Bounding χ by a Fraction of Δ for Graphs without Large Cliques

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Introduction

If G is a graph, then

- $\chi(G)$ = chromatic number of G
- $\Delta(G)$ = max degree of a vertex in G, and
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Brooks' Theorem (1941)

If $\Delta \geq 3$ and $\omega \leq \Delta$, then

$$\chi \leq \Delta.$$

Reed's Conjecture

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If $\omega \leq \Delta + 1 - 2k$, then

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Why $2k$?

Theorem (Spencer, 1977)

The off-diagonal Ramsey number satisfies

$$R(s, t) = \Omega\left((t/\ln t)^{\frac{s+1}{2}}\right)$$

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$$R(s,t) = \Omega\left((t/\ln t)^{\frac{s+1}{2}}\right)$$

Letting $s = 3$ implies that there exists G on n vertices with $\chi(G) \geq n/2$ and $\omega(G) = O(n^{\frac{1}{2}+o(1)})$.

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If $\omega \leq \Delta + 1 - 2k$, then
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Reed proved his conjecture for $\Delta \geq 10^8 \cdot k$.

Corollary

If $\omega \leq \Delta + 1 - 10^8 \cdot k$, *then*
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- In 2016, Bonamy, Perrett, and Postle proved it when $\omega \leq \Delta + 1 - 26 \cdot k$ and Δ is large (not depending on k).
- In 2017, Delcourt and Postle improved this to $\omega \leq \Delta + 1 - 13 \cdot k$.

Triangle-free Graphs ($\omega \leq 2$)

Theorem (Johansson, 1996)

If $\omega \leq 2$ (i.e. triangle-free), then

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Theorem (Pettie and Su, 2014)

If $\omega \leq 2$ (i.e. triangle-free), then

$$\chi \leq (4 + o(1))\frac{\Delta}{\ln \Delta}.$$

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- Molloy's bound matches Kim's bound from '95 for girth 5 graphs and implies Shearer's bound on $R(3, k)$.
- Random Δ-regular graphs can have $\omega = 2$ and $\chi \geq \frac{\Delta}{2 \ln \Delta}$.

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- For this and triangle-free result, Molloy used entropy compression – Bernshteyn found shorter proofs by sampling a partial coloring uniformly at random and using the Lovász Local Lemma.

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For Δ sufficiently large,

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Spencer's Ramsey-theory result implies there exists G on n vertices with $\omega(G) = O\left(n^{\frac{2}{c+2}+o(1)}\right)$ and $\chi(G) \geq n/c \geq \Delta/c$.

The state of the art

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$\ln \omega = o(\ln \Delta)$	$o(\Delta)$	BKNP
	$200\Delta\frac{\omega \ln\ln\Delta}{\ln\Delta}$, $72\Delta\sqrt{\frac{\ln\omega}{\ln\Delta}}$	Molloy, BKNP
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List-Coloring

For a graph G,

- $L = (L(v) : v \in V(G))$ is a list-assignment if each $L(v) \subset \mathbb{N}$ is a "list of colors",
- G is L-colorable if there is a proper coloring in which each $v \in V(G)$ receives a color from $L(v)$, and
- the list-chromatic number of G, denoted $\chi_\ell(G)$, is the smallest k such that G is L-colorable whenever $|L(v)| \geq k$ for all $v \in V(G)$.

Clearly,

$$\chi(G) \leq \chi_\ell(G)$$

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$$\omega(G) \leq \chi(G) \leq \chi_\ell(G) \leq \Delta(G) + 1.$$

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Clearly,

$$\omega(G) \leq \chi(G) \leq \chi_\ell(G) \leq \Delta(G) + 1.$$

What if $|L(v)|$ depends on local parameters, such as

- $d(v)$, the degree of v and
- $\omega(v)$, the size of a largest clique containing v?

The Local Paradigm

Theorem (Erdős, Rubin, Taylor, 1979)

Every connected graph G is L-colorable if $|L(v)| \geq d(v)$ for all $v \in V(G)$, unless every block of G is a clique or odd cycle.

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Every graph G is L-colorable if $|L(v)| \geq \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil$ for every $v \in V(G)$.

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What if

- $|L(v)| = \Omega\left(d(v)\sqrt{\frac{\ln \omega(v)}{\ln d(v)}}\right)$ or $\Omega\left(d(v)^{\frac{\omega(v) \ln \ln d(v)}{\ln d(v)}}\right)$, or
- $|L(v)| = \Omega(d(v)/\ln d(v))$ and G is triangle-free?

Our Local Versions

Theorem (BKNP 2018+)

If Δ is sufficiently large, $\Delta(G) \leq \Delta$, and for each $v \in V(G)$,

$$|L(v)| \geq 72d(v) \min\left\{\sqrt{\frac{\ln \omega(v)}{\ln d(v)}}, \frac{\omega(v) \ln \ln d(v)}{\ln d(v)}, \frac{\ln(\chi(G[N(v)]) + 1)}{\ln d(v)}\right\},$$

and $d(v) \geq \ln^2 \Delta$, then G is L-colorable.

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This simultaneously implies

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- $\chi \leq 72\Delta\sqrt{\ln \omega / \ln \Delta}$ (BKNP),
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- $\chi \leq 200\Delta\frac{\omega\ln\ln\Delta}{\ln\Delta}$ (Molloy), and
- $\chi \leq O(\Delta\ln r/\ln\Delta)$ if $\chi(G[N(v)]) < r$ for all $v \in V(G)$ (Johansson).

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and $d(v) \geq \ln^2 \Delta$, then G is L-colorable.

Theorem (BKNP, 2018+)

If Δ is sufficiently large, G has triangle-free and $\Delta(G) \leq \Delta$, and for each $v \in V(G)$,

$$|L(v)| \geq (4 + o(1))\frac{d(v)}{\log_2(d(v))}$$

and $d(v) \geq \ln^2 \Delta$, then G is L-colorable.

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Let $\varepsilon > 0$. Let $\alpha_{min}(v)$ be the minimum of $\overline{\alpha}(H)$ where $H \subseteq G[N(v)]$ and $i(H) \geq d(v)^{(1-\varepsilon)/2}$. If Δ sufficiently large, $\Delta(G) \leq \Delta$, and for each $v \in V(G)$,

$$|L(v)| \geq (1+\varepsilon)\frac{d(v)}{\alpha_{min}(v)}$$

and $d(v) \geq \ln^2 \Delta$, then G has an L-coloring.

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Lemma

$$\overline{\alpha}(H) \geq \begin{cases} \log(i(H))/(10\log(\chi(H)+1)), & (Alon) \\ \log(i(H))/(2\omega\log(\log(i(H)))), & (Shearer) \\ 24^{-1}\sqrt{\log(i(H))/\log(\omega)}. & (BKNP) \end{cases}$$

Thanks!