Beyond Degree-Choosability
Toward a Local Version of Reed's ω, Δ, χ conjecture

Tom Kelly
Joint work with Luke Postle

Department of Combinatorics & Optimization
University of Waterloo

CanaDAM, June 12, 2017

Coloring

Definition

A graph G is k-colorable if there is an assignment of the "colors" $1, \ldots, k$ to $V(G)$ such that adjacent vertices receive different colors.
The chromatic number of G, denoted $\chi(G)$, is the smallest k such that G is k-colorable.

- $\Delta(G)$ = max degree of a vertex in G, and
- $\omega(G)$ = max size of a clique in G.

Trivial bounds:

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1.$$

Coloring

Definition

A graph G is k-colorable if there is an assignment of the "colors" $1, \ldots, k$ to $V(G)$ such that adjacent vertices receive different colors.
The chromatic number of G, denoted $\chi(G)$, is the smallest k such that G is k-colorable.

- $\Delta(G)$ = max degree of a vertex in G, and
- $\omega(G)$ = max size of a clique in G.

Trivial bounds:

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1.$$

Theorem (Brooks (1941))

For every connected graph G that is not a clique or odd cycle,

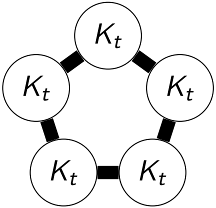
$$\chi(G) \leq \Delta(G).$$

Reed's ω, Δ, χ Conjecture

Reed's Conjecture (1998)

For every graph G,

$$\chi(G) \leq \left\lceil \frac{1}{2}\left(\Delta(G) + 1 + \omega(G)\right) \right\rceil .$$



5-cycle blowup

$$\Delta = 3t - 1$$
$$\omega = 2t$$
$$\lceil \tfrac{1}{2}(\Delta + 1 + \omega) \rceil = \lceil \tfrac{5t}{2} \rceil.$$
$$\alpha = 2.$$

Reed's ω, Δ, χ Conjecture

Reed's Conjecture (1998)

For every graph G,

$$\chi(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil.$$

As evidence, Reed proved his conjecture holds when $\Delta(G)$ is sufficiently large and

$$\omega(G) \geq \left(1 - 7 \cdot 10^{-7}\right) \Delta(G).$$

Corollary (Reed)

There exists $\varepsilon > 0$ such that for every graph G,

$$\chi(G) \leq (1 - \varepsilon)(\Delta(G) + 1) + \varepsilon\omega(G).$$

List-Coloring

Definition

For a graph G, $L = (L(v) : v \in V(G))$ is a list-assignment if each $L(v) \subset \mathbb{N}$ is a "list of colors", and G is L-colorable if there is an assignment of colors to $V(G)$ such that adjacent vertices receive different colors and each $v \in V(G)$ receives a color from $L(v)$.

The list-chromatic number of G, denoted $\chi_\ell(G)$, is the smallest k such that G is L-colorable whenever $|L(v)| \geq k$ for all $v \in V(G)$.

Clearly,

$$\chi(G) \leq \chi_\ell(G)$$

List-Coloring

Definition

For a graph G, $L = (L(v) : v \in V(G))$ is a list-assignment if each $L(v) \subset \mathbb{N}$ is a "list of colors", and G is L-colorable if there is an assignment of colors to $V(G)$ such that adjacent vertices receive different colors and each $v \in V(G)$ receives a color from $L(v)$.

The list-chromatic number of G, denoted $\chi_\ell(G)$, is the smallest k such that G is L-colorable whenever $|L(v)| \geq k$ for all $v \in V(G)$.

Clearly,

$$\omega(G) \leq \chi(G) \leq \chi_\ell(G) \leq \Delta(G) + 1.$$

List-Coloring

Definition

For a graph G, $L = (L(v) : v \in V(G))$ is a list-assignment if each $L(v) \subset \mathbb{N}$ is a "list of colors", and G is L-colorable if there is an assignment of colors to $V(G)$ such that adjacent vertices receive different colors and each $v \in V(G)$ receives a color from $L(v)$.

The list-chromatic number of G, denoted $\chi_\ell(G)$, is the smallest k such that G is L-colorable whenever $|L(v)| \geq k$ for all $v \in V(G)$.

Clearly,

$$\omega(G) \leq \chi(G) \leq \chi_\ell(G) \leq \Delta(G) + 1.$$

It is natural to ask:

List-Coloring

Definition

For a graph G, $L = (L(v) : v \in V(G))$ is a list-assignment if each $L(v) \subset \mathbb{N}$ is a "list of colors", and G is L-colorable if there is an assignment of colors to $V(G)$ such that adjacent vertices receive different colors and each $v \in V(G)$ receives a color from $L(v)$.

The list-chromatic number of G, denoted $\chi_\ell(G)$, is the smallest k such that G is L-colorable whenever $|L(v)| \geq k$ for all $v \in V(G)$.

Clearly,

$$\omega(G) \leq \chi(G) \leq \chi_\ell(G) \leq \Delta(G) + 1.$$

It is natural to ask:

- does Brooks' Theorem hold for χ_ℓ?

List-Coloring

Definition

For a graph G, $L = (L(v) : v \in V(G))$ is a list-assignment if each $L(v) \subset \mathbb{N}$ is a "list of colors", and G is L-colorable if there is an assignment of colors to $V(G)$ such that adjacent vertices receive different colors and each $v \in V(G)$ receives a color from $L(v)$.

The list-chromatic number of G, denoted $\chi_\ell(G)$, is the smallest k such that G is L-colorable whenever $|L(v)| \geq k$ for all $v \in V(G)$.

Clearly,

$$\omega(G) \leq \chi(G) \leq \chi_\ell(G) \leq \Delta(G) + 1.$$

It is natural to ask:

- does Brooks' Theorem hold for χ_ℓ?
- is Reed's Conjecture true for χ_ℓ?

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil .$$

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil.$$

ω	$\chi \leq$	proved by:	χ_ℓ?

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil.$$

ω	$\chi \leq$	proved by:	χ_ℓ?
$\geq (1 - 7 \cdot 10^{-7})\Delta$	$\lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$	Reed 98	N

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \le \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil.$$

ω	$\chi \le$	proved by:	χ_ℓ?
$\ge (1 - 7 \cdot 10^{-7})\Delta$	$\lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$	Reed 98	N
	$(1-\varepsilon)(\Delta+1)+\varepsilon\omega$, $\varepsilon = 1.4 \cdot 10^{-8}$	Reed 98	N

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil .$$

ω	$\chi \leq$	proved by:	χ_ℓ?
$\geq (1 - 7 \cdot 10^{-7})\Delta$	$\lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$	Reed 98	N
	$(1-\varepsilon)(\Delta+1)+\varepsilon\omega$, $\varepsilon = 20,000^{-1}$	Reed 98	N

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil.$$

ω	$\chi \leq$	proved by:	χ_ℓ?
$\geq (1 - 7 \cdot 10^{-7})\Delta$	$\lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$	Reed 98	N
	$(1-\varepsilon)(\Delta+1)+\varepsilon\omega$, $\varepsilon = 830^{-1}$	Bruhn & Joos 15+	N

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil.$$

ω	$\chi \leq$	proved by:	χ_ℓ?
$\geq (1 - 7 \cdot 10^{-7})\Delta$	$\lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$	Reed 98	N
	$(1-\varepsilon)(\Delta+1) + \varepsilon\omega$, $\varepsilon = 26^{-1}$	Bonamy, Perrett & Postle 16+	N

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil .$$

ω	$\chi \leq$	proved by:	χ_ℓ?
$\geq (1 - 7 \cdot 10^{-7})\Delta$	$\lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$	Reed 98	N
	$(1-\varepsilon)(\Delta+1)+\varepsilon\omega$, $\varepsilon = 13^{-1}$	Delcourt & Postle 17+	Y

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil .$$

ω	$\chi \leq$	proved by:	χ_ℓ?
$\geq (1 - 7 \cdot 10^{-7})\Delta$	$\lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$	Reed 98	N
	$(1-\varepsilon)(\Delta+1)+\varepsilon\omega$, $\varepsilon = 13^{-1}$	Delcourt & Postle 17+	Y
$\leq r$	$O(r)\frac{\Delta \ln\ln\Delta}{\ln\Delta}$	Johansson	Y

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil.$$

ω	$\chi \leq$	proved by:	χ_ℓ?
$\geq (1 - 7 \cdot 10^{-7})\Delta$	$\lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$	Reed 98	N
	$(1-\varepsilon)(\Delta+1)+\varepsilon\omega$, $\varepsilon = 13^{-1}$	Delcourt & Postle 17+	Y
$\leq r$	$O(r)\frac{\Delta \ln\ln \Delta}{\ln \Delta}$	Molloy 17+	Y

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil .$$

ω	$\chi \leq$	proved by:	χ_ℓ?
$\geq (1 - 7 \cdot 10^{-7})\Delta$	$\lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$	Reed 98	N
	$(1-\varepsilon)(\Delta+1) + \varepsilon\omega$, $\varepsilon = 13^{-1}$	Delcourt & Postle 17+	Y
$\leq r$	$O(r)\frac{\Delta \ln\ln\Delta}{\ln\Delta}$	Molloy 17+	Y
$= 2$	$O(1)\frac{\Delta}{\ln\Delta}$	Johansson	Y

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil.$$

ω	$\chi \leq$	proved by:	χ_ℓ?
$\geq (1 - 7 \cdot 10^{-7})\Delta$	$\lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$	Reed 98	N
	$(1-\varepsilon)(\Delta+1)+\varepsilon\omega$, $\varepsilon = 13^{-1}$	Delcourt & Postle 17+	Y
$\leq r$	$O(r)\frac{\Delta \ln\ln\Delta}{\ln\Delta}$	Molloy 17+	Y
$= 2$	$(67 + o(1))\frac{\Delta}{\ln\Delta}$	Jamall 11	Y

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil.$$

ω	$\chi \leq$	proved by:	χ_ℓ?
$\geq (1 - 7 \cdot 10^{-7})\Delta$	$\lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$	Reed 98	N
	$(1-\varepsilon)(\Delta+1)+\varepsilon\omega$, $\varepsilon = 13^{-1}$	Delcourt & Postle 17+	Y
$\leq r$	$O(r)\frac{\Delta \ln\ln \Delta}{\ln \Delta}$	Molloy 17+	Y
$= 2$	$(4 + o(1))\frac{\Delta}{\ln \Delta}$	Pettie & Su 15	Y

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil.$$

ω	$\chi \leq$	proved by:	χ_ℓ?
$\geq (1 - 7 \cdot 10^{-7})\Delta$	$\lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$	Reed 98	N
	$(1-\varepsilon)(\Delta+1)+\varepsilon\omega$, $\varepsilon = 13^{-1}$	Delcourt & Postle 17+	Y
$\leq r$	$O(r)\frac{\Delta \ln\ln\Delta}{\ln\Delta}$	Molloy 17+	Y
$= 2$	$(1+o(1))\frac{\Delta}{\ln\Delta}$	Molloy 17+	Y

The Local Paradigm

Theorem (Erdős, Rubin, Taylor, 1979)

Every connected graph G is L-colorable if $|L(v)| \geq d(v)$ for all $v \in V(G)$, unless every block of G is a clique or odd cycle.

The Local Paradigm

Theorem (Erdős, Rubin, Taylor, 1979)

Every connected graph G is L-colorable if $|L(v)| \geq d(v)$ for all $v \in V(G)$, unless every block of G is a clique or odd cycle.

Conjecture (Local Version of Reed's)

Every graph G is L-colorable if $|L(v)| \geq \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil$ for every $v \in V(G)$, where $\omega(v) = \omega(G[N(v) \cup \{v\}])$.

The Local Paradigm

Theorem (Erdős, Rubin, Taylor, 1979)

Every connected graph G is L-colorable if $|L(v)| \geq d(v)$ for all $v \in V(G)$, unless every block of G is a clique or odd cycle.

Conjecture (Local Version of Reed's)

Every graph G is L-colorable if $|L(v)| \geq \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil$ for every $v \in V(G)$, where $\omega(v) = \omega(G[N(v) \cup \{v\}])$.

Our main result:

Theorem (K, Postle (2017+))

For $\varepsilon \leq \frac{1}{52}$, if $\Delta(G)$ sufficiently large and for all $v \in V(G)$,

1. $|L(v)| \geq (1 - \varepsilon)(d(v) + 1) + \varepsilon\omega(v)$, *and*
2. $|L(v)| - \omega(v) \geq \log^{14}(\Delta(G))$,

then G has an L-coloring.

An Application

King's Conjecture (2009)

For every graph G,

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil.$$

An Application

King's Conjecture (2009)

For every graph G,

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil.$$

- In 2000, McDiarmid observed King's Conjecture is true fractionally.

An Application

King's Conjecture (2009)

For every graph G,

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil.$$

- In 2000, McDiarmid observed King's Conjecture is true fractionally.
- In 2013, Chudnovsky, King, Plumettaz, and Seymour proved King's Conjecture for quasi-line graphs.

An Application

King's Conjecture (2009)

For every graph G,

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil .$$

- In 2000, McDiarmid observed King's Conjecture is true fractionally.
- In 2013, Chudnovsky, King, Plumettaz, and Seymour proved King's Conjecture for quasi-line graphs.
- In 2015, King and Reed proved Reed's Conjecture for claw-free graphs and King's Conjecture for claw-free graphs with a three-colorable complement.

An Application

King's Conjecture (2009)

For every graph G,

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil .$$

Our result implies

Corollary (K,P)

Let $\varepsilon \leq \frac{1}{52}$. If $\Delta(G)$ sufficiently large and $\omega(G) \leq (1 - \varepsilon - o(1))\Delta(G)$, then

$$\chi_\ell(G) \leq \max_{v \in V(G)} (1 - \varepsilon)(d(v) + 1) + \varepsilon\omega(v).$$

A Naive Random Coloring Procedure

Let L be a list-assignment for a graph G, and set $L' = L$. For each $v \in V(G)$:

A Naive Random Coloring Procedure

Let L be a list-assignment for a graph G, and set $L' = L$. For each $v \in V(G)$:

1. Choose $\phi(v)$ uniformly at random from $L(v)$.

A Naive Random Coloring Procedure

Let L be a list-assignment for a graph G, and set $L' = L$. For each $v \in V(G)$:

1. Choose $\phi(v)$ uniformly at random from $L(v)$.
2. If $\exists u \in N(v)$ such that $|L(u)| \leq |L(v)|$ and $\phi(u) = \phi(v)$, "uncolor" v.

A Naive Random Coloring Procedure

Let L be a list-assignment for a graph G, and set $L' = L$. For each $v \in V(G)$:

1. Choose $\phi(v)$ uniformly at random from $L(v)$.
2. If $\exists u \in N(v)$ such that $|L(u)| \leq |L(v)|$ and $\phi(u) = \phi(v)$, "uncolor" v.
3. For each colored $u \in N(v)$, remove $\phi(u)$ from $L'(v)$.

A Naive Random Coloring Procedure

Let L be a list-assignment for a graph G, and set $L' = L$. For each $v \in V(G)$:

1. Choose $\phi(v)$ uniformly at random from $L(v)$.
2. If $\exists u \in N(v)$ such that $|L(u)| \leq |L(v)|$ and $\phi(u) = \phi(v)$, "uncolor" v.
3. For each colored $u \in N(v)$, remove $\phi(u)$ from $L'(v)$.

Let G' be the graph induced by G on "uncolored" vertices.

A Naive Random Coloring Procedure

Let L be a list-assignment for a graph G, and set $L' = L$. For each $v \in V(G)$:

1. Choose $\phi(v)$ uniformly at random from $L(v)$.
2. If $\exists u \in N(v)$ such that $|L(u)| \leq |L(v)|$ and $\phi(u) = \phi(v)$, "uncolor" v.
3. For each colored $u \in N(v)$, remove $\phi(u)$ from $L'(v)$.

Let G' be the graph induced by G on "uncolored" vertices.

Proposition: If G' is L'-colorable with non-zero probability, then G is L-colorable.

A Naive Random Coloring Procedure

Let L be a list-assignment for a graph G, and set $L' = L$. For each $v \in V(G)$:

1. Choose $\phi(v)$ uniformly at random from $L(v)$.
2. If $\exists u \in N(v)$ such that $|L(u)| \leq |L(v)|$ and $\phi(u) = \phi(v)$, "uncolor" v.
3. For each colored $u \in N(v)$, remove $\phi(u)$ from $L'(v)$.

Let G' be the graph induced by G on "uncolored" vertices.

Proposition: If G' is L'-colorable with non-zero probability, then G is L-colorable.

Key Fact: Every vertex is colored with some constant probability!

A Naive Random Coloring Procedure

Let L be a list-assignment for a graph G, and set $L' = L$. For each $v \in V(G)$:

1. Choose $\phi(v)$ uniformly at random from $L(v)$.
2. If $\exists u \in N(v)$ such that $|L(u)| \leq |L(v)|$ and $\phi(u) = \phi(v)$, "uncolor" v.
3. For each colored $u \in N(v)$, remove $\phi(u)$ from $L'(v)$.

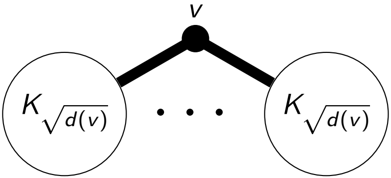
Let G' be the graph induced by G on "uncolored" vertices.

Proposition: If G' is L'-colorable with non-zero probability, then G is L-colorable.

Key Fact: Every vertex is colored with some constant probability!

Ideally we would show that in some instance of G' and L', for each $v \in V(G')$, $|L'(v)| > d_{G'}(v)$, but this doesn't work.

Local Difficulties

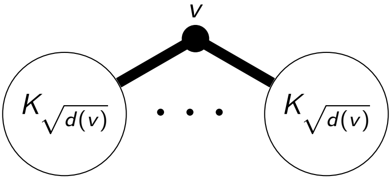


$$L = \{1, \ldots, \sqrt{d(v)}\}, \quad \{d(v) - \sqrt{d(v)}, \ldots, d(v)\}$$

A pathological case:

$$d_{G'}(v) - |L'(v)| \approx d_G(v) - |L(v)|.$$

Local Difficulties



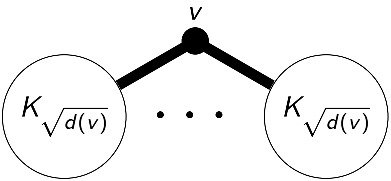
$$L = \{1, \ldots, \sqrt{d(v)}\}, \quad \{d(v) - \sqrt{d(v)}, \ldots, d(v)\}$$

A pathological case:

$$d_{G'}(v) - |L'(v)| \approx d_G(v) - |L(v)|.$$

Let $v \in V(G)$ and $u \in N(v)$. We say u is

Local Difficulties



$$L = \{1, \ldots, \sqrt{d(v)}\}, \quad \{d(v) - \sqrt{d(v)}, \ldots, d(v)\}$$

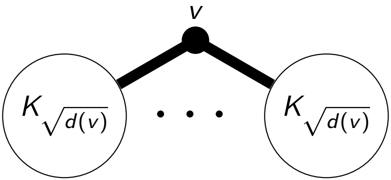
A pathological case:

$$d_{G'}(v) - |L'(v)| \approx d_G(v) - |L(v)|.$$

Let $v \in V(G)$ and $u \in N(v)$. We say u is

- a subservient neighbor of v if $|L(u)| < |L(v)|$,

Local Difficulties



$$L = \{1, \ldots, \sqrt{d(v)}\}, \quad \{d(v) - \sqrt{d(v)}, \ldots, d(v)\}$$

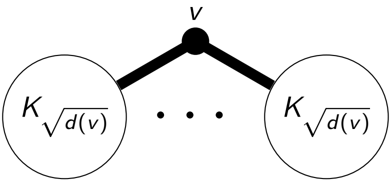
A pathological case:

$$d_{G'}(v) - |L'(v)| \approx d_G(v) - |L(v)|.$$

Let $v \in V(G)$ and $u \in N(v)$. We say u is

- a subservient neighbor of v if $|L(u)| < |L(v)|$,
- an egalitarian neighbor of v if $|L(u)| \in [|L(v)|, 1.4|L(v)|]$.

Local Difficulties



$$L = \{1, \ldots, \sqrt{d(v)}\}, \quad \{d(v) - \sqrt{d(v)}, \ldots, d(v)\}$$

A pathological case:

$$d_{G'}(v) - |L'(v)| \approx d_G(v) - |L(v)|.$$

Let $v \in V(G)$ and $u \in N(v)$. We say u is

- a subservient neighbor of v if $|L(u)| < |L(v)|$,
- an egalitarian neighbor of v if $|L(u)| \in [|L(v)|, 1.4|L(v)|]$.

Say v is lordly if v has many subservient neighbors.

New Ways to Save

To deal with lordly vertices, color G' in the order of original list size, i.e. color vertices before their subservient neighbors.

New Ways to Save

To deal with lordly vertices, color G' in the order of original list size, i.e. color vertices before their subservient neighbors.

We say a vertex v is:

New Ways to Save

To deal with lordly vertices, color G' in the order of original list size, i.e. color vertices before their subservient neighbors.

We say a vertex v is:

- egalitarian-sparse if it has many non-edges between egalitarian neighbors

New Ways to Save

To deal with lordly vertices, color G' in the order of original list size, i.e. color vertices before their subservient neighbors.

We say a vertex v is:

- egalitarian-sparse if it has many non-edges between egalitarian neighbors
- aberrant if it has many neighbors u with $L(u)$ significantly different from $L(v)$:
  - some neighbors with much bigger list, or
  - many neighbors with somewhat bigger list.

New Ways to Save

To deal with lordly vertices, color G' in the order of original list size, i.e. color vertices before their subservient neighbors.

We say a vertex v is:

- egalitarian-sparse if it has many non-edges between egalitarian neighbors
- aberrant if it has many neighbors u with $L(u)$ significantly different from $L(v)$:
  - some neighbors with much bigger list, or
  - many neighbors with somewhat bigger list.

Key Fact: If a vertex is lordly, egalitarian-sparse, or aberrant, the expected number of "saved" colors is $\Omega(d(v) - \omega(v))$.

Proof Overview

Theorem (K, Postle (2017+))

For $\varepsilon \leq \frac{1}{52}$, *if* Δ *sufficiently large,* $\Delta(G) \leq \Delta$, *and for all* $v \in V(G)$,

1. $|L(v)| \geq (1 - \varepsilon)(d(v) + 1) + \varepsilon\omega(v)$, *and*
2. $|L(v)| - \omega(v) \geq \log^{14}(\Delta)$,

then G *has an* L*-coloring.*

Proof Overview

Theorem (K, Postle (2017+))

For $\varepsilon \leq \frac{1}{52}$, if Δ sufficiently large, $\Delta(G) \leq \Delta$, and for all $v \in V(G)$,

1. $|L(v)| \geq (1-\varepsilon)(d(v)+1) + \varepsilon\omega(v)$, *and*
2. $|L(v)| - \omega(v) \geq \log^{14}(\Delta)$,

then G has an L-coloring.

Main Structural Lemma

Every vertex of a minimum counterexample is either aberrant, lordly, or egalitarian-sparse.

Proof Overview

Theorem (K, Postle (2017+))

For $\varepsilon \leq \frac{1}{52}$, if Δ sufficiently large, $\Delta(G) \leq \Delta$, and for all $v \in V(G)$,

1. $|L(v)| \geq (1-\varepsilon)(d(v)+1) + \varepsilon\omega(v)$, *and*
2. $|L(v)| - \omega(v) \geq \log^{14}(\Delta)$,

then G has an L-coloring.

Main Structural Lemma

Every vertex of a minimum counterexample is either aberrant, lordly, or egalitarian-sparse.

Main Probabilistic Lemma

If a vertex v is either aberrant, lordly, or egalitarian-sparse, then with high enough probability,

$$|L'(v)| > |\{u \in N(v) \cap V(G') : |L(u)| \geq |L(v)|\}|.$$

Thanks!