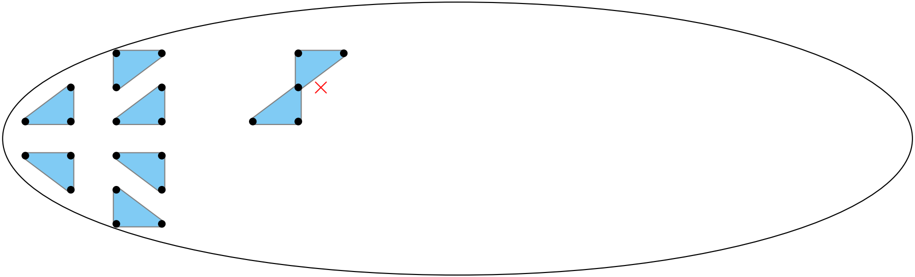

The “nibble” for 3-uniform nearly perfect matching

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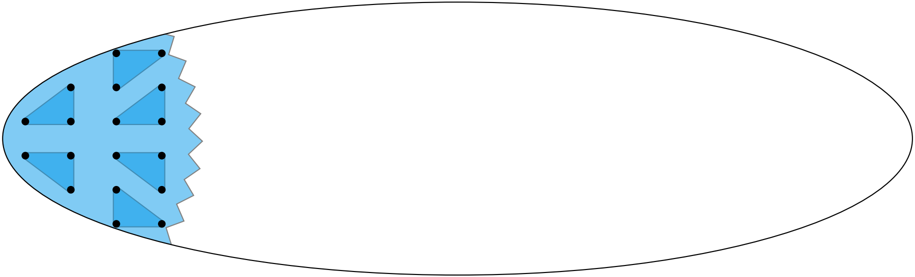

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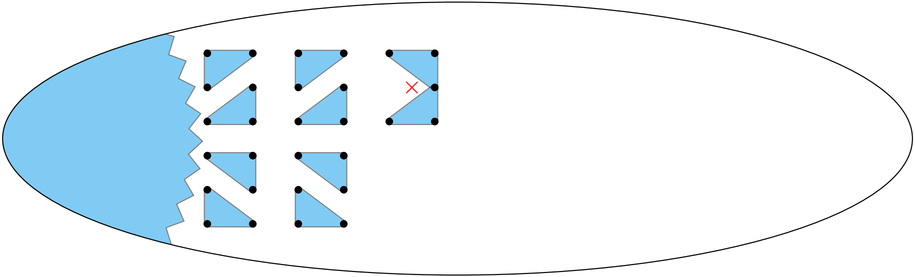

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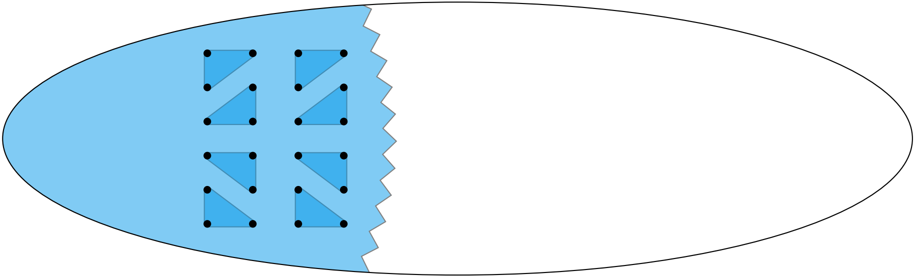

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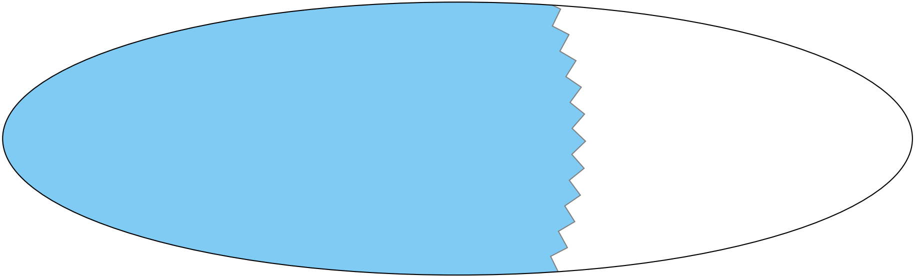

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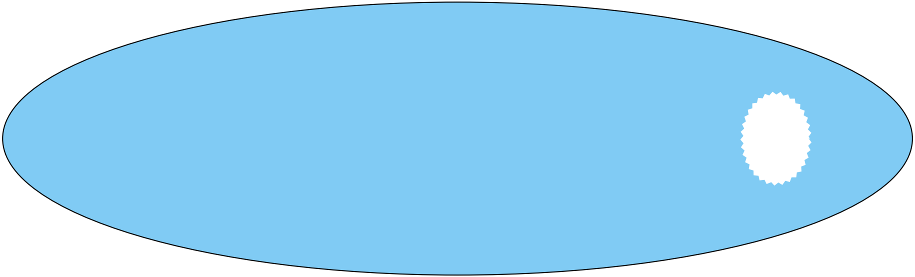


The "nibble" for 3-uniform nearly perfect matching

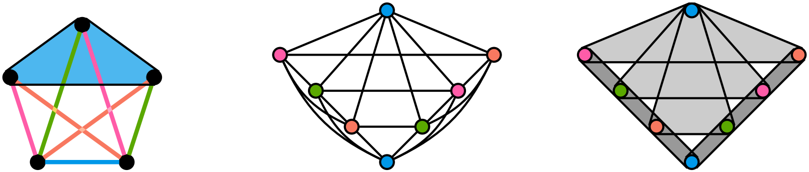
Part III

The Erdős–Faber–Lovász conjecture

Erdős–Faber–Lovász conjecture (reformulated)

The Erdős–Faber–Lovász conjecture (1972)

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.



Line graph:

- edges $\rightarrow$ vertices: edges that share a vertex are adjacent
- proper edge-coloring $\rightarrow$ proper vertex-coloring

The previous formulation is equivalent:

If $G_1, \ldots, G_n$ are complete graphs, each on at most n vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^{n} G_i) \leq n$.

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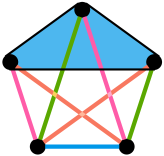
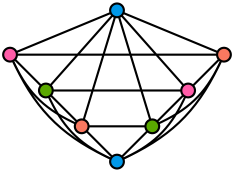
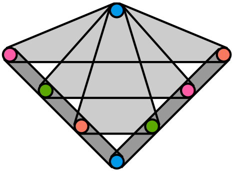






Hypergraph duality:

- edges $\rightarrow$ vertices and vertices $\rightarrow$ edges
- linearity is preserved

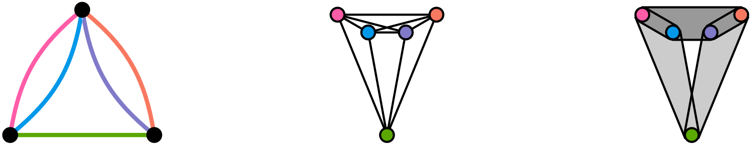
The previous formulation is equivalent:

If $G_1, \ldots, G_n$ are complete graphs, each on at most n vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^{n} G_i) \leq n$.

The dual of Erdős' question

Question (Erdős, 1977)

If $\mathcal{H}$ is an n-vertex hypergraph of maximum degree at most n and codegree at most t, what is the max possible value of $\chi'(\mathcal{H})$?



- max degree of $\mathcal{H} = \max |V(G_i)|$
- codegree of $\mathcal{H} = \max_{i \neq j} |V(G_i) \cap V(G_j)|$

The previous formulation is equivalent:

If $G_1, \ldots, G_n$ are complete graphs, each on at most n vtcs, such that every pair shares at most t vtcs, what is the max possible value of $\chi(\bigcup_{i=1}^{n} G_i)$?

Extremal examples for EFL

The Erdős–Faber–Lovász conjecture (1972)

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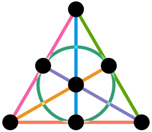
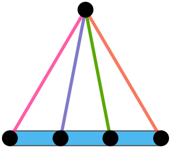
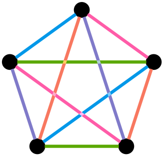
Extremal examples:







Finite projective plane of order k: $(k+1)$-uniform intersecting linear hypergraph with $n = k^2 + k + 1$ vertices and edges

Degenerate plane / near pencil: intersecting linear hypergraph with $n-1$ size-two edges and one size-$(n-1)$ edge

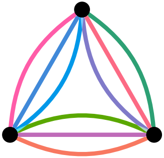
Complete graph: $\binom{n}{2}$ size-two edges; if $\chi' < n$, then color classes are perfect matchings $\Rightarrow$ n is even

Extremal examples for $t \geq 2$

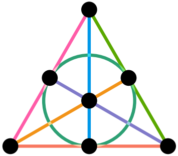
The "t-EFL" conjecture

If $\mathcal{H}$ is an n-vertex codegree-t hypergraph of max degree $\leq n$, then

$$\chi'(\mathcal{H}) \leq tn.$$



3-fold order-1 projective plane



1-fold Fano plane

t-fold projective plane: replace each edge with t repeated edges

Extremal examples: t-fold projective planes of order k for $t \leq k$:

- codegree t
- max degree $t(k+1)$ (and $t(k+1) \leq n$ if $t \leq k$)

Part IV

Results

Previous results

The Erdős–Faber–Lovász conjecture (1972)

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

Direct approaches:

Trivial: $\chi'(\mathcal{H}) \leq 2n - 3$ (color greedily, in order of size)

Chang–Lawler (1989): $\chi'(\mathcal{H}) \leq \lceil 3n/2 - 2 \rceil$

Previous results

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If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

Relaxed parameters:

de Bruijn–Erdős (1948): true for intersecting hypergraphs

Seymour (1982): $\exists$ a matching of size at least $|\mathcal{H}|/n$

Kahn–Seymour (1992): fractional chromatic index is at most n

Previous results

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Probabilistic "nibble" approach:

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- $\Rightarrow$ EFL if $|e| \in [3, k]$ $\forall e \in \mathcal{H}$ and $n \gg k$ (since $\Delta(\mathcal{H}) \le n/2$)
- $\Rightarrow$ EFL "asymptotically" if $|e| \le k$ $\forall e \in \mathcal{H}$ and $n \gg k$ ($\Delta(\mathcal{H}) \le n$)
- $\Rightarrow$ "t-EFL" for $t \ge 2$ if $|e| \le k$ $\forall e \in \mathcal{H}$ and $n \gg k$

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- $\Rightarrow$ "t-EFL" for $t \geq 2$ if $|e| \leq k$ $\forall e \in \mathcal{H}$ and $n \gg k$

Faber–Harris (2019): EFL is true if $|e| \in [3, c\sqrt{n}]$ $\forall e \in \mathcal{H}$ ($c \ll 1$)

Kahn (1992): $\chi'(\mathcal{H}) \leq (1 + o(1))n$

Both use "list coloring" generalization (proved by Kahn) of PS-theorem

Our results

We confirm the EFL conjecture for all but finitely many hypergraphs:

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

For sufficiently large n, every n-vertex linear hypergraph has chromatic index at most n.

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We also prove a stability result, predicted by Kahn:

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

$\forall \delta > 0$, $\exists \sigma > 0$ such that the following holds for n sufficiently large. If $\mathcal{H}$ is an n-vertex linear hypergraph such that

- $\Delta(\mathcal{H}) \leq (1 - \delta)n$ and
- at most $(1 - \delta)n$ edges have size $(1 \pm \delta)\sqrt{n}$,

then $\chi'(\mathcal{H}) \leq (1 - \sigma)n$.

Our results II

We confirm t-EFL for $t \geq 2$ for all but finitely many hypergraphs:

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

$\forall \varepsilon > 0$, the following holds for n sufficiently large and $t \in \mathbb{N}$.
If $\mathcal{H}$ is an n-vertex hypergraph with codegree at most t and maximum degree at most $(1-\varepsilon)tn$, then $\chi'_\ell(\mathcal{H}) \leq tn$. Moreover, if $\chi'_\ell(\mathcal{H}) = tn$, then $\mathcal{H}$ is a t-fold projective plane.

Strengthens answer to Erdős question in three ways:

- allows relaxed maximum degree assumption (except when $t = 1$)
- characterizes extremal examples
- holds for list coloring

Our results II

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We also generalize our stability result and the de Bruijn–Erdős theorem:

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

If $\mathcal{H}$ is an n-vertex intersecting hypergraph with codegree at most t, then $\mathcal{H}$ has at most tn edges, and if equality holds, then $\mathcal{H}$ is either

- a t-fold projective plane or
- a t-fold near-pencil.

Part V

Proof ideas

Roadmap to the proofs

KKKMO (2021+): If $\mathcal{H}$ is an n-vertex hypergraph of maximum degree at most n and codegree at most t, then $\chi'(\mathcal{H}) \leq tn$. $(n \gg 1)$

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1: "Small" edge case: $|e| \le k \; \forall e \in \mathcal{H}$ (Kahn asked in '94 for $k = 3$)

- The Pippenger–Spencer theorem (i.e. "nibble") implies the case $t \ge 2$ and implies $\chi'(\mathcal{H}) \le n + o(n)$ for $t = 1$
- Using absorption, reduce $t = 1$ case to a graph coloring problem

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2: "FPP-extremal case": $|e| \geq (1 - \delta)\sqrt{n}\ \forall e \in \mathcal{H}$ (for $\delta \ll 1$)

- Delicate argument – includes when $\mathcal{H} \approx t$-fold proj. plane
- Can also prove $\chi'_\ell(\mathcal{H}) < tn$ unless $\mathcal{H}$ is intersecting

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- Greedy coloring in order of size $\Rightarrow \chi'(\mathcal{H}) \leq (1 + 2/r)tn$.
- "Reordering lemma" finds highly structured $\mathcal{W} \subseteq \mathcal{H}$ – either $\mathcal{W} \approx t$-fold proj. plane or line graph of $\mathcal{W}$ is "locally sparse"

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KKKMO (2021+): If $\mathcal{H}$ is an n-vertex hypergraph of maximum degree at most n and codegree at most t, then $\chi'(\mathcal{H}) \le tn$. $(n \gg 1)$

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4: Merge cases

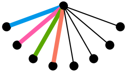
- Color large edges first, with special properties
- Stability result: can use fewer colors in non-FPP-extremal case

Vizing-reduction strategy for bounded edge-sizes

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \in \{2, 3\}$ $\forall e \in \mathcal{H}$.

- Fix $0 < \gamma \ll \varepsilon \ll 1$, and let $U := \{v \in V(\mathcal{H}) : d(v) > (1-\varepsilon)n\}$.

Low degree: more flexibility



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Vizing-reduction: Using $k := \lfloor (1/2 + \gamma)n \rfloor$ colors, color $\mathcal{H}$ such that:

- all size-3 edges are colored;
- $\geq (1/2 - \gamma)$-proportion of graph edges at each vtx are colored;
- every color class covers U (**perfect coverage** of U).

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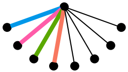

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Proof that $\chi'(\mathcal{H}) \leq n$ (assuming Vizing-reduction)

- vertices in U have leftover degree $\leq (n - 1) - k < n - k$;
- vertices not in U have leftover degree $\leq (1/2 + \gamma)(1 - \varepsilon)n < n - k$.

Uncolored edges comprise a **graph** of max degree $< n - k$. $(\star)$

Finish with Vizing's theorem! □

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- every color class covers U (**perfect coverage** of U).

Perfect coverage of U not always possible (e.g. K_n for n odd).
Instead, find coloring with **nearly perfect coverage:**

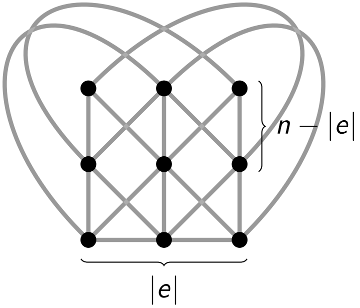
- every color class covers all but one vertex of U and
- each vertex of U is covered by all but one color class.

Works with one extra color; additional ideas needed to prove $\chi' \leq n$.

Coloring the large edges

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \geq r \ \forall e \in \mathcal{H}$, where $r \gg 1$.

Trivial: $\forall e \in \mathcal{H}$, at most $|e|(n - |e|)/(|e| - 1) \leq n + 2n/r$ edges of size at least $|e|$ intersect e.



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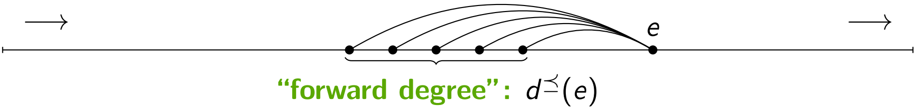
Corollary: $\chi'(\mathcal{H}) \leq n + o(n)$: color greedily.



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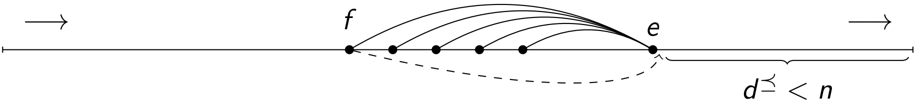

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If reordering "finishes", then $d^{\preceq}(e) < n \; \forall e \in \mathcal{H}$, so $\chi'(\mathcal{H}) \leq n$.

"Reordering lemma" (informal)

If reordering "gets stuck", then there is a highly structured $\mathcal{W} \subseteq \mathcal{H}$: either

- $\mathcal{W} \approx$ projective plane (i.e. its line graph is close to complete), or
- line graph of $\mathcal{W}$ is **locally sparse** (i.e. nbrhoods far from complete).

Use structure to color $\mathcal{H}$ with $\leq n$ colors (via graph theoretical techniques)

Part VI

Conclusion

Summary

The "t-EFL" conjecture

If $\mathcal{H}$ is an n-vertex codegree-t hypergraph of max degree $\leq n$, then

$$\chi'(\mathcal{H}) \leq tn.$$

Combining our results resolves the "t-EFL" conjecture for large n:

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

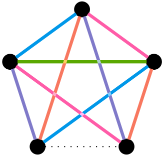
For sufficiently large n, every n-vertex hypergraph of maximum degree at most n and codegree at most t has chromatic index at most tn.

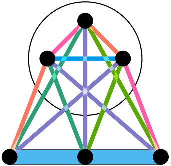
- The case $t = 1$ confirms the Erdős–Faber–Lovász conjecture for all but finitely many hypergraphs
- For $t \geq 2$, we characterize extremal examples and prove bounds hold for list coloring and with relaxed max degree assumption
- We also prove stability results and a generalization of the de Bruijn–Erdős theorem

More extremal examples

Overfull graph: $> \Delta\lfloor n/2 \rfloor$ edges, where $\Delta =$max degree and $n = \#$ vtcs

"Blowup" of degenerate plane: replace pencil point with a clique





Additional extremal examples for EFL:

- overfull graphs with maximum degree $n - 1$
- "odd blowups" of a degenerate plane

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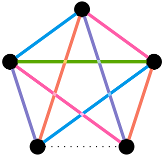

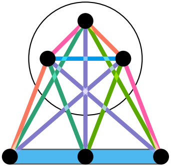

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Conjecture

If $\mathcal{H}$ is an n-vertex linear hypergraph of chromatic index n, then either

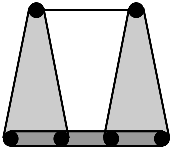
- $\mathcal{H}$ has more than $(n-1)^2/2$ size-two edges and n is odd,
- $\mathcal{H}$ is a finite projective plane (of order k, where $n = k^2 + k + 1$), or
- $\mathcal{H}$ is an odd blowup of a degenerate plane.

Open problems

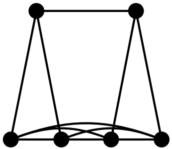
Conjecture (Berge, 1989; Füredi, 1986; Meyniel)

If $\mathcal{H}$ is a linear hypergraph, then $\chi'(\mathcal{H}) \leq \max_{v \in V(\mathcal{H})} |\bigcup_{e \ni v} e|$.

- common generalization of Vizing's theorem and EFL



$\max_v |\bigcup_{e \ni v} e| = 5$



$\Delta(\text{“shadow”}) + 1 = 5$

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The List EFL conjecture (Faber, 2017)

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\mathcal{H}$ has list chromatic index $\leq n$.

I.e. if $C(e)$ is a "list of colors" such that $|C(e)| \geq n \ \forall e \in \mathcal{H}$, then $\mathcal{H}$ can be properly edge-colored s.t. every e is assigned a color from $C(e)$.

- Implies EFL if $C(e) = \{1, \ldots, n\} \ \forall e \in \mathcal{H}$.

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"Restricted" Larman's conjecture, 1981

If $\mathcal{H}$ is an n-vertex **intersecting** hypergraph, then $\mathcal{H}$ can be decomposed into $\mathcal{F}_1, \ldots, \mathcal{F}_n \subseteq \mathcal{H}$ such that $|F \cap F'| \geq \mathbf{2} \ \forall \ F, F' \in \mathcal{F}_i$ and $i \in [n]$.

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Thanks for listening!