A proof of the Erdős-Faber-Lovász conjecture

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Joint work with:
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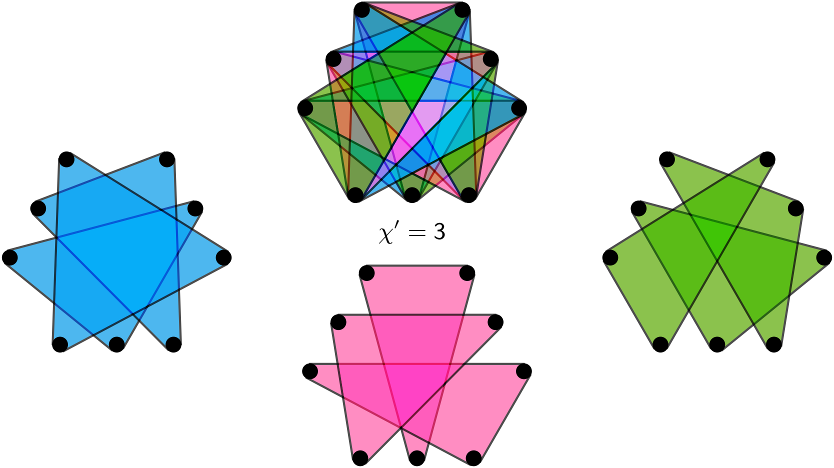
Hypergraph coloring

(proper) edge-coloring: no two edges of same color share a vertex

chromatic index: min # colors used in proper edge-coloring, denoted χ'



$\chi' = 3$

The Erdős-Faber-Lovász conjecture

linear hypergraph: every pair of vertices contained in at most one edge

Erdős-Faber-Lovász conjecture (1972)

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

One of Erdős' "three most favorite combinatorial problems":

- Erdős initially offered \$50 for a solution, raised to \$500.

  Faber, Lovász and I made this harmless looking conjecture at a party in Boulder Colorado in September 1972. Its difficulty was realised only slowly. I now offer 500 dollars for a proof or disproof. (Not long ago I only offered 50; the increase is not due to inflation but to the fact that I now think the problem is very difficult. Perhaps I am wrong.) *–Paul Erdős, 1981*

The Erdős-Faber-Lovász conjecture

linear hypergraph: every pair of vertices contained in at most one edge

Erdős-Faber-Lovász conjecture (1972)

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

- Graphs are linear hypergraphs
- Linear hypergraphs with n vertices have maximum degree $\leq n - 1$.

Vizing's theorem (1964): If G is a graph of maximum degree Δ, then $\chi'(G) \leq \Delta + 1$.

Corollary: EFL is true for graphs.

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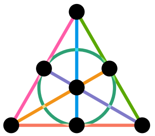
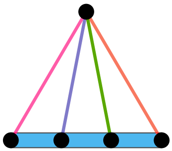
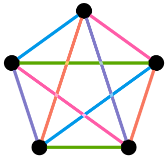
Extremal examples:







Finite projective plane of order k: $(k+1)$-uniform intersecting linear hypergraph with $n = k^2 + k + 1$ vertices and edges

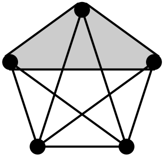
Degenerate plane / near pencil: intersecting linear hypergraph with $n-1$ size-two edges and one size-$(n-1)$ edge

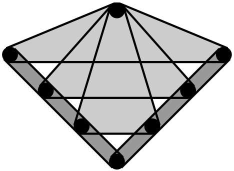
Complete graph: $\binom{n}{2}$ size-two edges; if $\chi' < n$, then color classes are perfect matchings $\Rightarrow$ n is even

Dual versions

Erdős-Faber-Lovász conjecture ("dual")

If $\mathcal{H}$ is an n-uniform, n-edge, linear hypergraph, then the vertices of $\mathcal{H}$ can be n-colored such that every edge contains a vertex of every color.





Hypergraph duality:

- edges $\rightarrow$ vertices and vertices $\rightarrow$ edges
- linearity is preserved
- proper edge-coloring $\leftrightarrow$ vertex-coloring where no edge contains two vertices of same color

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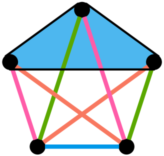

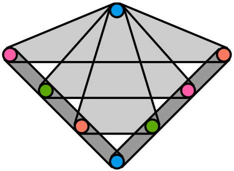

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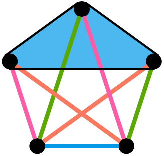
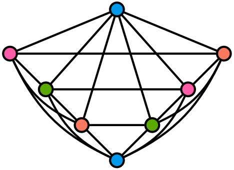
Erdős-Faber-Lovász conjecture ("graphic")

If G is the union of n complete graphs, each on at most n vertices, such that every pair shares at most one vertex, then $\chi(G) \leq n$.





Line graph:

- edges $\rightarrow$ vertices: edges that share a vertex are adjacent
- proper edge-coloring $\rightarrow$ proper vertex-coloring (no monochromatic edge)

Previous results

Erdős-Faber-Lovász conjecture (1972)

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

Relaxed parameters:

de Bruijn-Erdős (1948): true for intersecting hypergraphs

Seymour (1982): $\exists$ a matching of size at least $|\mathcal{H}|/n$

Kahn-Seymour (1992): fractional chromatic index is at most n

Previous results

Erdős-Faber-Lovász conjecture (1972)

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\chi'(\mathcal{H}) \leq n$.

Probabilistic "nibble" approach:

Faber-Harris (2019): EFL is true if $|e| \in [3, c\sqrt{n}]$ $\forall e \in \mathcal{H}$ $(c \ll 1)$

Kahn (1992): $\chi'(\mathcal{H}) \leq (1 + o(1))n$

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Both use "list coloring" generalization (proved by Kahn) of:

Pippenger-Spencer theorem (1989)

If $\mathcal{H}$ is a linear hypergraph with bounded edge-sizes and maximum degree at most Δ, then $\chi'(\mathcal{H}) \leq \Delta + o(\Delta)$.

- $\Rightarrow$ EFL if $|e| \in [3, k]$ $\forall e \in \mathcal{H}$ and $n \gg k$ (since $\Delta(\mathcal{H}) \leq n/2$)
- $\Rightarrow$ EFL "asymptotically" if $|e| \leq k$ $\forall e \in \mathcal{H}$ and $n \gg k$ ($\Delta(\mathcal{H}) \leq n$)

Our results

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

For sufficiently large n, every n-vertex linear hypergraph has chromatic index at most n.

I.e., we confirm the EFL conjecture for all but finitely many hypergraphs.

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For sufficiently large n, every n-vertex linear hypergraph has chromatic index at most n.

I.e., we confirm the EFL conjecture for all but finitely many hypergraphs.
We also prove a stability result, predicted by Kahn:

Theorem (Kang, K., Kühn, Methuku, and Osthus, 2021+)

$\forall \delta > 0$, $\exists \sigma > 0$ *such that the following holds for n sufficiently large. If $\mathcal{H}$ is an n-vertex linear hypergraph such that*

- $\Delta(\mathcal{H}) \leq (1-\delta)n$ *and*
- *at most $(1-\delta)n$ edges have size $(1 \pm \delta)\sqrt{n}$,*

then $\chi'(\mathcal{H}) \leq (1-\sigma)n$.

Overview of the proof

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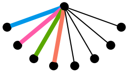
Upshot: Reduce to the "right" **graph** coloring problem in each case.

Vizing-reduction strategy for bounded edge-sizes

- Let $\mathcal{H}$ be a linear hypergraph such that $|e| \in \{2, 3\}$ $\forall e \in \mathcal{H}$.
- Fix $0 < \gamma \ll \varepsilon \ll 1$, and let $U := \{v \in V(\mathcal{H}) : d(v) > (1 - \varepsilon)n\}$.

Low degree: more flexibility



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Vizing-reduction: Using $k := \lfloor (1/2 + \gamma)n \rfloor$ colors, color $\mathcal{H}$ such that:

- all size-3 edges are colored;
- $\geq (1/2 - \gamma)$-proportion of graph edges at each vtx are colored;
- every color class covers U (**perfect coverage** of U).

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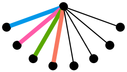

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- all size-3 edges are colored;
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- every color class covers U (**perfect coverage** of U).

Proof that $\chi'(\mathcal{H}) \leq n$ (assuming Vizing reduction)

- vertices in U have leftover degree $\leq (n - 1) - k < n - k$;
- vertices not in U have leftover degree $\leq (1/2 + \gamma)(1 - \varepsilon)n < n - k$.

Uncolored edges comprise a **graph** of max degree $< n - k$. $(\star)$

Finish with Vizing's theorem! □

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- all size-3 edges are colored;
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- every color class covers U (**perfect coverage** of U).

Perfect coverage of U not always possible (e.g. K_n for n odd).
Instead, find coloring with **nearly perfect coverage:**

- every color class covers all but one vertex of U and
- each vertex of U is covered by all but one color class.

Works with one extra color; additional ideas needed to prove $\chi' \leq n$.

Simplified proof with one extra color

Recall: $U = \{v \in V(\mathcal{H}) : d(v) > (1-\varepsilon)n\}$ $(0 < \gamma \ll \varepsilon \ll 1)$

Aim: Using $k = \lfloor (1/2 + \gamma)n \rfloor$ colors, color $\mathcal{H}$ such that:

- all size-3 edges are colored;
- for each vertex, nearly half of graph edges containing it are colored;
- the color classes have **nearly perfect coverage** of U.

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- for each vertex, nearly half of graph edges containing it are colored; ✓
- the color classes have **nearly perfect coverage** of U. ✗

Proof (sketch) of $\chi' \leq n+1$

Put each graph edge in a "reservoir" R independently with probability $1/2$;

- with high probability $\Delta(\mathcal{H} \setminus R) \leq (1/2 + o(1))n$, so $\chi'(\mathcal{H} \setminus R) \leq (1/2+\gamma)n$ by the Pippenger-Spencer theorem.

To obtain nearly perfect coverage, "re-run" Pippenger-Spencer proof (**nibble**) but apply **absorption** for each color class.

Nibble: Randomly construct matching in $\mathcal{H} \setminus R$ covering $\approx (1-\gamma)n$ vtcs.

Absorption: Augment with matching in R covering remaining U-vtcs.

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Proof (sketch) of $\chi' \leq n+1$

Put each graph edge in a "reservoir" R independently with probability $1/2$;

Nibble + absorption: using $k = (1/2+\gamma)n$ colors, color some $\mathcal{H}' \supseteq \mathcal{H} \setminus R$ with **nearly perfect coverage** of U:

- vertices in U have leftover degree $\leq (n-1)-(k-1) \leq n-k$;
- vertices not in U have leftover degree $\leq (1-\varepsilon)n/2 + o(n) < n-k$.

Thus $\mathcal{H} \setminus \mathcal{H}'$ is a **graph** and $\Delta(\mathcal{H} \setminus \mathcal{H}') \leq n-k$, so by Vizing's thm

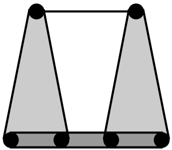
$$\chi'(\mathcal{H}) \leq \chi'(\mathcal{H}') + \chi'(\mathcal{H} \setminus \mathcal{H}') \leq k + (n-k+1) = n+1. \qquad \square$$

Open problems

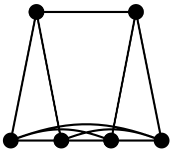
Conjecture (Berge '89, Füredi '86, Meyniel (unpublished))

If $\mathcal{H}$ is a linear hypergraph, then $\chi'(\mathcal{H}) \leq \max_{v \in V(\mathcal{H})} |\bigcup_{e \ni v} e|$.

- common generalization of Vizing's theorem and EFL



$\max_v |\bigcup_{e \ni v} e| = 5$



$\Delta(\text{"shadow"}) + 1 = 5$

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List EFL

If $\mathcal{H}$ is an n-vertex linear hypergraph, then $\mathcal{H}$ has list chromatic index $\leq n$.

I.e. if $C(e)$ is a "list of colors" such that $|C(e)| \geq n$ $\forall e \in \mathcal{H}$, then $\mathcal{H}$ can be properly edge-colored s.t. every e is assigned a color from $C(e)$.

- Implies EFL if $C(e) = \{1, \ldots, n\}$ $\forall e \in \mathcal{H}$.

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"Restricted" Larman's conjecture '81

If $\mathcal{H}$ is an n-vertex **intersecting** hypergraph, then $\mathcal{H}$ can be decomposed into $\mathcal{F}_1, \ldots, \mathcal{F}_n \subseteq \mathcal{H}$ such that $|F \cap F'| \geq \mathbf{2}$ $\forall$ $F, F' \in \mathcal{F}_i$ and $i \in [n]$.

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Nibble + absorption

- $U = \{v \in V(\mathcal{H}) : d(v) > (1-\varepsilon)n\}$ $\qquad (0 < \gamma \ll \varepsilon \ll 1)$
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Alternate applications of "nibble" & "absorption"; construct k matchings

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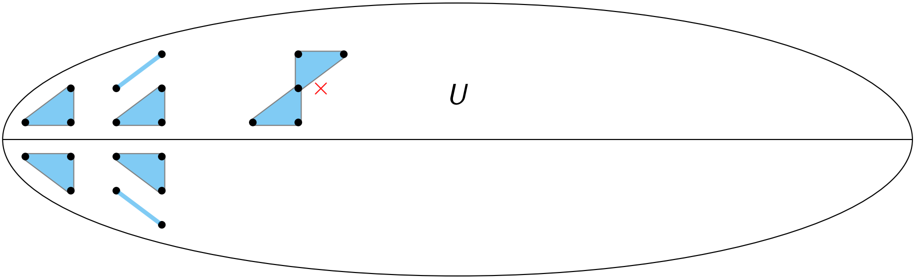

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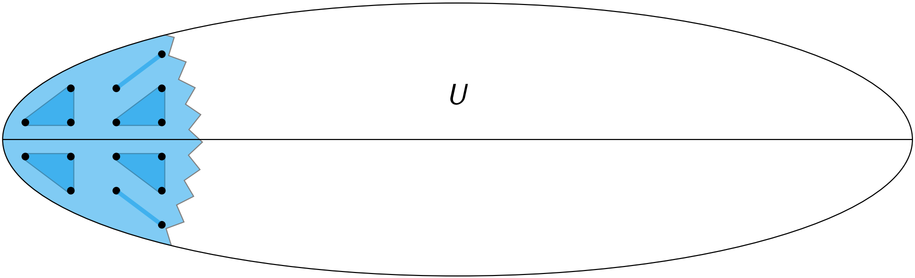

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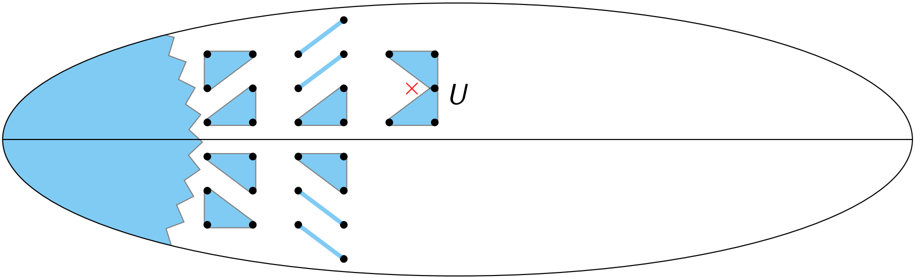

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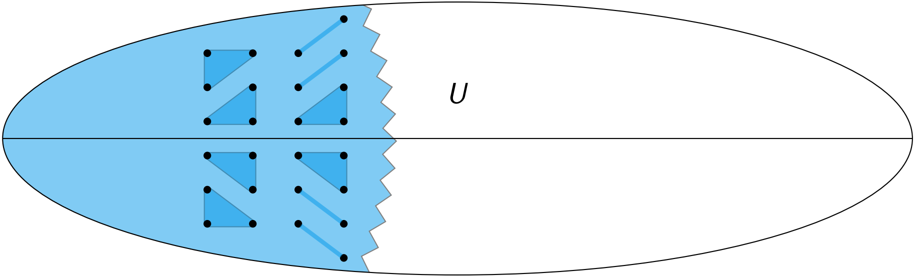

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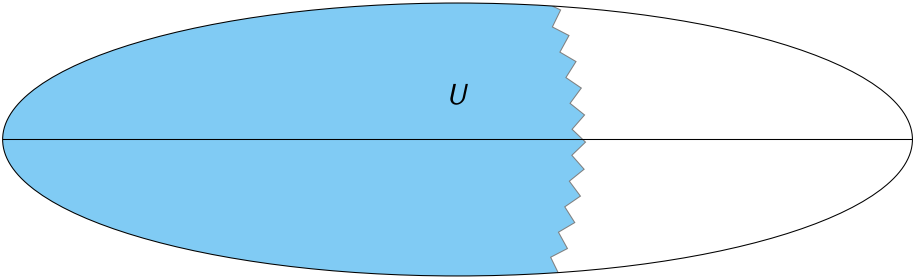

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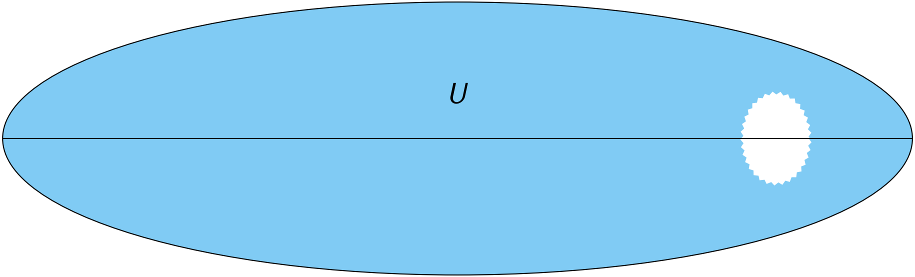

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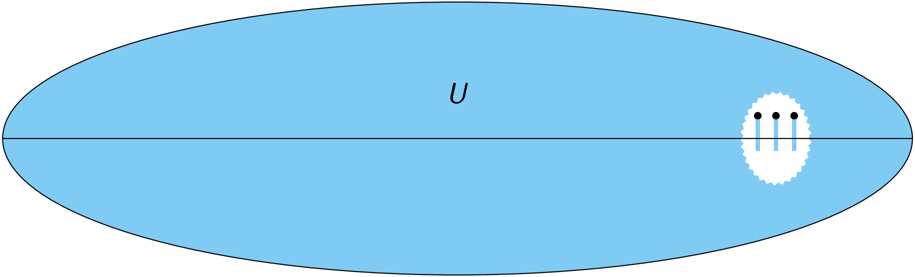

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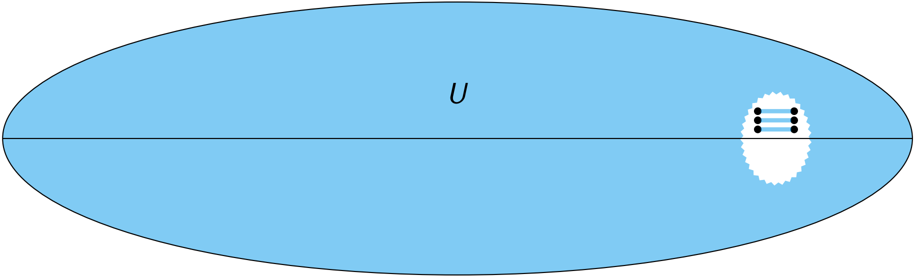

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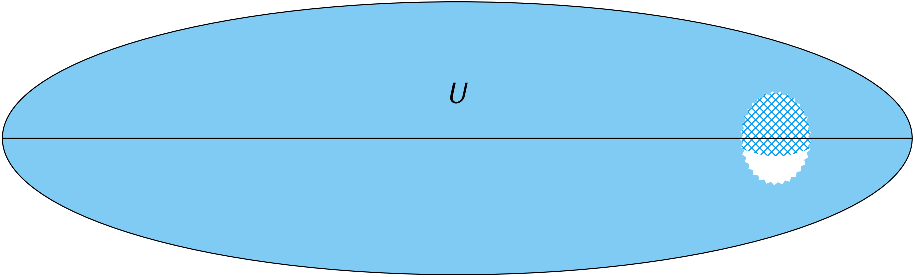

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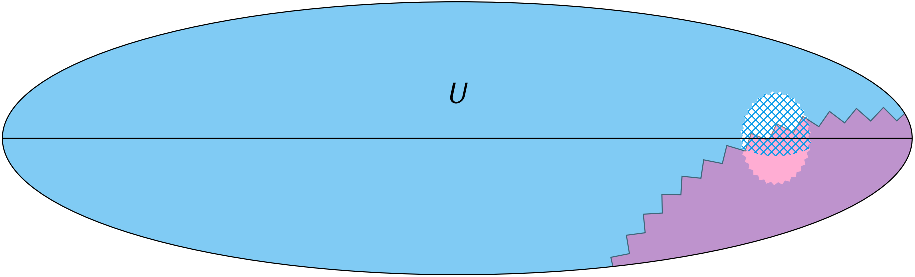

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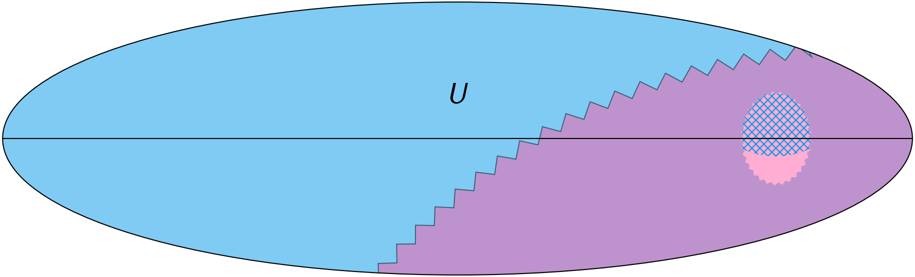

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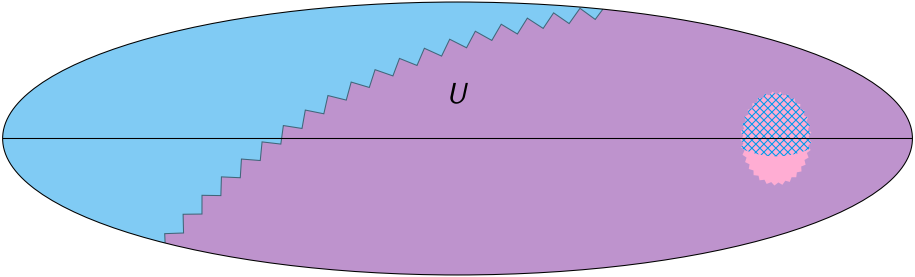

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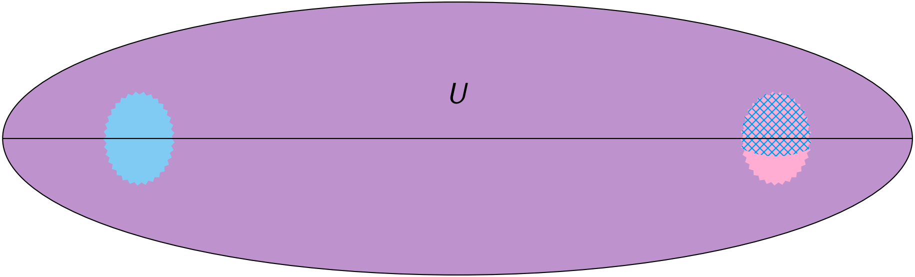

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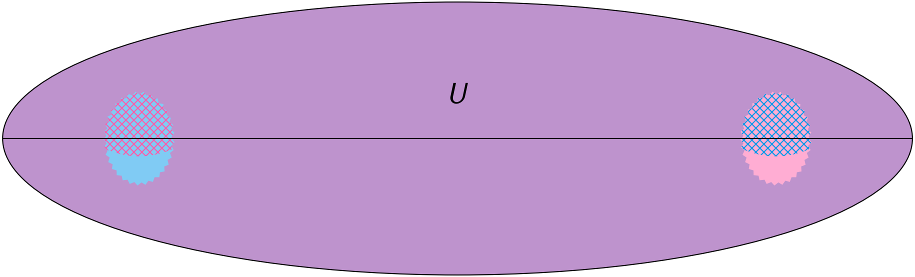
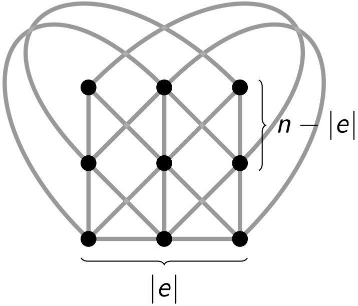


Coloring the large edges

Let $\mathcal{H}$ be a linear hypergraph such that $|e| \geq r\ \forall e \in \mathcal{H}$, where $r \gg 1$.

Trivial: $\forall e \in \mathcal{H}$, at most $|e|(n - |e|)/(|e| - 1) \leq n + o(n)$ edges of size at least $|e|$ intersect e.



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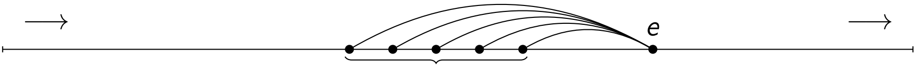
Corollary: $\chi'(\mathcal{H}) \leq n + o(n)$: color greedily.



"forward degree": $d^{\preceq}(e)$

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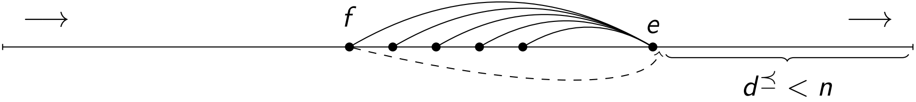

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If reordering "finishes", then $d^{\preceq}(e) < n\ \forall e \in \mathcal{H}$, so $\chi'(\mathcal{H}) \leq n$.

"Reordering lemma" (informal)

If reordering "gets stuck", then there is a highly structured $\mathcal{W} \subseteq \mathcal{H}$: either

- $\mathcal{W} \approx$ projective plane (i.e. its line graph is close to complete), or
- line graph of $\mathcal{W}$ is **locally sparse** (i.e. nbrhoods far from complete).

Use structure to color $\mathcal{H}$ with $\leq n$ colors (via graph theoretical techniques)